Regularity of free boundaries in obstacle problems

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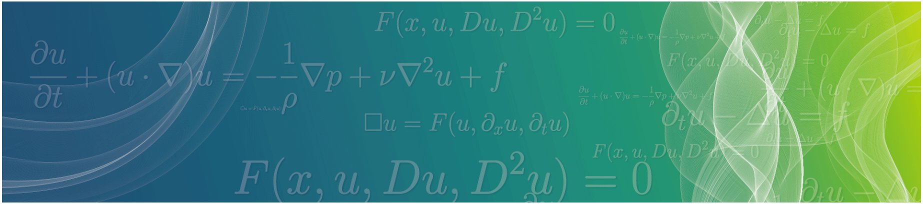


Regularity theory for elliptic PDEs

"Are all solutions to a given PDE smooth, or they may have singularities?"

Hilbert XIX problem

- We consider minimizers u of convex functionals in $\Omega \subset \mathbb{R}^n$

$$\mathcal{E}(u) := \int_\Omega L(\nabla u)dx, \qquad\qquad u = g \ \text{ on } \ \partial\Omega$$

- The Euler-Lagrange equation of this problem is a *nonlinear elliptic PDE*.
- Question (Hilbert, 1900): If L is smooth and uniformly convex, is $u \in C^\infty$?
- First results (1920's and 1940's): If $u \in C^1$ then $u \in C^\infty$
- De Giorgi - Nash (1956-1957): YES, u is always C^1 ! (and hence C^∞)

Fully nonlinear elliptic PDEs

$$\boxed{F(D^2u) = 0} \qquad \text{or, more generally,} \qquad F(D^2u, \nabla u, u, x) = 0$$

- Question: If F is smooth and uniformly elliptic, is $u \in C^\infty$?
- First results (1930's and 1950's): If $u \in C^2$ then $u \in C^\infty$
- Dimension $n = 2$ (Nirenberg, 1953): In $\mathbb{R}^2$, u is always C^2 (and hence C^∞)
- Krylov-Safonov (1979): u is always C^1
- Evans - Krylov (1982): If F is *convex*, then u is always C^2 (and hence C^∞)
- Counterexamples (Nadirashvili-Vladut, 2008-2012): In dimensions $n \geq 5$, there are solutions that are **not** C^2 !
- OPEN PROBLEM: What happens in $\mathbb{R}^3$ and $\mathbb{R}^4$?

What are free boundary problems?

- Any PDE problem that exhibits apriori unknown (free) interfaces or boundaries
- They appear in Physics, Industry, Finance, Biology, and other areas

  - Most classical example:

    Stefan problem (1831)

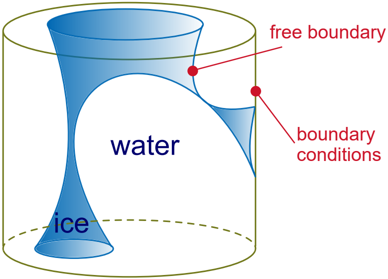
    It describes the melting of ice.

  - If $\theta(t, x)$ denotes the temperature,

$$\theta_t = \Delta\theta \quad \text{in} \quad \{\theta > 0\}$$

  - Free boundary determined by:

$$|\nabla_x \theta|^2 = \theta_t \quad \text{on} \quad \partial\{\theta > 0\}$$



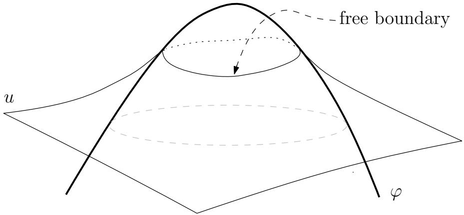
Another important free boundary problem

The obstacle problem

Given $\varphi \in C^\infty$, minimize

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx$$

with the constraint $\boxed{u \geq \varphi}$



The obstacle problem is

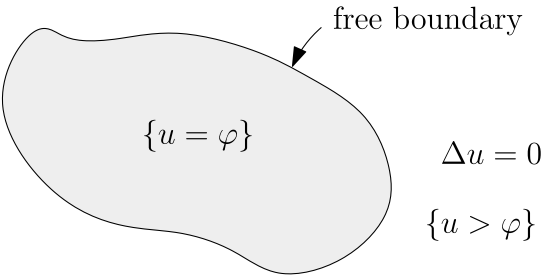
$$\boxed{\begin{cases} u \geq \varphi & \text{in} \quad \Omega \\ \Delta u = 0 & \text{in} \quad \{x \in \Omega \,:\, u > \varphi\} \\ \nabla u = \nabla \varphi & \text{on} \quad \partial\{u > \varphi\}, \end{cases}}$$

(usually with boundary conditions $u = g$ on $\partial\Omega$)

$$\begin{cases} u \geq \varphi & \text{in} \quad \Omega, \\ \Delta u = 0 & \text{in} \quad \{x \in \Omega : u > \varphi\} \\ \nabla u = \nabla \varphi & \text{on} \quad \partial\{u > \varphi\}. \end{cases}$$

Unknowns: solution u & the contact set $\{u = \varphi\}$

The free boundary (FB) is the boundary $\partial\{u > \varphi\}$



Free boundary problems

- Various free boundary problems appear in Physics, Industry, Finance, Biology, and other areas
- in Sciences: Fluid mechanics; elasticity; pricing of options; interacting particle systems, etc.
- in Mathematics: Optimal stopping (Probability), Quadrature domains (Complex Analysis, Potential Theory), Random matrices, Minimal surfaces (Geometry), etc.
- All these examples give rise to the obstacle problem or Stefan problem !
- Moreover, Stefan problem $\longleftrightarrow$ (evolutionary) obstacle problem !
- Thus, we want to understand better the obstacle problem

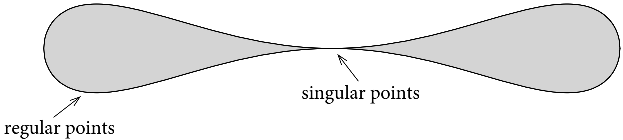
The obstacle problem

Fundamental question:

Is the Free Boundary smooth?

- First results (1960's & 1970's): Solutions u are $C^{1,1}$, and this is optimal.
- Kinderlehrer-Nirenberg (1977): If the FB is C^1, then it is C^∞
- Caffarelli (Acta Math. 1977): The FB is C^1 (and thus C^∞),

  possibly outside a certain set of singular points



- Similar results hold for the Stefan problem

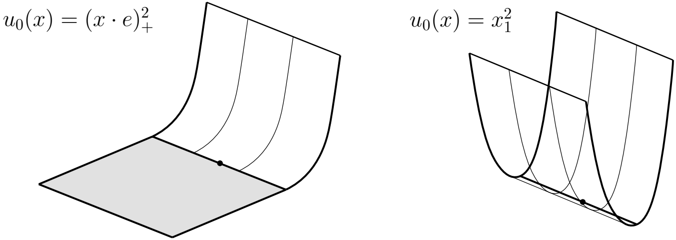
To study the regularity of the FB, one considers blow-ups

$$u_r(x) := \frac{(u-\varphi)(x_0+rx)}{r^2} \longrightarrow u_0(x) \qquad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n)$$

The key difficulty is to **classify blow-ups**:

$$\text{regular point} \quad \Longrightarrow \quad u_0(x) = (x\cdot e)_+^2 \quad \text{(1D solution)}$$

$$\text{singular point} \quad \Longrightarrow \quad u_0(x) = \sum \lambda_i x_i^2 \quad \text{(paraboloid)}$$



$$\text{regular point} \quad \Longrightarrow \quad u_0(x) = (x \cdot e)_+^2 \quad \text{(1D solution)}$$

$$\text{singular point} \quad \Longrightarrow \quad u_0(x) = \sum \lambda_i x_i^2 \quad \text{(paraboloid)}$$

Finally, once the blow-ups are classified, we transfer the information from u_0 to u , and prove that the free boundary is C^1 near regular points.

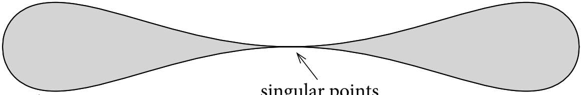
- This strategy is very related to the study of minimal surfaces in $\mathbb{R}^n$!
- In minimal surfaces, **blow-ups** are **cones**

$$\frac{1}{r}E \longrightarrow E_0 \ \text{(cone)} \quad \text{as} \quad r \to 0$$

- Area-minimizing cones are flat (half-spaces) up to dimension $n \leq 7$ (Simons, 1968)
- Minimal surfaces are smooth in dimensions $n \leq 7$ (De Giorgi 1961 + Simons 1968)

Singular points

Question: What can one say about singular points?



- Schaeffer (1970's): Several (quite ugly!) examples
- Caffarelli (1998): Singular points are contained in a $(n-1)$-dimensional C^1 manifold.

  Moreover, at each singular point x_0 we have

$$u(x) - \varphi(x) = p(x) + o(|x - x_0|^2)$$

- Weiss (1999): In dimension $n = 2$, singular points are contained in a $C^{1,\alpha}$ manifold.
- Figalli-Serra (2017): Outside a small set of lower dimension, singular points are contained in a $C^{1,1}$ manifold.

Open problems

Open problems in the field

- It is a very active area of research, with several open questions, generalizations of the obstacle problem, etc.
- Important open problem in the field: prove **generic regularity**
- This is an open problem in many nonlinear PDE's

Conjecture (Schaeffer 1974)

For generic obstacles, the free boundary in the obstacle problem is C^∞ (with no singular points).

- Theorem (Monneau 2002): True in $\mathbb{R}^2$!
- For minimal surfaces: Similar result valid in $\mathbb{R}^8$ (Smale 1993)
- Nothing known in higher dimensions!

In a forthcoming work, we prove the following:

Theorem (Figalli-R.-Serra '18)

Let u_λ be the solution to the obstacle problem in $\mathbb{R}^3$, with obstacle $\varphi + \lambda$.

Then, for almost every constant λ, the free boundary is C^∞ (with no singular points).

- This proves the Conjecture in $\mathbb{R}^3$!
- In fact, we can take $\varphi + \lambda\Psi$ $(\Psi > 0)$, and for a.e. λ there are no singular points.
- What happens in higher dimensions?

Theorem (Figalli-R.-Serra '18)

Let u_λ be the solution to the obstacle problem in $\mathbb{R}^n$, with obstacle $\varphi + \lambda$.

Then, for almost every λ, the singular set has Hausdorff dimension (at most) $n - 4$.

For almost every obstacle, the singular set is very small!

Stefan problem

- We also prove a related result for the evolutionary obstacle problem!
- That is, we study the generic regularity in the Stefan problem.

Theorem (Figalli-R.-Serra '18)

Let $u(t, x)$ be the solution to the Stefan problem in $\mathbb{R}^3$.

Then, for almost every time t, the free boundary is C^∞ (with no singular points).

Furthermore, the set of "singular times" has Hausdorff dimension $\leq \frac{2}{3}$.

- This result is new even in $\mathbb{R}^2$!
- More or less: "When ice melts, its does not create too many singularities"

Thank you!