UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Assignatura:

Estudiant/a: [redacted]

Data:

$4+6=10$

Full 1 de 5

QUALIFICACIÓ
4

Teoria

1. Segons el enunciat, rnorm(m) permet generar una mostra pseudo-aleatoria de m observacions N(0,1)

(a) Recordem que definim χ^2_K com la variable aleatòria de la suma ~~del quadrat~~ de K normals N(0,1) independents entre sí al quadrat

És a dir $\chi^2_K = \sum_{i=1}^{K} Z_i^2$ on $Z_i \sim N(0,1)$ i independents.

Llavors, el codi seria (per una observació)

$\boxed{\texttt{chisq\_K[i] = sum(rnorm(K)^2)}}$ $i=1:n$

ja que calcula la suma del quadrat de K normals N(0,1) independents entre sí. Per n observacions repetim el procés n vegades.

(b) Recordem que t_K és definida com la variable aleatòria de

$$t_K = \frac{Z}{\sqrt{U/K}}$$ on $Z \sim N(0,1)$ i $U \sim \chi^2_n$. independents entre si.

Llavors, el codi seria (per una observació) (fent servir anterior)

$\boxed{\texttt{t\_K = rnorm(1)/sqrt(sum(rnorm(K)^2)/K)}}$

Per n observacions, repetim el procés n vegades

```
t_k[i] = rnorm(1)/sqrt(sum(rnorm(k)^2)/k)
```

i = 1:n.

(c) Finalment, recordem que definim F_{k_1,k_2} com la variable aleatòria

$$F_{k_1,k_2} = \frac{U/k_1}{V/k_2} \quad \text{on } U \sim \chi^2_{k_1},\ V \sim \chi^2_{k_2} \text{ independents.}$$

Llavors, el codi seria.

```
F_k1-k2[i] = (sum(rnorm(k1)^2)/k1)/(sum(rnorm(k2)^2)/k2)
```

per i = 1:n, ja que es fa la divisió de dos χ^2 entre els seus graus.

Teoria 2.

$$X \sim N(0, \sigma^2) \qquad f(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

(a). Tenim el prior

$$\pi(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}$$

Volem calcular el posterior. $\pi(\sigma^2|\underline{x})$

$$\left[f(\underline{x}|\sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}} \right]$$

Sabem que el posterior és (T. Bayes)

$$\pi(\sigma^2|\underline{x}) = \frac{f(\underline{x}, \sigma^2)}{f(\underline{x})} = \frac{f(\underline{x}|\sigma^2)\,\pi(\sigma^2)}{f(\underline{x})}$$

Anem a calcular el producte

$$f(\underline{x}|\sigma^2)\,\pi(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{(2\pi)^{n/2}} (\sigma^2)^{-\alpha-1-n/2} e^{-\left(\beta + \frac{\sum_{i=1}^n x_i^2}{2}\right)/\sigma^2}$$

Veiem que

$$\pi(\sigma^2|\underline{x}) \propto C(\underline{x}) \cdot (\sigma^2)^{-\alpha-1-n/2} e^{-\left(\beta + \frac{\sum^n x_i^2}{2}\right)/\sigma^2}$$

llavors, sabem que

$$\boxed{\pi(\sigma^2|\underline{x}) \sim IG\left(\alpha + n/2,\ \beta + \frac{\sum_{i=1}^n x_i^2}{2}\right)} \quad (1)$$

$\Rightarrow$ la distribució inversa ~~conjugada~~ Gamma es prior conjugada per $N(0,\sigma^2)$ ✓

(b) Sabem (1), anem a treure la densitat.

$$\boxed{\pi(\sigma^2|\underline{x}) = \frac{\left(\beta + \left(\sum_{i=1}^n x_i^2/2\right)\right)^{\alpha+n/2}}{\Gamma(\alpha + n/2)} (\sigma^2)^{-\alpha-n/2-1} e^{-\left(\beta + \left(\sum_{i=1}^n x_i^2/2\right)\right)/\sigma^2}}$$

a través de la fòrmula de $IG(\alpha,\beta)$ ✓

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Assignatura:

Estudiant/a: [redacted]

Data:

Tecria 3.

$X_1, \dots X_n$ m.a.s. $X \sim Po(\lambda)$.

$$\hat{\lambda} = K \sum_{i=1}^{n} i X_i$$

(a) $$\mathbb{E}[\hat{\lambda}] = K\,\mathbb{E}\left[\sum_{i=1}^{n} i X_i\right] = K \sum_{i=1}^{n} i\, \overbrace{\mathbb{E}[X_i]}^{\lambda}.$$

$$= K\lambda \sum_{i=1}^{n} i = K\lambda \frac{n(n+1)}{2}.$$

Si volem que no tingui biaix.

$$\mathbb{E}[\hat{\lambda}] = K\lambda \frac{n(n+1)}{2} = \lambda \Rightarrow \boxed{K = \frac{2}{n(n+1)}}$$

(b). Cal calcular primer la seva variància.

$$\left[\mathrm{Var}[\hat{\lambda}] = K^2 \mathrm{Var}\left[\sum_{i=1}^{n} i X_i\right] \overset{X_i \text{ independents}}{=} K^2 \sum_{i=1}^{n} \mathrm{Var}[i X_i]\right.$$

$$= K^2 \sum_{i=1}^{n} i^2 \underbrace{\mathrm{Var}[X_i]}_{\lambda} = K^2 \lambda \sum_{i=1}^{n} i^2 = K^2 \lambda \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{4}{n^2(n+1)^2} \frac{\cancel{n}\cancel{(n+1)}(2n+1)}{6} \lambda^2$$

$$= \frac{2}{3} \frac{(2n+1)}{n(n+1)} \lambda^2$$

- Anem a buscar la $CRLB_n(\lambda)$.

Sabem $P(x|\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$L(\lambda|x) = \frac{e^{-\lambda}\lambda^x}{x!} \Rightarrow \ell(\lambda|x) = -\lambda + x\log(\lambda) - \log(x!)$$

$$\frac{\partial \ell}{\partial \lambda} = -1 + \frac{x}{\lambda} \qquad \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{x}{\lambda^2}$$

$$I_1(\lambda) = +\mathbb{E}\left[\frac{x}{\lambda^2}\right] = \frac{1}{\lambda^2} \cdot \lambda = \frac{1}{\lambda}$$

$$I_n(\lambda) = n\, I_1(\lambda) = \frac{n}{\lambda}$$

$$\Rightarrow \boxed{CRLB_n(\lambda) = \frac{1}{I_n(\lambda)} = \frac{\lambda}{n}}$$

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Assignatura: [redacted]

Data:

Veiem que $\boxed{Var[\hat{\lambda}] > CRLB_n(\lambda)}$

$$\frac{2}{3} \frac{(2n+1)\cancel{\lambda}}{\cancel{n}(n+1)} > \frac{\cancel{\lambda}}{\cancel{n}}$$

$$4n+2 > 3n+3$$

$$\boxed{n > 1}$$

Llavors, tenim que per $n>1$ no és eficient.

Per $n=1$ tenim $\hat{\lambda} = X_i$ i sí és eficient.

$\boxed{n=1}$

$$\boxed{Var[\hat{\lambda}_1] = \lambda = CRLB_n(\lambda)} \checkmark$$

Teoria 4.

(a) Com la mesura inicial i mesura final no són independents ja que medim ~~sobre~~ la mateixa persona, llavors són dues mostres aparellades, ~~per que~~ (X_i i Y_i són del mateix individu.)

Llavors, farem servir el test de mostres aparellades

on $\boxed{D_i = X_i - Y_i} \sim N(\mu_D, \sigma_D^2)$.

$\Rightarrow \bar{D} \sim N(\mu_D, \sigma_D^2/n)$

$$\Rightarrow \quad T = \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} \sim t_{n-1}$$

~~Les hipòtesis~~ El test consisteix en

$$\begin{cases} H_0: \mu_D \geq 5 \\ H_1: \mu_D < 5 \end{cases}$$ és un test unilateral.

L'estadístic serà $\boxed{T = \frac{\bar{D} - 5}{S_D/\sqrt{n}} \mid H_0 \sim t_{n-1}}$

La regió crítica és

$$RC = \{ T \leq A \}$$ on A és tal que

$$P(T \leq A \mid H_0) = \alpha$$

(b) Anem a calcular T.

$$\left[T = \frac{\bar{D} - 5}{S_D/\sqrt{n}} = \frac{5{,}27 - 5}{0{,}88/\sqrt{15}} = 1{,}1883 \right]$$

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QUALIFICACIÓ

Assignatura:

Estudiant/a: [redacted]

Data:

Anem a calcular el p-valor. $n = 15 - 1 = 14$

$$P(T \leq 1{,}1883 \mid H_0) = P(t_{14} \leq 1{,}1883)$$

$$\geqslant 0{,}75$$

Llavors, p-valor $\geq 0{,}05$ podem concloure que la H_0 és certa i podem afirmar que estadísticament hi ha una reducció ~~de~~ superior a 5 kg.

Amb regió crítica:, en una $t_{n-1} = t_{14}$

tenim $A = t_{14}^{0{,}05} = -t_{14}^{0{,}95} = -1{,}761$

Com $1{,}1883 \geq -1{,}761$ podem acceptar H_0

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Assignatura: [redacted]

Data:

Problema 1.

$$P_\theta(Y=y) = \binom{y-1}{r-1} \theta^r (1-\theta)^{y-r}$$

on $y = r, r+1, r+2, \dots$, $r \in \mathbb{N}$,

$\theta \in (0,1)$

$$E_\theta[Y] = \frac{r}{\theta} \qquad V_\theta[Y] = \frac{r(1-\theta)}{\theta^2}$$

(a) ~~...~~ Com Y_i son independents

$$P_\theta(\underline{Y} = \underline{y}) = \prod_{i=1}^n P_\theta(Y_i = y_i) = \theta^{rn} (1-\theta)^{\sum_{i=1}^n y_i - rn} \prod_{i=1}^n \binom{y_i - 1}{r-1}$$

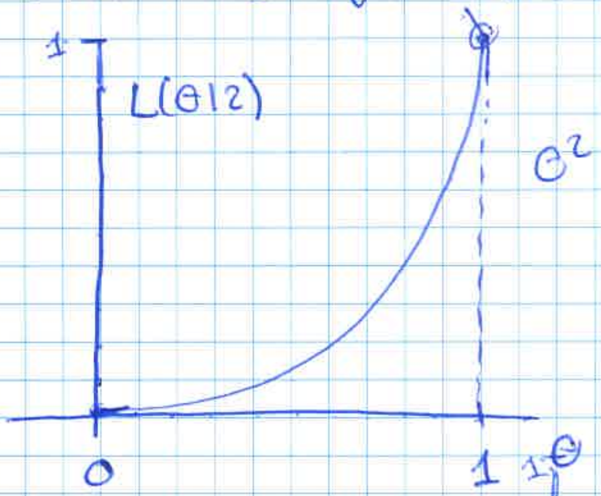
llavors $\boxed{L(\theta | \underline{y}) = \theta^{rn} (1-\theta)^{\sum_{i=1}^n y_i - rn} \prod_{i=1}^n \binom{y_i-1}{r-1}}$

$$\boxed{\ell(\theta|\underline{y}) = \log L(\theta|\underline{y}) = rn \log\theta + \left(\sum_{i=1}^n y_i - rn\right) \log(1-\theta) + \sum_{i=1}^n \log\binom{y_i-1}{r-1}.}$$

on $\boxed{\theta \in (0,1)}$!! recordem per $r=2$ $n=1$ $y=2$

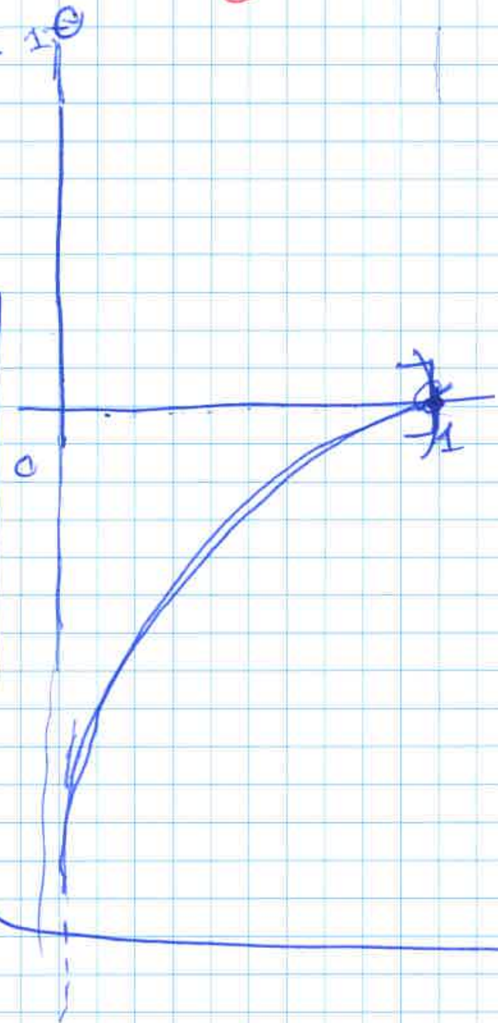
$$L(\theta \mid 2) = \theta^2 (1-\theta)^0 \cdot \binom{2-1}{2-1} = \theta^2$$

Llavors



$\theta \in (0,1)$

$$\ell(\theta \mid 2) = 2 \log \theta.$$



(b)

(i) $E_\theta(Y) = \dfrac{r}{\theta}$

$\Rightarrow m_1 = \bar{y} = \dfrac{r}{\hat{\theta}_1}$

$\Rightarrow \boxed{\hat{\theta}_1 = \dfrac{r}{\bar{y}}}$

(ii) $E_\theta(Y^2) = E_\theta(Y)^2 + V_\theta(Y) = \dfrac{r(1-\theta)}{\theta^2} + \dfrac{r^2}{\theta^2} =$

$= \dfrac{r^2 + r - r\theta}{\theta^2}$

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Data:

$$\Rightarrow \quad m_2 = \overline{y^2} = \frac{r^2 + r - r\hat{\theta}_2}{\hat{\theta}_2^2}$$

$$m_2 \hat{\theta}_2^2 + r\hat{\theta}_2 - r^2 - r = 0.$$

$$\hat{\theta}_2 = \frac{-r \pm \sqrt{r^2 + 4m_2(r^2+r)}}{2m_2}.$$

Com $\theta \in (0,1)$, descartem negatiu.

$$\boxed{\hat{\theta}_2 = \frac{\sqrt{r^2 + 4m_2(r^2+r)} - r}{2m_2}}$$

on $m_2 = \overline{y^2}$

(iii). $V_\theta(Y) = \dfrac{r(1-\theta)}{\theta^2} \Rightarrow m_2' = \dfrac{r(1-\hat{\theta}_3)}{\hat{\theta}_3^2}.$

$$\Rightarrow \quad m_2' \cdot \hat{\theta}_3^2 + r\hat{\theta}_3 - r = 0$$

$$\hat{\theta}_3 = \frac{-r \pm \sqrt{r^2 + 4m_2' r}}{2m_2'}$$

Com $\theta \in (0,1)$,

$\theta > 0$

$$\boxed{\hat{\Theta}_3 = \frac{\sqrt{r^2 + 4 m_2' r} - r}{2 m_2'}}$$

on $m_2' = \frac{1}{n} \sum_{i=1}^{n} (\cancel{x}_i - \bar{\cancel{x}})^2$

- cas $r = 2$ $n = 2$ $Y = 2, 3.$

(i). $\bar{Y} = \frac{2+3}{2} = \frac{5}{2}$ $\boxed{\hat{\Theta}_1 = \frac{2}{5/2} = \frac{4}{5}}$

(ii). $m_2 = \overline{Y^2} = \frac{4+9}{2} = \frac{13}{2}$

$$\left[\hat{\Theta}_2 = \frac{\sqrt{4 + 4\frac{13}{2}(4+2)} - 2}{\cancel{2} \cdot \frac{13}{\cancel{2}}}\right.$$

$$= \left.\frac{\sqrt{4+156} - 2}{13} = 0,81916\right]$$

(iii)

$$\left[m_2' = \frac{1}{2}\left((2-2,5)^2 + (3-2,5)^2\right) = 0,25\right]$$

$$\left[\hat{\Theta}_3 = \frac{\sqrt{4 + 4 \cdot 1/4 \cdot 2} - 2}{2 \cdot 0,25} = \frac{\sqrt{6} - 2}{0,5} = 0,89898\right]$$

Assignatura:

Estudiant/a: [redacted]

Data:

(c). Anem a busca el màxim de la versemblança.

$$\frac{\partial \ell(\theta|y)}{\partial\theta} = \frac{rn}{\theta} - \left(\left(\sum_{i=1}^{n} y_i\right) - rn\right)\frac{1}{1-\theta}$$

$$\Rightarrow \frac{rn}{\hat{\theta}_{MV}} = \left(\left(\sum_{i=1}^{n} y_i\right) - rn\right)\frac{1}{1-\hat{\theta}_{MV}}$$

$$(1-\hat{\theta}_{MV}) = \left(\frac{\sum_{i=1}^{n} y_i}{rn} - 1\right)\hat{\theta}_{MV}$$

$$\hat{\theta}_{MV} \frac{\sum_{i=1}^{n} y_i}{rn} = 1 \qquad \sum_{i=1}^{n} y_i = \frac{rn}{\hat{\theta}_{MV}}$$

$$\left[\hat{\theta}_{MV} = \frac{rn}{\sum_{i=1}^{n} y_i} = \frac{r}{\bar{Y}}\right] \qquad \hat{\theta}_{MV} \in (0,1)$$

(ens dona igual el de moments amb $E_\theta(Y)$)

És màxim?

$$\frac{\partial^2 \ell}{\partial\theta^2} = \frac{-rn}{\theta^2} - \left(\sum_{i=1}^{n} y_i - rn\right)\frac{1}{(1-\theta)^2}$$

$$\Rightarrow \frac{-rn}{\hat{\theta}_{MV}^{2}} - \left(\frac{rn}{\hat{\theta}_{MV}} - rn\right)\frac{1}{(1-\hat{\theta}_{MV})^{2}}$$

$$\|$$

$$\frac{-rn}{\hat{\theta}^{2}_{MV}} - \frac{rn}{\hat{\theta}_{MV}}\,\frac{\cancel{(1-\hat{\theta}_{MV})}}{(1-\hat{\theta}_{MV})^{\cancel{2}}}$$

$$\|$$

$$-\frac{rn}{\hat{\theta}_{MV}}\left(+\frac{1}{\hat{\theta}_{MV}} + \frac{1}{1-\hat{\theta}_{MV}}\right) = \frac{-rn}{\underset{>0}{\hat{\theta}_{MV}}}\;\frac{1}{\underset{>0}{\hat{\theta}_{MV}}\,(\underset{>0}{1-\hat{\theta}_{MV}})}$$

$-1+\hat{\theta}_{MV}+\hat{\theta}_{MV}$

$? < 0$

llavors és màxim ✓

és ML: $\boxed{\hat{\theta}_{MV} = \frac{r}{\bar{y}}}$

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Assignatura:

Estudiant/a: [redacted]

Data:

Anem a calcular la $CRLB_n(\theta)$.

Recordem $\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{rn}{\theta^2} - \left(\sum_{i=1}^{n} y_i - rn\right)\frac{1}{(1-\theta)^2}$

$$\left[I_n(\theta) = -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right] \right] = +\left(+\frac{rn}{\theta^2} + \frac{1}{(1-\theta)^2}\left(\frac{nr}{\theta} - nr\right)\right)$$

on $\left[E\left[\sum_{i=1}^{n} y_i\right] = \frac{nr}{\theta}\right]$

$$= +\left(\frac{rn}{\theta^2} + \frac{nr}{(1-\theta)^2\theta}\cancel{(1-\theta)}\right) =$$

$$= \left(\frac{rn}{\theta^2} + \frac{nr}{(1-\theta)\theta}\right) = \frac{nr}{\theta}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)$$

$$= \frac{nr}{\theta}\left(\frac{1-\theta - \theta}{\theta(1-\theta)}\right) = \frac{nr}{\theta^2}\,\frac{\cancel{\quad}}{(1-\theta)}$$

Llavors $\boxed{CRLB_n(\theta) = \frac{\theta^2(1-\theta)}{nr} = \frac{1}{I_n(\theta)}}$

Per tant, la distribució assimptòtica serà (fent servir la que sabem del ML)

$$\boxed{\hat{\Theta}_{MV} \xrightarrow[n\to\infty]{} N\left(\theta, \frac{\theta^2(1-\theta)}{nr}\right)}$$

o

$$\boxed{\frac{\hat{\theta}_{MV} - \theta}{\theta\sqrt{1-\theta}}\sqrt{nr} \;\xrightarrow[n\to\infty]{}\; N(0,1).}$$

(d). Si n prou gran, podem assumir:

$$\hat{\Theta}_{MV} = \frac{r}{\bar{y}} \sim N\left(\theta, \frac{\theta^2(1-\theta)}{nr}\right)$$

$$\cancel{\frac{\frac{r}{\bar{y}} - \theta}{\theta\sqrt{1-\theta}}\sqrt{nr} \sim N(0,1)}.$$

Fem un nivell aproximat llavors podem assumir

$$\hat{\Theta}_{MV} = \frac{r}{\bar{y}} \sim N\left(\theta, \frac{\hat{\theta}^2_{MV}(1-\hat{\theta}_{MV})}{nr}\right)$$

$$=$$

Fem servir l'aproximació de la variància per el MV.

$$N\left(\theta, \underbrace{\frac{\frac{r^2}{\bar{y}^2}\left(1-\frac{r}{\bar{y}}\right)}{nr}}_{=\sigma^2}\right)$$

Assignatura: [redacted]
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Llavors on $\sigma^2 = \dfrac{r}{n\bar{y}^2}\left(\dfrac{\bar{y}-r}{\bar{y}}\right)$

$$\frac{\frac{r}{\bar{y}} - \theta}{\sigma} \sim N(0,1),$$

Per tant,

$$P\left(Z^{\alpha/2} \le \frac{\frac{r}{\bar{y}} - \theta}{\sigma} \le Z^{1-\alpha/2}\right) = 1-\alpha$$

$$\parallel$$

$$P\left(\sigma Z^{\alpha/2} \le \frac{r}{\bar{y}} - \theta \le \sigma Z^{1-\alpha/2}\right) = 1-\alpha$$

$$\parallel$$

$$P\left(\frac{r}{\bar{y}} - \sigma Z^{1-\alpha/2} \le \theta \le \frac{r}{\bar{y}} - \sigma Z^{\alpha/2}\right)$$

$$\Rightarrow \boxed{IC_{1-\alpha}(\theta) = \left(\frac{r}{\bar{y}} - \sigma Z^{1-\alpha/2}, \frac{r}{\bar{y}} + \sigma Z^{1-\alpha/2}\right) = \left(\frac{r}{\bar{y}} - \frac{r}{n\bar{y}^2}\left(\frac{\bar{y}-r}{\bar{y}}\right) Z^{1-\alpha/2}, \frac{r}{\bar{y}} + \frac{r}{n\bar{y}^2}\left(\frac{\bar{y}-r}{\bar{y}}\right) Z^{1-\alpha}\right)}$$

(e) $\begin{cases} H_0: \theta \geq 0.5 \\ H_1: \theta < 0{,}5 \end{cases}$

Anem a calcular el quocient de versemblances.

$$\frac{L(\theta_1|\underline{y})}{L(\theta_0|\underline{y})} = \frac{\theta_1^{rn}(1-\theta_1)^{\sum y_i - rn} \cancel{\prod \binom{y_i-1}{r-1}}}{\theta_0^{rn}(1-\theta_0)^{\sum y_i - rn} \cancel{\prod \binom{y_i-1}{r-1}}}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{rn} \left(\frac{1-\theta_1}{1-\theta_0}\right)^{\sum y_i - rn}$$

$\hookrightarrow$ creixent.!!

~~Veiem que alesores~~ Com $\theta_1 < \theta_0$

$\Rightarrow 1-\theta_0 < 1-\theta_1 \Rightarrow \boxed{1 < \frac{1-\theta_1}{1-\theta_0}}$

$\Rightarrow$ veiem que el quocient depen de $\boxed{T = \sum y_i}$ un estadístic suficient de manera exacta, llavors

(Karley-Rubin)

Pel criteri de versemblança monòtona, la RC serà

$$RC = \{\underline{y} : \sum_{i=1}^{n} y_i \geq A\} = \{\underline{y} : \bar{y} \geq A'\}$$

Assignatura:

Estudiant/a [redacted]

Data:

~~Aleshores~~ On A' es tal que

$$\boxed{P(\bar{y} \geq A' \mid \theta = 0{,}5) \leq 0{,}05}$$

(f) $\begin{cases} H_0 : \theta = 0{,}5 \\ H_1 : \theta \neq 0{,}5 \end{cases}$ Farem la raó de versemblança generalitzada.

$$\Lambda(\underline{y}) = \frac{\max_{\theta \in \Omega_0} L(\theta \mid \underline{y})}{\max_{\theta \in \Omega_1} L(\theta \mid \underline{y})} = \frac{\theta_0^{rn} (1-\theta_0)^{\sum y_i - rn} \cancel{\prod \binom{y_i - 1}{r-1}}}{\quad}$$

Anem a aplicar el ML. $\hat{\theta}_{ML} = \frac{r}{\bar{y}}$

$$L(\hat{\theta}_{ML} \mid \underline{y}) = \left(\frac{r}{\bar{y}}\right)^{rn} \left(1 - \frac{r}{\bar{y}}\right)^{\sum y_i - rn} \cancel{\prod \binom{y_i - 1}{r-1}}$$

$$\Rightarrow \left[\Lambda(\underline{y}) = \left(\frac{\bar{y}\,\theta_0}{r}\right)^{rn} \left(\frac{1-\theta_0}{1-\frac{r}{\bar{y}}}\right)^{\sum y_i - rn} \right]$$

Llavors la regió crítica serà

$$RC = \left\{ \underline{y} : \left(\frac{\bar{y}\,\theta_c}{r}\right)^{rn} \left(\frac{1-\theta_c}{1-\frac{r}{\bar{y}}}\right)^{n\bar{y}-rn} \leq A \right\}$$

on A és tal que $\left[P(\underline{Y} \in RC \mid \theta = 0{,}5) = 0{,}05\right]$

(podiem intentar aplicar $Q = -2\log(\Lambda(\underline{y})) \sim \chi^2_d$ però no arreglo massa).

(g) Anem primer a calcular $\hat{\theta}_{MV}$.

$$\hat{\theta}_{MV} = \frac{r}{\bar{y}} \overset{r=1}{=} \frac{1}{\bar{y}}$$

$$\bar{y} = \frac{\sum y_i \cdot freq(y_i)}{127} = \frac{45+62+60+36+30+30}{127}$$

$$+ \frac{28+16+9+10+22+12}{}$$

$$= \frac{360}{127}.$$

$$\Rightarrow \left[\hat{\theta}_{MV} = \frac{127}{360} = 0{,}352\bar{7}\right]$$

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Estudiant/a: [redacted]

Data:

Anem a fer servir el (d).

Havíem vist $\hat{\Theta}_{MV} = \frac{r}{\bar{y}} \overset{n\to\infty}{\sim} N\left(\Theta, \frac{\Theta^4(1-\Theta)}{nr}\right)$

Aproximem $\frac{\Theta^2(1-\Theta)}{nr} = \frac{\hat{\Theta}^2_{MV}(1-\hat{\Theta}_{MV})}{nr} =$

$= 0{,}00063423782 = \sigma^2$.

llavors $\frac{r}{\bar{y}} \sim N(\Theta, \sigma^2)$

L'interval és $\quad z^{0{,}975} = 1{,}96$

$\left[IC_{95\%}(\Theta) = \left(\frac{r}{\bar{y}} - z^{0{,}975}\sigma, \frac{r}{\bar{y}} + z^{0{,}985}\sigma \right) \right.$

$= \left(\hat{\Theta}_{MV} - z^{0{,}975}\sigma, \hat{\Theta}_{MV} + z^{0{,}975}\sigma \right)$

$= \left. (0{,}30342, 0{,}40214) \right]$

(h). Farem ~~test~~ un test de bondat d'ajust.

El test serà $\begin{cases} H_0: Y_i \sim Y \quad \forall i \text{ independents.} \\ H_1: Y_i \nsim Y \end{cases}$

Per tal de fer-ho, farem una fragmentació de l'espai de sortida. El dividirem per intervals, (els quals intentarem que hi hagi com a mínim 5 observacions) per què vagi bé.

En el nostre cas

Y	Observats	Esperats
1	45 O_1	E_1
2	31 O_2	E_2
3	20 O_3	E_3
4	9 O_4	E_4
5	6	E_5
6	5	E_6
$[7,8]$	6	$E_{[7,8]}$
$[9,+\infty)$	4	$E_{[9,+\infty)}$

Per cada interval, calcularem el nombre esperat d'observacions sota l'H_0. En el nostre cas, com no ens fixa θ, haurem d'estimar θ amb el $\boxed{\hat{\theta}_{MV} = \frac{r}{\bar{Y}}}$ (!!!)

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Full 8 de 8

QUALIFICACIÓ

Assignatura:

Estudiant/a: [redacted]

Data:

Per calcular els esperats, on Y és la del enunciat amb $\hat{\Theta}_{MV}$ calculem $\boxed{E_k = P(Y = k \mid \hat{\Theta}_{MV}) \cdot N}$

és a dir $E_1 = P(Y = 1 \mid \hat{\Theta}_{MV}) \cdot N$

$E_2 = P(Y=2 \mid \hat{\Theta}_{MV}) \cdot N$

$\vdots$

$E_{[7,8]} = P(Y=7 \cup Y=8 \mid \hat{\Theta}_{MV}) \cdot N$

$E_{[9,+\infty)} = P(Y \geq 9 \mid \hat{\Theta}_{MV}) \cdot N.$

Un cop tenim els esperats calculem

$$\chi^2 = \sum_{\text{interval } k} \frac{(O_k - E_k)^2}{E_k} \sim \chi^2_{\#\text{intervals} - 1 - 1}$$

perquè hem estimat 1 paràmetre

en el nostre cas

$\#\text{intervals} = 8 \cancel{-1} \Rightarrow \boxed{\chi^2_6}$

La regió crítica serà $RC = \{\chi^2 \geq A\}$

on $A = \chi^2_{6,\,0.95} = 12,592$

Si tenim $\chi^2 \geq 12{,}592$, llavors rebutgem H_0
si passa al revés, acceptem H_0 i diem que el
model s'ajusta estadísticament.

P1
a) ok
b) ok

P2
a) ok
b) ok