SOLUCIÓ EXAMEN ASI (ADVANCED STATISTICAL INFERENCE) ME810 UPC-UB

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PROBLEM 1 (3p)

Y r.v. with pdf $f(y;p,\lambda) = \begin{cases} p+(1-p)e^{-\lambda} & \text{if } y=0 \\ (1-p)\dfrac{e^{-\lambda}\lambda^{y}}{y!} & \text{if } y=1,2,\ldots \end{cases}$

$\downarrow$ discrete

(1) Likelihood function for sample $(y_1 \ldots y_n) = \underset{\sim}{y}$

$$L(p,\lambda;\underset{\sim}{y}) = \prod_{i=1}^{n} f(y_i;p,\lambda) = \prod_{i=1}^{n} \{p+(1-p)e^{-\lambda}\}^{1\{y_i=0\}} \cdot \prod_{i=1}^{n} \left\{(1-p)e^{-\lambda}\frac{\lambda^{y_i}}{y_i!}\right\}^{1\{y_i\neq 0\}}$$

Denote by $N_0 = \sum_{i=1}^{n} 1\{y_i=0\} = \#$ observations $=0$

$$L(p,\lambda;\underset{\sim}{y}) = \left[p+(1-p)e^{-\lambda}\right]^{N_0}\left[(1-p)e^{-\lambda}\right]^{n-N_0}\frac{\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

For the sample $y_0 = (1,0,2,0,6,3)$ we have

$$L(p,\lambda;y_0) = \left(p+(1-p)e^{-\lambda}\right)^{2}\left[(1-p)e^{-\lambda}\right]^{4}\frac{\lambda^{12}}{\underbrace{1!\,2!\,6!\,3!}_{8640}}$$

(2) Assume λ known. Find $\hat{p}_{MLE}$

$$\log L(p,\underset{\text{known}}{\lambda};\underset{\sim}{y}) = N_0 \log\left(p+(1-p)e^{-\lambda}\right) + (n-N_0)\underbrace{\log\left((1-p)e^{-\lambda}\right)}_{\log(1-p)-\lambda} + \left(\sum_{i=1}^{n} y_i\right)\log\lambda - \log\prod_{i=1}^{n} y_i!$$

$$\frac{\partial}{\partial p}\log L(p,\lambda;\underset{\sim}{y}) = N_0\frac{1-e^{-\lambda}}{p+(1-p)e^{-\lambda}} + (n-N_0)\frac{-1}{1-p}$$

$$= \frac{N_0(1-e^{-\lambda})}{p+(1-p)e^{-\lambda}} - \frac{n-N_0}{1-p} = N_0\left[\frac{1-e^{-\lambda}}{p+(1-p)e^{-\lambda}} + \frac{1}{1-p}\right] - \frac{n}{1-p}$$

$$\frac{\partial}{\partial p} \log L(p,\lambda|\underset{\sim}{y}) = 0 \Rightarrow \frac{n}{1-p} = N_0 \left[\frac{1-e^{-\lambda}}{p+(1-p)e^{-\lambda}} + \frac{1}{1-p} \right]$$

$\Rightarrow$ after some simplification

$$\hat{p}_{MLE} = \frac{N_0 - n e^{-\lambda}}{n(1-e^{-\lambda})}$$ after we check that $\frac{\partial^2}{\partial p^2} \log L(p,\lambda|\underset{\sim}{y}) < 0$

$$\frac{\partial^2}{\partial p^2} \log L(p,\lambda|\underset{\sim}{y}) = \frac{-N_0(1-e^{-\lambda})^2}{[p+(1-p)e^{-\lambda}]^2} - \frac{n-N_0}{(1-p)^2} < 0$$ always

$$\hat{p}_{MLE}(\underset{\sim}{y}_0) = \frac{2-6e^{-2}}{6(1-e^{-2})} = 0,228$$

(3) Find MLE for $P(Y=0)$.

$P(Y=0) = p + (1-p)e^{-\lambda}$

By the invariance principle of the MLE, the MLE of $P(Y=0)$ is given by $\hat{p}_{MLE} + (1-\hat{p}_{MLE})e^{-\lambda}$

Substituting

$$\hat{p}_{MLE} + (1-\hat{p}_{MLE})e^{-\lambda} = \frac{N_0 - ne^{-\lambda}}{n(1-e^{-\lambda})} + \frac{(n(1-e^{-\lambda}) - N_0 + ne^{-\lambda})e^{-\lambda}}{n(1-e^{-\lambda})}$$

$$= \frac{N_0 - ne^{-\lambda} + (n - \cancel{ne^{-\lambda}} - N_0 + \cancel{ne^{-\lambda}})e^{-\lambda}}{n(1-e^{-\lambda})} = \frac{N_0 - \cancel{ne^{-\lambda}} + \cancel{ne^{-\lambda}} - N_0 e^{-\lambda}}{n(1-e^{-\lambda})}$$

$$= \frac{N_0(1-e^{-\lambda})}{n(1-e^{-\lambda})} = \frac{N_0}{n}$$ and we obtain that the MLE for the $P(Y=0)$ is the proportion of 0's in the sample.

PROBLEM 2 (3p)

$X_1 — X_{15} \sim \exp(\mu = 1/\lambda)$

(1) $H_0: \lambda = 1/5$ vs $H_1: \lambda < 1/5$ use $X_{(1)} = \min\{X_1 — X_{15}\}$

Reject H_0 if $X_{(1)} \geq 1$

a) $P(\text{Type I error}) = P(X_{(1)} \geq 1 \mid \lambda = \frac{1}{5}) =$

$= P(X_1 \geq 1, \ldots, X_{15} \geq 1 \mid \lambda = 1/5) = \left(P(X \geq 1 \mid \lambda = 1/5)\right)^{15}$

$= \left(e^{-\lambda \cdot 1}\right)^{15}\Big|_{\lambda = \frac{1}{5}} = e^{-15/5} = e^{-3} = 0.049$

b) Power for $\lambda = 1/25$

$P(X_{(1)} \geq 1 \mid \lambda = \frac{1}{25}) = \left(P(X \geq 1 \mid \lambda = \frac{1}{25})\right)^{15}$

$= \left(e^{-\lambda \cdot 1}\right)^{15}\Big|_{\lambda = \frac{1}{25}} = e^{-\frac{15}{25}} = e^{-0.6} = 0.548$

(2) UMP for $H_0: \lambda = 1/5$ vs $H_1: \lambda < 1/5$

Denote $\lambda_0 = 1/5$ and λ_1 a fixed value $< \lambda_0$

the most powerful test for $H_0: \lambda = \lambda_0$ vs $H_1: \lambda = \lambda_1$ has as rejection region

$C = \left\{ x \in \mathcal{X}^n : \frac{L(\lambda_1 \mid x)}{L(\lambda_0 \mid x)} \geq A \right\}$ where $L(\lambda \mid x)$ is the likelihood

function, that is

$L(\lambda \mid x) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

$\frac{L(\lambda_1 \mid x)}{L(\lambda_0 \mid x)} = \left(\frac{\lambda_1}{\lambda_0}\right)^n e^{-(\lambda_1 - \lambda_0) \sum_{i=1}^{n} x_i}$

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QUALIFICACIÓ

$$\frac{L(\lambda_1|\underline{x})}{L(\lambda_0|\underline{x})} \geq A \iff \left(\frac{\lambda_1}{\lambda_0}\right)^n e^{-(\lambda_1-\lambda_0)\sum x_i} \geq A$$

Since $0 < \frac{\lambda_1}{\lambda_1} < 1$ and $-(\lambda_1-\lambda_0) > 0$, last inequality is $\iff$

$$-(\lambda_1-\lambda_0)\sum_{i=1}^n x_i \geq A' \iff \sum_{i=1}^n x_i \geq A''$$

Hence the rejection region only depends of $\sum_i x_i$ and not of the particular value λ_1, as long as, $\lambda_1 < \lambda_0$.

therefore for $\alpha = P_{\lambda_0}(\sum_{i=1}^n x_i \geq A'')$, where $\sum_{i=1}^n x_i$ follows a gamma distribution, the most powerful rejection region

$A''(\alpha)$ such that

is given by $\sum_i x_i \geq A''(\alpha)$

Since we have solved b) for an arbitrary value $\lambda_1 < \lambda_0$, we can specify $\lambda_1 = 1/7$ to get the result, and since this also proves that the test is UMP.

If $X \sim \exp(1/\lambda)$, then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\lambda)$

In part b), given α, $P_{\lambda_0}(\sum_i X_i \geq A') = \alpha \iff$

$\iff P(\text{Gamma}(n,5) \geq A'') = \alpha$ which $\Rightarrow$

$$\alpha = \int_{A''}^{\infty} \frac{y^{n-1} e^{-\frac{1}{5}y}}{5^n (n-1)!}\,dy$$ and A'' has to be solved numerically

PROBLEM 1 (cont)

$$P(Y=0\,;\,y_0) = 0.228 + (1-0.228)e^{-2} = \frac{1}{3}$$
$$\lambda = 2$$

(4) Moment estimator for (p,λ)

$$E(Y) = \sum_{y=0}^{\infty} y\,P(Y=y) = \lambda(1-p) \quad \text{after basic sums}$$

$$E(Y^2) = \sum_{y=0}^{\infty} y^2 P(Y=y) = \lambda(1-p)(\lambda+1)$$

$$Var(Y) = \lambda(1-p)(1-\lambda p)$$

let $\overline{Y}, S^2$ the sample mean and variance, we obtain the moment estimators equating

$$\left.\begin{aligned} \overline{Y} &= \lambda(1-p) \\ S^2 &= \lambda(1-p)(1-\lambda p) \end{aligned}\right\} \quad \lambda = \frac{\overline{Y}}{1-p} \Longrightarrow$$

$$S^2 = \frac{\overline{Y}}{\cancel{1-p}}\cancel{(1-p)}\left(1 - \frac{\overline{Y}}{1-p}p\right) \Rightarrow \frac{S^2}{\overline{Y}} = 1 - \overline{Y}\frac{p}{1-p}$$

$$\Rightarrow 1 - \frac{S^2}{\overline{Y}} = \overline{Y}\frac{p}{1-p} \Longrightarrow \frac{\overline{Y}-S^2}{\overline{Y}^2} = \frac{p}{1-p} \Longrightarrow$$

$$\Rightarrow \hat{p}_{MM} = \frac{\frac{\overline{Y}-S^2}{\overline{Y}^2}}{\frac{\overline{Y}-S^2}{\overline{Y}^2}+1} = \frac{\overline{Y}-S^2}{\overline{Y}+\overline{Y}^2-S^2} \longrightarrow$$

$y = (1,0,2,0,6,3)$

$\Rightarrow \overline{Y} = \frac{12}{6} = 2$

$\overline{Y}^2 = 4$

$S^2 = \frac{1}{5}26$

and $$\hat{\lambda}_{MM} = \frac{\overline{Y}^2}{\overline{Y}+\overline{Y}^2-S^2}$$

$$\hat{\lambda}_{MM}(y_0) = \frac{4}{2+4-26/5} = 5\,;\quad \hat{p}_{MM}(y_0) = 1 - \frac{\overline{Y}}{\hat{\lambda}} = 1 - \frac{2}{5} = \frac{3}{5}$$

Exercise 3 $\quad X_1 - X_n \overset{iid}{\sim} f(x;\theta) = \frac{2x}{\theta} e^{-x^2/\theta}, x>0$

A) X_1^2 is an unbiased estimator of θ

$W = X^2 \qquad \begin{cases} w(x) = x^2 \\ x(w) = w^{1/2} = \sqrt{w} \\ \quad (w>0) \end{cases} \quad x'(w) = \frac{-1}{2\sqrt{w}}$

El teorema de canvi de variable estableix que

$$f_Y(y) = f_X(x(y)) \cdot \left|\frac{dx(y)}{dy}\right| =$$

$$= \frac{\cancel{2\sqrt{w}}}{\theta} e^{-(\sqrt{w})^2/\theta} \cdot \left|\frac{-1}{\cancel{2\sqrt{w}}}\right| = \frac{1}{\theta} e^{-w/\theta}, w>0$$

Aixo es la densitat d'una llei exponencial de parametre θ y $\boxed{E(W) = \theta}$ com es sabut o fent.

$$\int_0^\infty w \frac{1}{\theta} e^{-w/\theta} d\theta = w \cdot (-e^{-w/\theta}) - \int_0^\infty -e^{-w/\theta} dw =$$

$\int u\, dv \qquad = u \cdot v$

$$= -w \cdot e^{-w/\theta}\Big|_0^\infty - \theta e^{-w/\theta}\Big|_0^\infty$$

$$= \theta \quad - (0-\theta) = \underline{\theta}$$

Es a dir $E(X^2) = \theta \Rightarrow$ Es un e.s.b. de θ.

2) Calcula la CCR per la variancia d'un e.s.b. de θ

Demanen calcular $\dfrac{1}{I_n(\theta)}$ on $I_n(\theta) = n\, I(\theta)$

i $I(\theta) = -E\left(\dfrac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)$

$$f(x;\theta) = \frac{2x}{\theta} e^{-x^2/\theta}$$

$$\ln(x;\theta) = \ln 2x - \ln\theta - \frac{x^2}{\theta}$$

$$\frac{\partial \ln(x;\theta)}{\partial\theta^2} = -\frac{1}{\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 \ln(x,\theta}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x^2}{\theta^3}$$

$$-E\left(\frac{\partial^2 \ln(x;\theta)}{\partial\theta^2}\right) = -\frac{1}{\theta^2} + \frac{2}{\theta^3} E(X^2) = \frac{2-1}{\theta^2} = \frac{1}{\theta^2}$$

$$CCR = [n\, I(\theta)]^{-1} = \boxed{\frac{\theta^2}{n}}$$

3.3) Troba l'UMVUE

Podem factoritzar la funció Score:

$$\frac{\partial \ln L(\theta; \underline{x})}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i^2}{\theta^2} = \frac{\sum x_i^2 - n\theta}{\theta^2} =$$

$$= \underbrace{\frac{n}{\theta^2}}_{I_n(\theta) = CCR^{-1}} \Big(\underbrace{\frac{\sum x_i^2}{n}}_{\text{UMVUE}} - \theta \Big)$$

També es fàcil veure que $Var\left(\frac{\sum X_i^2}{n}\right) = \left(\frac{n}{\theta^2}\right)^{-1} = \frac{\theta^2}{n}$

$X_i^2 \sim \xi(1/\theta) \quad Var(X_i^2) = \theta^2 \quad Var(\sum X_i^2/n) = \frac{n\theta^2}{n^2} =$

És a dir la Var d'un estimador s.b. que assoleix la CCR és l'UMVUE

Una darrera forma d'establir-ho es a partir de l'argument que

(i) Es una família exponencial (evident per la fdd) regular (espai de paràmetres: R^+, obert)

(ii) $\sum X_i^2/n$ es funció de l'estadístic suficient i un e.s.b $\xrightarrow{\text{complet}}$ UMVUE

Cal verificar?

Soluciones Ex. 3

1) $W = X_1^2 \underset{\cdots}{\rightarrow} f_W(w) = \frac{1}{\theta} e^{-w/\theta} \Rightarrow \underline{E(W) = \theta}$

2) $CCR = \left[-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right)\right]^{-1} = \theta^2/n$

3) UMVUE $\quad \frac{\partial \ln L(\theta; \underline{x})}{\partial \theta} = \frac{n}{\theta^2}\left(\frac{\sum x_i^2}{n} - \theta\right)$

4) (4.1) LRT: $\Lambda(\underline{x}) = \left(\frac{\hat{\theta}}{\theta_0}\right)^n \exp\left\{n - n\frac{\hat{\theta}}{\theta_0}\right\}$

$$U = -2\ln\Lambda(\underline{x}) = -2n\left(\ln\frac{\hat{\theta}}{\theta_0} - \frac{\hat{\theta}}{\theta_0} + 1\right) \sim \chi^2_{(1)}$$

(4.2) $W = (\hat{\theta} - \theta_0)^2 \cdot I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{n}{\hat{\theta}^2} \sim \chi^2_{(1)}$

(4.3) $SC = [SC(\theta_0)]^2 \cdot [I(\theta_0)]^{-1} = \frac{(\hat{\theta} - \theta_0)^2 \cdot n}{\theta_0^2} \sim \chi^2_{(1)}$

5) $n = 200 \quad \theta_0 = 3.5 \quad \hat{\theta} = 2.5$

$$\left.\begin{array}{l} U = 20.3 \\ W = 32 \\ SC = 16.3 \end{array}\right\} > \underset{\chi^2_{1,0.95}}{3.81} \Rightarrow \text{Rebutjem } H_0$$