Graph theory. Fall 2018

Planar graphs

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1. Preliminaries on graph theory

Sections 1.1 to 1.6 from *Graph Theory*, by R. Diestel, available online, are required preliminaries.

Given a graph G, we denote by $G - e$ the graph obtained by removing the edge e, and by G/e the graph obtained by contracting e (multiple edges are removed after contraction).

An ear decomposition of a graph G is a decomposition $E(G) = C \cup P_1 \cup \cdots \cup P_k$, where C is a cycle and each P_i is a path having only the two endpoints in common with $C \cup P_1 \cup \cdots \cup P_{i-1}$.

Lemma 1.1. *A graph is 2-connected iff it admits an ear decomposition.*

We also need Menger's theorem (Theorem 3.3.1 from the same reference). We quote it for completeness.

Theorem 1.2 (Menger)**.** *Let $A, B \subset V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A-B paths in G.*

The following consequence is often used.

Corollary 1.3. *Let u, v be non-adjacent vertices in G. Then the minimum number of vertices separating u from v in G is equal to the maximum number of internally disjoint u-v paths in G.*

2. Planar graphs

A graph is *planar* if it can be embedded in the plane without edge crossings. A *plane graph* is an embedding of a plane graph. The faces of an embedding of $G = (V, E)$ are the connected components of the complement of $V \cup E$ in the plane. There is exactly one unbounded face. The degree of a face is the number of edges in its boundary, where isthmus are counted twice.

We let $n = |V|$ and $m = |E|$. By double counting we have

$$\sum_{F \text{ face}} \deg(F) = 2m.$$

Proposition 2.1 (Euler's formula)**.** *Let f be the number of faces in an embedding of a connected planar graph. Then*

$$n - m + f = 2$$

Corollary 2.2. *If G is a planar graph then*

$$m \leq 3n - 6.$$

If G is planar graph whose faces all have degree at least d, then

$$m \leq \frac{d(n-2)}{d-2}.$$

In particular, if G is bipartite and planar, then $m \leq 2n - 4$.

Kuratwoski's graphs. The graphs K_5 and $K_{3,3}$ are non planar.

Given a plane embedding of $G = (V, E)$, its plane dual $G^* = (F, E^*)$ has the faces F of G as vertices, and for every edge e of G, there is an edge e^* joining the two faces (which may be the same) on the two sides of e. Notices that $E \leftrightarrow E^*$ is a bijection between the edges of G and those of G^*.

A graph is *maximal* planar if $m = 3n - 6$. This is equivalent to being a *triangulation*, that is, each face has degree 3. In the case of bipartite graphs, the condition for maximality is $m = 2n - 4$; equivalently, the graph is a *quadrangulation* (every boundary face is a cycle of degree 4).

Proposition 2.3. *Let G be a planar graph.*

(1) *In a 2-connected plane graph, every face is bounded by a cycle.*
(2) *In a 3-connected plane graph, the face boundaries are characterized as its non-separating induced cycles.*

Corollary 2.4 (Whitney)**.** *A 3-connected planar graph has a unique embedding in the plane (up to the choice of the unbounded face).*

Theorem 2.5 (5-colour theorem)**.** *Every planar graph is 5-colourable.*

Much more difficult is to prove the following; the only known proof requires to consider hundreds of cases and to check each case with a computer program.

Theorem 2.6 (4-colour theorem)**.** *Every planar graph is 4-colourable.*

3. The theorem of Kuratowski and planar separators

A *subdivision* of a graph G is any graph H obtained from G by replacing the edges by induced paths. A subdivision of G is planar if and only if G is planar.

Theorem 3.1 (Kuratowski)**.** *A graph is planar if and only if does not contain a subdivision of K_5 and $K_{3,3}$.*

The proof uses the following:

Lemma 3.2. *If G is 3-connected (not necessarily planar) with $n > 4$ vertices, then there is an edge e such that G/e is 3-connected.*

The proof of Kuratowski's theorem also gives

Theorem 3.3 (Tutte)**.** *A 3-connected planar graph admits an embedding in which the boundary of every face is a strictly convex polygon.*

Corollary 3.4 (Fáry)**.** *Every planar graph admits an embedding in which the edges are straight line segments.*

A *separator* in a graph G is a set of vertices S such $V(G) - S$ is partitioned into sets A and B, and there is no edge joining a vertex in A to a vertex in B. It is an α-balanced $f(n)$-separator if $|S| \le f(n)$ and $|A| \le \alpha n$, $|B| \le \alpha n$.

Theorem 3.5 (Planar separator)**.** *Every planar graph has a 2/3-balanced $\sqrt{8n}$-separator.*

4. Exercises

(1) An st-ordering of a graph G is a numbering $V = \{v_1, \dots, v_n\}$ or the vertex set such that v_1 and v_n are adjacent, and for each $i = 2, \dots, n-1$ the are v_j and v_k with $j < i < k$, and v_i adjacent to both v_j and v_k. Show that G is 2-connected iff admits an st-ordering. Moreover, any edge st of G can be chosen with $v_1 = s$ and $v_2 = t$.

(2) a triangulation is a plane graph in which every face is a triangle. Show that a triangulation with $n \geq 4$ vertices is 3-connected.

(3) Show that a triangulation is 3-colorable if and only if it is Eulerian (all vertices have even degree).

(4) Show that the 4-color theorem is equivalent to the following fact: every bridgeless cubic planar graph is 3-edge colourable.

(5) Show that if G is planar and $n \geq 11$, then the complement G^c is nonplanar.

(6) Let $G = (V, E)$ be a plane graph and $G^* = (E^*, F)$ its plane dual. Let T be the edge set of a spanning tree of G. Show that $E^* \setminus T^*$ is a spanning tree of G^*.

(7) Show that the skeleton of a polytope in $\mathbb{R}^3$ is a plane 3-connected graph.

(8) Show that the five platonic solids are the only possible combinatorial types of regular polytopes in $\mathbb{R}^3$.

(9) Show that every graph can be embedded in $\mathbb{R}^3$ with all edges straight. Show that every graph can be embedded in the surface formed by two intersecting planes in $\mathbb{R}^3$.

(10) Show that every planar graph is 5-list-colourable (follow the short Thomassen's proof in Diestel's book).

(11) Show that the only non-planar 3-connected graph not containing a subdivision of $K_{3,3}$ is K_5.

(12) A graph is outerplanar if it admits a plane embedding in which all vertices are in the outer face. Show that a graph is outerplanar if and only if does not contain a subdivision of K_4 or $K_{2,3}$.

(13) Show that a tree T has always a vertex v such that all connected components of $T - v$ have size at most $2n/3$.

(14) Show that a $2/3$-balanced separator of the $n \times n$ grid has size at least $c\sqrt{n}$ for some constant $c > 0$.

Graph theory

Graphs on surfaces

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1. Embedding graphs in surfaces

Let $\mathbb{S}_g$ be the orientable surface of genus g, and $\mathbb{N}_h$ the non-orientable surface of non-orientable genus h. Remark that $\mathbb{S}_0$ is the sphere. Faces are defined as for planar graphs. In the sequel, n is the number of vertices of a graph and m is the number if edges.

Theorem 1.1. *A compact surface (without boundary) is isomorphic to either $\mathbb{S}_g$ for some $g \geq 0$, or to $\mathbb{N}_h$ for some $h > 0$.*

An embedding of G on surface S is a drawing of G in S without edge crossings. An embedding is cellular if every face is contractible (isomorphic to a disk).

Proposition 1.2 (Euler's formula)**.** *If G has embedding in $\mathbb{S}_g$, then*

$$n - m + f = 2 - 2g.$$

If G has embedding in $\mathbb{N}_h$, then

$$n - m + f = 2 - h.$$

Corollary 1.3. *If a graph embeds in $\mathbb{S}_g$, then $m \leq 3n - 6 + 6g$.*

Embeddings can be defined in purely combinatorial terms; here we restrict to the orientable case. Given a graph $G = (V, E)$, a *rotation system* consists of giving a cyclic ordering π_v of the edges around each vertex v. That is, a collection of circular permutations $\{\pi_v\}_{v \in V}$. This gives an embedding whose face boundaries are obtained as follows: start at a vertex v, traverse an edge vu, then take the next edge after uv in the cyclic order around u, until returning to the initial edge. This procedure gives a number f of faces. The genus of the surface hosting the embedding is given by Euler's formula as $g = (2 - n + m - f)/2$.

Exanple. $G = K_4, V = \{1, 2, 3, 4\}$. The rotation system $\{\pi_1 = \{12, 13, 14\}, \pi_2 = \{21, 24, 23\}, \pi_3 = \{31, 32, 34\}, \pi_4 = \{41, 43, 42\}\}$ produces an embedding in the sphere, whereas $\{\pi_1 = \{12, 14, 13\}, \pi_2 = \{21, 24, 23\}, \pi_3 = \{32, 31, 34\}, \pi_4 = \{41, 43, 42\}\}$ produces an embedding in the torus.

The (orientable) genus of a graph G is the minimum $g = g(G)$ such that G embeds in $\mathbb{S}_g$.

Lemma 1.4. *For any graph G,*

$$g(G) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

When $G = K_n$ this gives

$$g(K_n) \geq \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor, \quad n \geq 3. \tag{1}$$

A deep result is that the former inequality is in fact an equality (a similar result holds for non-orientable surfaces). The proof (Ringel-Youngs) consists of producing an embedding of K_n for each n on the corresponding surface. The construction for $n \equiv 7 \mod 12$ is the simplest one: given $n = 12k + 7 (k \geq 0)$, one constructs an embedding of K_n in the surface of genus $12k^2 + 7k + 1$.

Theorem 1.5 (Heawood)**.** *If a graph G has genus $g > 0$ then*

$$\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor.$$

Notice that the case $g = 0$ would be the 4-colour Theorem. Equality is achieved by K_n with $\left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$: this is equivalent to (1) being an equality.

2. Exercises

(1) The cube Q^n is the graph having as vertices $V = \{0,1\}^n$, and two words are adjacent if they differ in exactly one position. Show that
   (a) Q^n is planar iff $n \leq 3$.
   (b) $g(Q^n) \geq (n-4)2^{n-3} + 1$.
   (c) $g(Q^4) = 1$.

(2) Find an embedding of K_7 and of $K_{4,4}$ in the torus. Find an embedding of the Petersen graph in the projective plane, and identify the dual graph.

(3) Complete the details for embedding K_n with $n = 12k+7 (k \geq 0)$, in the surface of genus $12k^2+7k+1$. [Mohar and Thomassen. Graphs on surfaces, pp. 115–116]

(4) Show that the genus of $K_{m,n}$ is equal to $\lceil \frac{((n-2)(m-2)}{4} \rceil$. [Mohar, Thomassen, Graphs on surfaces pp. 117–118]

Topological graphs and minors.

Session 4: Graphs minors

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1. Basics

A graph H is a minor of G if H can be obtained from a subgraph of G by deleting and contracting edges.

Proposition 1.1. *In order to obtain a minor from a graph it does not matter the order in which deletions and contractions are performed.*

Proposition 1.2. *Let H be a graph with k vertices. G contains H as a minor if and only if there exist disjoints sets of vertices V_x in G, one for each vertex x in H, such that the subgraph induced by V_x is connected and there is an edge between V_x and V_y whenever there x and y are joined by an edge in H.*

2. Characterizations

A 2-connected graph if series-parallel if it can be obtained starting with a single edge by repeated applications of the following two operations: (1) subdivide and edge; (2) duplicate and edge. A graph is series-parallel if all its blocks are series-parallel.

Theorem 2.1. *A graph is series-parallel if and only if does not contain K_4 as a minor.*

A class $\mathcal{G}$ of graphs is minor closed if whenever a graph is in $\mathcal{G}$, so are all its minors. The class of planar graphs is minor closed, and so is the class of graphs embeddable in a fixed surface.

Theorem 2.2 (Wagner)**.** *A graph is planar if and only if does not contain either K_5 or $K_{3,3}$ as minors.*

Let W_8 be the cubic graph obtained from a cycle of length 8 by joining opposite vertices.

Theorem 2.3 (Wagner)**.** *If G is a edged-maximal graph not containing K_5 as a minor, then G can be obtained from planar triangulations and the exceptional graph W_8 by gluing along triangles and edges, in such a way that two triangulations are never glued along an edge.*

Conjecture 2.1 (Hadwiger)**.** *A graph not containing K_{r+1} as a minor can be r-colored.*

Hadwiger conjecture has been proved for $r \leq 5$.

An analogous conjecture was made by Hajos for graphs not containing a subdivision of K_{r+1}, but was disproved.

3. Excluded minors and well quasi-ordering

Let $\mathcal{G}$ be a minor-closed class. H is an *excluded minor* for $\mathcal{G}$ if $H \notin \mathcal{G}$ but every proper minor of H is in $\mathcal{G}$.

Lemma 3.1. *Let $\mathcal{H}$ be the collection of excluded minors of $\mathcal{G}$. Then a G graph is in $\mathcal{G}$ if and only if G does not contain any graph of $\mathcal{H}$ as a minor.*

The main result in the area is

Theorem 3.2 (Robertson-Seymour)**.** *Every proper minor-closed class has a **finite** number of exclude minors.*

An alternative formulation is in terms of well quasi-orders. A quasi-order on a set X is a reflexive and transitive binary relation $\leq$ defined on X. A quasi-order X is a well quasi-order (WQO) if for every infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ there is a pair $i < j$ such that $x_i \leq x_j$.

We will need the following

Lemma 3.3. *If $(X, \leq)$ is a WQO, then every infinite sequence has an increasing infinite subsequence.*

If $\leq$ is a quasi-order in X, for finite subsets A and B we write $A \leq B$ if there is an injection $f \colon A \to B$ such that $a \leq f(a)$ for all $a \in A$.

Lemma 3.4. *IF $(X; \leq)$ is a WQO, so is the quasi-order on finite subsets of X.*

The Robsertson-Seymiour theorem is equivalent to saying that graphs are well quasi-ordered by the minor relation.

We say that H is topological minor of G if H is a subdivision of a subgraph of G. We prove

Theorem 3.5 (Kruskal)**.** *The class of finite trees is well quasi-ordered by the topological minor relation.*

4. Exercises

(1) Prove Proposition 1.1.
(2) Show that a series-parallel graph with n vertices has at most $2n - 3$. Show that a series-parallel graph is 3-colorable. edges.

(3) A graph no containing K_5 as a minor has at most $3n - 6$ edges and is 4-colorable (use the Four Color Theorem).
(4) Call a graph G apex if there exists a vertex v such that $G-v$ is planar. Show that apex graphs form a minor-closed class. Find at least two excluded minors for the class of apex graphs.
(5) Given a graph H prove the following. There is a finite list of graphs $H_1, H_2, \ldots, H_k$ such that a graph contains H as a minor if and only if contains one of the H_i as a subdivision.
(6) Prove Hadwiger conjecture for $r = 2, 3, 4$.

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Spectra of Graphs

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1. Notation

Let $G = (V, E)$ be a simple graph (no loops nor multiple edges) with order $n = |V|$ and size $m = |E|$. Given an ordering $v_1, \dots, v_n$ of the vertices, the *adjacency matrix* $A(G) = (a_{ij})$ of G has entry $a_{ij} = 1$ when $v_i \sim v_j$ and $a_{ij} = 0$ if v_i and v_j are not adjacent.

The *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix and is denoted by $\phi(G, x) = \det(A - x \cdot I)$. Since $A(G)$ is real and symmetric, it has n (not necessarily distinct) eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$. The set of eigenvalues (or the set of *distinct* eigenvalues together with their multiplicities) is the *spectrum* of G.

2. Basic results

Proposition 2.1 (Spectrum and walks)**.** *Let G be a graph and $A = A(G)$ its adjacency matrix. Then the entry $a_{ij}^{(k)}$ of A^k counts the number of walks of length k between vertices v_i and v_j.*

In particular,

(i) $\sum_{i=0}^{n-1} \lambda_i = Tr(A) = 0$.

(ii) $\sum_{i=0}^{n-1} \lambda_i^2 = Tr(A^2) = 2m$.

(iii) $\sum_{i=0}^{n-1} \lambda_i^3 = Tr(A^3) = 6T$*, where T is the number of triangles in G.*

Proposition 2.2 (Spectrum and degree)**.** *Let G be an r–regular connected graph. Then $\lambda_1 = r$ and has multiplicity one. Moreover $|\lambda_i| < r$ for $i > 1$.*

Proposition 2.3 (Spectrum and diameter)**.** *Let G be a connected graph with k pairwise distinct eigenvalues and diameter D. Then $D \leq k - 1$.*

3. Computing Spectra or characteristic polynomials

Proposition 3.1 (Circulant graphs)**.** *Let G be a graph with adjacency matrix $A = circ(a_1, \dots, a_n)$, where row j is a cyclic shift of row $j-1$ and $(a_1, \dots, a_n)$ is the first row. Then the i–th eigenvalue of G is*

$$\lambda_k = \sum_{j=2}^{n} a_j \omega^{(j-1)k},$$

where $\omega = e^{2\pi i/n}$ is a primitive n–th root of unity.

In particular,

(i) $Spec(K_n) = (n-1, -1, \ldots, -1)$,

(ii) $Spec(C_n) = (2, 2\cos(2\pi/n), 2\cos(4\pi/n), \ldots, 2\cos((n-1)\pi/n)$.

Proposition 3.2 (Complement of a graph)**.** *Let G be an r–regular graph. Then*

$$\phi(\overline{G}, x) = (-1)^n \left(\frac{x+r+1-n}{x+r+1}\right) \phi(G, -x-1),$$

where $\overline{G}$ denotes the complement of G. In particular the $Spec(\overline{G}) = \{n-r-1, -\lambda_n-1, \ldots, -\lambda_2-1\}$, where $Spec(G) = \{r, \lambda_2, \cdots, \lambda_n\}$.

Proposition 3.3 (Line graphs)**.** *Let $L(G)$ denote the line graph of G ($V(L(G)) = E(G)$ and two vertices are adjacent in $L(G)$ if the corresponding edges are incident in G). Then, if G is r–regular,*

$$\phi(L(G), x) = (x+2)^{m-n}\phi(G, x+2-r).$$

4. Exercises

(1) Let H_s be the hyperoctahedral graph, obtained from the complete graph K_{2s} by removing a perfect matching. By using Proposition 3.1, compute the spectrum of H_{2s}.

(2) Show that the Petersen graph is the complement of the line graph of K_5. Using Proposition 3.2 and Proposition 3.3, compute the spectrum of the Petersen graph.

(3) Check that K_{10} contains two edge–disjoint copies of a Petersen graph. Show that K_{10} does not contain there edge–disjoint copies of the Petersen graph.

[Hint: Consider the spectra of the Petersen graph and its eigenvalue 1.]

(4) A graph G is *strongly regular* with parameters (n, r, λ, μ) if it is a r-regular graph with n vertices, such that every two adjacent (non-adjacent) vertices have exactly λ (resp. μ) common neighbours. Show that the spectrum consists on 3 different eigenvalues.

(5) Let G be a bipartite graph.

(i) Show that the spectrum of G is symmetric.

(ii) Compute the spectrum of the bipartite complete graph $K_{r,s}$.

(6) Show that if G has two connected components G_1, G_2 then $Spec(G) = Spec(G_1) \cup Spec(G_2)$ (adding multiplicities).

(7) Give an interpretation of the coefficients of x^{n-1}, x^{n-2} and x^{n-3} of the characteristic polynomial $\phi(G, x)$ of a graph G with n vertices, in terms of properties of the graph.

(8) Let G be a graph with n^+ positive eigenvalues and n^- negative eigenvalues. Let $G = B_1 \oplus \cdots \oplus B_k$ be an edge–decomposition of G into k bipartite graphs. Show that $k \geq \max\{n^+, n^-\}$. Deduce that any decomposition of K_n into bipartite graphs needs at least $n-1$ of them.

[Hint: The adjacency matrix of a complete bipartite subgraph can be written as uv^T for u the characteristic vector of one stable set and v the characteristic vector of the second one.]

(9) Let v be a vertex of degree one in G and let u the vertex incident with v. Show that

$$\phi(G, x) = x\phi(G-v, x) - \phi(G-u-v, x).$$

Compute the characteristic polynomial of the path P_n.

[Hint: For the last part, you may identify the a defining recurrence for $\phi(P_n, x)$ and look for Xhebyshev polynomials.]

(10) Let $G - v$ be the subgraph of G obtained by deleting the vertex $v \in V = V(G)$. Show the following formula for the derivative of the characteristic polynomial:

$$\phi'(G, x) = \sum_{v \in V} \phi(G - v, x).$$

[Hint: A simplifying trick is to consider a polynomial on n variables defined as $\phi(x_1, \ldots, x_n) = det(A - (x_1, \ldots, x_n)I)$ and then evaluate at $x = x_1 = \cdots = x_n$.]

(11) Let $G = (X \cup Y, E)$ be a bipartite regular graph such that every stable set has even order. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $|X_1| = |X_2| = |Y_1| = |Y_2|$ be partitons of X and Y. Consider the graphs G_1 and G_2 obtained from G by adding one vertex z to G adjacent to all vertices of $X_1 \cup Y_1$ in G_1 and to all vertices of of $X_2 \cup Y_2$ in G_2. Show that G_1 and G_2 are cospectral graphs.

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Theorems on matrices. Laplacian of a graph

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1. Theorems on Matrices

Proposition 1.1 (Simultaneous diagonalization). *Let A, B be two $n \times n$ symmetric matrices. If $AB = BA$ then there is a common basis of eigenvectors to both matrices.*

Simple examples of classes of matrices where the Propsition applies are the class of circulant matrices or the algebra generated by a matrix.

The Perron-Frobenius theorem plays a significant role in spectral graph theory.

Theorem 1.2 (Perron-Frobenius). *Let A be a real matrix with nonnegative entries such that A^k has all entries positive for some k. Then*

(i) There is a real eigenvalue λ_1 of A wich is the spectral radius of A ($|\lambda| \le \lambda_1$ for each eigenvalue λ of A.)

(ii) λ_1 has algebraic and geometric multiplicity one.

(ii) There is an eigenvector $\mathbf{v}$ belonging to λ_1 with all coordinates positive.

Theorem 1.2 applies to adjacency matrices of graphs. One has to consider the bipartite case separatedly by applying the theorem to $I + A$.

Corollary 1.3. *If G is a connected non bipartite graph then λ_1 is the spectral radius of G and it is a simple eigenvalue. Moreover*

(i) there is an eigenvalue of λ_1 with all coordinates positive.

(ii) if G is regular and $(x_1, \ldots, x_n)$ is an eigenvector of a different eigenvalue than λ_1, then $\sum_i x_i = 1$.

The Courant–Fisher inequalities are another important tool in spectral graph theory.

Theorem 1.4 (Courant-Fischer inequalities). *Let A be a real symmetric matrix with eigenvalues $\lambda_1 \ge \lambda_1 \cdots \ge \lambda_n$. Then*

$$\lambda_k = \max_{V \in \mathcal{V}_k} \min_{x \in V, \|x\|=1} x^T A x,$$

where $\mathcal{V}_k$ denotes the family of all k–subspaces of $\mathbb{R}^n$. Similarly,

$$\lambda_k = \min_{V \in \mathcal{V}_{n-k}} \max_{x \in V_{n-k}, \|x\|=1} x^T A x.$$

Corollary 1.5. *Let G be a connected graph and $A = A(G)$ its adjacency matrix. Then, for each vector y with positive entries,*

$$\frac{y^T A y}{y^t y} \leq \lambda_1 \leq \max_j \frac{e_j^T A y}{e_j^T y}.$$

In particular,

$$2m/n \leq \lambda_1 \leq \Delta(G),$$

and if one equality holds then both hold and G is regular.

Our last result in matrices is the Interlacing Theorem.

Theorem 1.6 (Cauchy Interlacing Theorem)**.** *Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be an orthogonal projection. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $B = PAP^T$. Then*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \mu_{n-1} \geq \lambda_n.$$

The conditions of Theorem 1.6 are satisfied when $A = A(G)$ is the adjacency matrix of a graph G and $B = A(G - v_1)$.

2. The Laplacian of a graph

Definition 2.1 (Laplacian)**.** *The* Laplacian *matrix of graph G is the $n \times n$ matrix $L(G) = (l_{ij})$ defined as*

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \sim v_j \\ d(v_i), & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with entries the degrees of vertices.

Proposition 2.2 (Basic relations)**.** *Let $Q = (q_{ij})$ be the $n \times m$* incidence *matrix of graph G defined as follows. Orient arbitrarily each edge in G and label the edges $e_1, \ldots, e_m$. Denote by $i(e)$ and $f(e)$ the initial and final vertices of the oriented edge e. The ij–entry of Q is*

$$q_{ij} = \begin{cases} -1, & \text{if } v_i = i(e_j) \\ 1, & \text{if } v_i = f(e_j) \\ 0, & \text{otherwise.} \end{cases}$$

Then $L(G) = QQ^T$. Moreover, for each vector $\mathbf{x} \in \mathbb{R}^n$,

$$x^T L x = \sum_{v_i \sim v_j} (x_i - x_j)^2.$$

Proposition 2.3 (Laplacian spectrum)**.** *Let $\mu_0 \leq \mu_1 \leq \cdots \mu_n$ be the spectrum of the Laplacian $L(G)$ of graph G. Then*

(i) $\mu_0 = 0$ and has eigenvector $(1, 1, \ldots, 1)$.

(ii) If G is connected then $\mu_1 > 0$.

(iii) If G is r–regular then $\mu_i = r - \lambda_i$ where λ_i is the i–th eigenvalue of $A(G)$ (listed in nonincreasing order).

3. Exercices

(1) Prove that the chromatic number verifies

$$\chi(G) \leq 1 + \lambda_1.$$

[Hint: Choose a critical subgraph $H \subset G$ with chromatic number $\chi(H) = \chi(G)$. What is its minimum degree?]

(2) Let G be a connected graph with adjacency matrix $A = A(G)$. Show that there is a positive integer k such that $A^k > 0$ if an donly if G is non bipartite.

(3) If H is a subgraph of G then $\lambda_1(H) \leq \lambda_1(G)$. Moreover, if G is connected, show that equality holds if and only if $G = H$.

(4) Show that if $\lambda_1(G) > \sqrt{m}$ then G contains a triangle.

(5) Show that a connected graph G with maximum eigenvalue λ_1 is bipartite if and only if $-\lambda_1$ is an eigenvalue of G.

(6) Let d_i denote the degree of vertex v_i in a connected graph G. Show that

$$\frac{1}{m} \sum_{v_i \sim v_j} \sqrt{d_i d_j} < \lambda_1(G) < \max_i \frac{1}{d_i} \sum_{v_i \sim v_j} \sqrt{d_i d_j}.$$

(7) Let G be a graph with n vertices such that every pair of vertices has a unique common neighbour. Prove that there is a vertex with degree $n-1$.

[Hint: Prove first what should happen if G is regular by using spectral techniques. If G is nonregular show that G has no 4–cycles and deduce the result.]

(8) Let T be a tree with n vertices. Denote by m_k the number of matchings of T with k edges. Show that the coefficient of λ^{n-2k} in the characteristic polynomial of T is $(-1)^k m_k$.

(9) Let G be a r–regular graph with n vertices. Let G' be obtained from G by adding a vertex and joining it to all vertices in G. Show that

$$\phi(G', x) = (x^2 - rx - n)\phi(G, x)/(x - r).$$

(10) Let $G \Box H$ denote the cartesian product of G and H. The vertex set is $V(G) \times V(H)$ and $(x, y) \sim (z, t)$ if one of the coordinates agree and the other one is a pair of adjacent vertices.

(a) Show that the Laplace eigenvalues of $G \Box H$ are precisely $\mu_i(G) + \mu_j(H)$, for all i, j.

(b) The n–cube Q_n is defined as $Q_1 = K_2$ and $Q_n = K_2 \Box Q_{n-1}$ for $n \geq 2$. Determine $\mu_2(Q_n)$.

(c) Show that $i(Q_n) = 1$.

(11) Consider the following additional types of graph products in addition to the Cartesian product. The set of vertices in each case is the cartesian product of vertex sets of factors.

(i) The categorical product $G \times H$ has edge set $(x, y) \sim (x', y') \Leftrightarrow x \sim x'$ and $y \sim y'$.

(ii) The strong product $G \otimes H$ has edge set the union of categorical and Cartesian products, namely $(x, y) \sim (x', y')$ if either $x \sim x'$ and $y = y'$ or $y \sim y'$, or $x = x'$ and $y \sim y'$.

(ii) The lexicographic product $G[H]$ has edge set $(x, y) \sim (x', y')$ if either $x \sim x'$ and $y \neq y'$ or $x = x'$ and $y \sim y'$ (note that this one is not commutative).

Describe the eigenvectors and eigenvalues of each of the above products in terms of the eigenvectors and eigenvalues of the factors.

(12) Let G be an r–regular graph with eigenvalues $r = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$. Show that the stability number $\alpha(G)$ (cardinality of the largest clique in the complement of G) satisfies

$$\alpha(G) \leq \frac{-n\lambda_{n-1}}{r - \lambda_{n-1}}.$$

Find an analogous inequality for $\omega(G)$ the cardinality of the largest clique of G.

(13) Let G be a graph with maximum eigenvalue of the Laplacian μ_n. Show that

$$\mu_n \leq \max_{x \sim y}(d(x) + d(y)).$$

If the graph is connected, then equality holds only if G is bipartite regular (or semiregular: from each side).

[Hint: Let $L_1 = L + 2A$, where L is the Laplacian and A the adjacency matrix. If $\theta_1 \geq \cdots \geq theta_n$ are the eigenvalues of L_1, show that $\mu_n \leq \theta_1$ and show that θ_1 is at most the largest eigenvalue of the adjacency matrix of the line graph $L(G)$ plus 2.]

(14) Let G be connected with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that

$$1 + \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \mu_i.$$

[Hint: Concentrate on the set Z of vertices with degree at most m and on its neighbours, $N(Z)$. Use the partition given by these two sets and use interlacing. For the quotient matrix use the trace.]

(15) Prove that the Petersen graph is not Hamiltonian by using the observation that a Hamiltonian cycle in the Petersen graph would give an induced cycle of length 10 in its line graph. Consider the spectrum of this line graph and use interlacing.

(16) Show that a regular graph G with $\mu_2 \geq 1$ has a perfect matching.

[Hint: One has to use the Tutte theorem and a clever argument. You may want to check the page entitled Eigenvalues and Perfect Matchings in the webpage of the course.]

(17) Let $\mathcal{G}$ be the class of graphs which can be obtained from an isolated vertex by any sequence of operations (i) add an isolated vertex, and (ii) take the complement. Show that, for a graph $G \in \mathcal{G}$,

$$\mu_{n-j} \geq |\{x \in V(G) : d(x) \geq j\}|.$$

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Cuts and Diameter

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1. Algebraic Connectivity

The second smallest eigenvalue μ_2 of the Laplacian matrix of a graph G has strong connections with connectivity properties G and it is often called the *algebraic connectivity* of the graph.

By the Courant–Fisher inequalities, one has

$$\mu_2(G) = \min\{x^T L x : x \perp \mathbf{1}, \|x\| = 1\}.$$

A relation which avoids restricting on the the subspace orthogonal to $\mathbf{1}$ is given in the next Proposition by Fiedler:

Proposition 1.1.

$$\mu_2(G) = 2n \min \left\{ \frac{x^T L x}{\sum_{i,j}(x_i - x_j)^2} : x \neq \lambda \mathbf{1} \right\}.$$

Proof. We observe that both $x^T L x$ and $\sum_{i,j}(x_i - x_j)^2$ are invariant by substitution of x by $x + c\mathbf{1}$. For the first one because $\mathbf{1}$ is both a right and left eigenvector of L with eigenvalue 0, and for the second one by an obvious reason.

Now, if $x^T \mathbf{1} = c$ then $x - (c/n)\mathbf{1}$ is orthogonal to $\mathbf{1}$. Therefore, the minimum is the same as restricted to the orthogonal space to $\mathbf{1}$. In this subspace we have

$$\sum_{i,j}(x_i - x_j)^2 = 2\sum_{i,j} x_i^2 - 2\sum_{i,j} x_i x_j = 2n x^T x - 2(x^T \mathbf{1})^2 = 2n x^T x.$$

□

We observe that μ_2 (and in fact each μ_i) is monotone on subgraphs.

Proposition 1.2. *Let G and H be two edge–disjoint graphs on the same set of vertices and $G \oplus H$ the graph with the union of the edge sets of G and H. Then*

$$\mu_2(G \oplus H) \geq \mu_2(G) + \mu_2(H).$$

Proof. We have $L(G \oplus H) = L(G) + L(H)$ and the Courant–Fisher inequalities give

$$\begin{aligned} \mu_2(G \oplus H) &= \min\{x^T L(G \oplus H) x : x \perp \mathbf{1}, \|x\| = 1\} \\ &\geq \min\{x^T L(G) x : x \perp \mathbf{1}, \|x\| = 1\} + \min\{x^T L(H) x : x \perp \mathbf{1}, \|x\| = 1\} \\ &= \mu_2(G) + \mu_2(H). \end{aligned}$$

□

The name algebraic connectivity for μ_2 is partly suggested by the following propositions.

Proposition 1.3. *For a graph G with connectivity $\kappa(G)$,*

(1) *G is connected if and only if $\mu_2(G) > 0$,*
(2) *If $G \neq K_n$ then $\mu_2(G) \leq \kappa(G)$.*

Proof. From $x^T Lx = \sum_{ij\in E(G)}(x_i - x_j)^2$, we see that $x^T Lx = 0$ if and only if x is constant on connected components of G. In particular, if G is connected then the eigenvalue 0 has multiplicity one and $\mu_2 > 0$ (all eigenvalues are nonnegative because the quadratic form $x^T Lx$ is positive semidefinite).

For the second part, let S be a separating set of G and let G' be the graph obtained from G by adding all edges from S to $V \setminus S$. Let C_1 be a connected component of $G[V \setminus S]$ and let $C_2 = V \setminus (C_1 \cup S)$. Let G' be the graph obtained from G by adding all edges between S and $V \setminus S$. Let x be a vector which is constant on C_1 with value $1/|C_1|$, constant on C_2 with value $-1/|C_2|$ and 0 on S. Then $x^T\mathbf{1} = 0$ and $x^T x = \frac{1}{|C_1|} + \frac{1}{|C_2|}$. On the other hand

$$x^T LX = \sum_{ij\in E(G)} (x_i - x_j)^2 = \frac{1}{|C_1|^2}|S||C_1| + \frac{1}{|C_2|^2}|C_2||S|.$$

By monotonicity of μ_2 and the Courant–Fisher inequalities, $\mu_2(G) \leq \mu_2(G') \leq x^T Lx/(x^T x) = |S|$. The result follows by taking S a minimal separating set. □

2. Cuts in graphs

Let A, B be disjoint sets of vertices in graph G. We denote by $E(A, B)$ the set of edges joining vertices in A with vertices in B, and by $e(A, B)$ its number. In particular, $e(A, V \setminus A)$ is an edge–cut.

Proposition 2.1 (Spectral Bounds on edge cuts)**.** *For a subset $A \subset V$ the following holds,*

$$\frac{\mu_1}{n} \leq \frac{e(A, V \setminus A)}{|A|(n - |A|)} \leq \frac{\mu_n}{n}.$$

Note that the above bounds indicate that in a graph with small spectral radius, the size of edge cuts depend basically on the cardinality of A.

The *bisection width* $bw(G)$ of graph G is the minimum edgecut separating the graph into two equal parts (or parts differing in one unity if n is odd).

Proposition 2.2 (Spectral bounds for bisection width)**.** *If n is even then*

$$bw(G) \geq \frac{n}{4}\mu_1(G).$$

If n is odd then

$$bw(G) \geq \frac{n^2 - 1}{4n}\mu_1(G).$$

The *maxcut*, $mc(G)$, of graph G is the maximum of the edgecuts in the graph.

Proposition 2.3 (Spectral bounds for maxcut)**.** *We have*

$$mc(G) \leq \frac{n}{4}\mu_n(G).$$

The *isoperimetric number*, $i(G)$, of graph G is

$$i(G) = \min\{\frac{e(A, V \setminus A)}{|A|}, \quad A \subset V, 0 < |A| \leq n/2\}.$$

Theorem 2.4 (Spectral bound for the isoperimetric number). *We have*

$$i(G) \geq \mu_1(G)/2.$$

On the other hand, if $G \neq K_n$ then

$$i(G) \leq \sqrt{2\mu_1} \text{ (Cheeger inequality).}$$

The bisection width, maxcut and isoperimetric number are NP-hard problems in general.

3. Spectral bounds on the Diameter

Recall that the diameter $\text{diam}(G)$ of an r–regular graph G satisfies the Moore bound $diam(G) \geq \log_{r-1}(1 + (n-1)(r-2)/r) = \Omega(\ln n)$.

Theorem 3.1 (Spectral bound for the diameter). *Let G be a nonbipartite connected r–regular graph and $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$, where $r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ is the spectra of G. Then*

$$diam(G) \leq \frac{\log(n-1)}{\log(r/\lambda)} + 1.$$

One can obtain a general bound which includes the bipartite graphs by considering instead the matrix $M(G) = (1/2d)A(G) + (1/2)I$:

Theorem 3.2 (Bound on diameter). *Let G be a d–regular graph of order n. Let $\mu = \max\{|\mu_1|, |\mu_{n-1}|\}$ be the second largest eigenvalue of $M = M(G)$ in absolute value. Then*

$$diam(G) \leq \frac{\ln n}{1 - \mu}.$$

The spectral gap is an important property of a graph. As seen by the spectral estimations of the isoperimetric number or the diameter, the larger is the spectral gap the more expanding properties the graph has. The following result gives an upper bound for the length of this gap in regular graphs.

Theorem 3.3 (Alon). *Let G be an r–regular graph with diameter* $\text{diam}(G) \geq 2b + 2 \geq 4$. *Then the second largest eigenvalue $\lambda = \lambda_2$ of the adjacency matrix of G satisfies*

$$\lambda \geq 2\sqrt{r-1} - \frac{2\sqrt{r-1} - 1}{b}.$$

Our last result in this session concerns the so–called 'doubling distance'. Let $S \subset V$ be a set of vertices of the d-regular graph G of order n and $|S| \leq n/4$. The *ball* of S with radios k, $B_k(S)$, is the set of vertices at distance at most k from some vertex in S. The *doubling distance* $doubl(G)$ is the smallest k such that $|B_k(S)| \geq 2|S|$ for each subset $S \subset V$ of cardinality at most $n/4$.

Theorem 3.4 (Bound for the Doubling distance)**.** *Let G be a d–regular graph and let μ_1 be the second smallest eigenvalue of its Laplacian matrix. Then the doubling distance of G verifies*

$$doubl(G) \leq \sqrt{\frac{8d}{\mu_1}}.$$

4. Exercices

(1) The k–dimensional torus C_n^k is the cartesian product of k copies of the cycle on n vertices. Compute the doubling distance of this graph and give upper and lower bounds for its isoperimetric number.

(2) The Paley graph $Pay(p)$ is defined as follows. Take a prime p whith $p \cong -1 \pmod 4$. Consider the graph with vertex set the integers modulo p and two vertices i, j form an edge if $i - j$ is a square in $\mathbb{Z}/p\mathbb{Z}$. Give upper and lower bounds for the isoperimetric number of $Pay(p)$ and give asymptotic expressions for this number when $p \to \infty$.

(3) Deduce the spectral bound for the diameter in Theorem 3.2.
[Hint: mimic the proof of Theorem 3.1 with the matrix $M(G) = (1/2d)A(G) + (1/2)I$. Note that this matrix is positive semidefinite and $\mathbf{1}$ is an eigenvector.]

(4) Let k be the largest integer such that, for every set S of cardinality at most $n/6$, the k–th ball of S is three times as large as S, $|B_k(S)| \geq 3|S|$. Give a bound for this parameter for a r–regular graph in terms of the second smallest eigenvalue of the Laplacian matrix of G.

(5) Let G be a circulant graph, namely, its adjacency matrix is circulant (ech row is a cyclic shift of the preceding one). Show that $i(G) \leq 10rn^{-1/r}$, where r is the degree of G.
Hint: L. Lovász, Combinatorial Problems and exercises, Prob 11.32 (a).

(6) Let G be a vertex symmetric graph with diameter D. Sow that $i(G) \geq 1/(2D)$.
Hint: L. Lovász, Combinatorial Problems and exercises, Prob 11.32 (b).

Master on Applied Mathematics and Mathematical Engineering

Graph Theory.

Some applications of the Laplacian of a graph

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1. Matrix Tree Theorem

A spanning tree of a connected graph G of order n is a connected acyclic subgraph of order n. The number of spanning trees of a connected graph can be derived from the spectrum of its Laplacian matrix.

Proposition 1.1. *The incidence matrix of a graph G is totally unimodular, namely, the determinant of each square submatrix belongs to $\{-1, 0, 1\}$.*

Proof. Let N be the $n \times m$ incidence matrix of an orientation of a graph. By induction on the size ℓ of the submatrix. If there is a row or column with all-zero entries then the determinant is zero. This is also the case if every column has entries 1 and -1, because in this case the sum of the rows is zero. Otherwise there is a column with an only entry different from zero. By expanding the determinant by this column we get the determinant of an $(\ell - 1) \times (\ell - 1)$ submatrix which is in $\{-1, 0, 1\}$ by induction. □

Proposition 1.2. *The rank of the incidence matrix of a connected graph G with n vertices is $n-1$.*

Proof. If G is connected then the eigenvalue $\mu_1 = 0$ of the Laplacian has multiplicity one. This means that $L = NN^T$ has rank $n - 1$ and therefore so does N. □

Theorem 1.3 (Matrix tree Theorem)**.** *The number $\tau(G)$ of spanning trees of a graph G is*

$$\tau(G) = \det(L_{(i,i)}) = \frac{1}{n}\mu_2 \cdots \mu_n,$$

where $L_{(i,i)}$ denotes the minor of the Laplacian matrix L of G obtained by deleting row i and column i and $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ denote the eigenvalues of L.

Proof. One proof uses the Cauchy–Binet formula, which is worth knowing.

Lemma 1.4. *Let A be an $n \times m$ matrix, B a $m \times n$ matrix, $m \leq n$, and $D = diag(d_1, \ldots, d_m)$. Then*

$$\det(ADB) = \sum_{S \in \binom{[m]}{n}} \det(A_S) \det(B^S) \prod_{i \in S} d_i$$

where A_S and B^S denote the $n \times n$ submatrices of A and B obtained by selecting the lines specified in S.

Proof. Treat the entries of D as indeterminates.

Then $(ADB)_{ij} = \sum_{k=1}^m a_{ik}b_{kj}d_k$. Therefore $\det(ADB) = P(d_1,\dots,d_m)$ is an homogeneous polynomial of degree n in the variables $d_1,\dots,d_m$.

By evaluating the polynomial with some $d_i = 0$ then we get zero for any values of the remaining variables, as in this case the rank of ADB is not full. It means that the coefficients of monomials $d_1^{t_1}\cdots d_m^{t_m}$ with some $t_i > 1$ are all zero.

Hence, the only nonzero coefficients are the ones of terms with $t_i \in \{0,1\}$. By evaluating P with the zero entries of t equated to zero we can read off the coefficient of that term, which is precisely $\det(A_S B^S)$. □

We now proceed with the proof. The matrix N has rank $n-1$ as the graph G is connected. Let N' be the matrix obtained from N by deleting a row. Since teh sum of all rows is zero, we still get a matrix with rank $n-1$. Then, by the Cauchy–Binet formula

$$\det L' = \det N'N'^T = \sum_{S\in\binom{[m]}{n-1}} N'_S(N'^T)^S = \sum_{S\in\binom{[m]}{n-1}} (\det N_S)^2.$$

By the unimodularity of N, all determinants $\det N_S$ are in $\{-1,0,1\}$, and they are nonzero if and only if the set of edges selected in S form a spanning tree of the graph.

We note that $\det L' = \frac{1}{n}\mu_2\cdots\mu_n$. This is so because all the n matrices L' obtained by deleting one row and one column have the same determinant. □

2. Shannon capacity

Theorem 2.1 (Hoffman). *Let G be an r–regular graph. Then the staibility number of G satisfies*

$$\alpha(G) \le \frac{-\lambda_n}{d-\lambda_n}$$

Proof. Choose a maximal stable set S and let 1_S the characteristic function of S. Consider the vector $x = 1_S - (|S|/n)\cdot 1$, which is orthogonal to 1. We have

$$x^T L x = 1_s^T L 1_S = r|S|.$$

On the other hand, by the Courant–Fisher inequalities, $\mu_n \le x^T Lx/x^T x$. we have $\|x\| = |S|(n-|S|)/n$. We should get it. □

The *strong* product $G\otimes H$ of two graphs G and H has the cartesian product $V(G)\times V(H)$ as vertex set and there is an edge $(x,y)\simeq(x',y')$ whenever $x = x'$ or $x\simeq x'$ in G and $y = y'$ or $y \simeq y'$ in H.

Denote by G^n the strong n–th power of G.

Lemma 2.2. *We have $\alpha(G\otimes H) \ge \alpha(G)\alpha(H)$. In particular, $\lim_{n\to\infty}\alpha(G^n)^{1/n}$ exists.*

Proof. The cartesian product of two stable sets is stable in the strong product. That the limit exists is a consequence of the Lemma of Fekete, which states that a function $f:\mathbb{N}\to\mathbb{N}$ satisfying $f(m+n)\ge f(m)f(n)$ has that limit (apply to $f(n) = \alpha(G^n)$).

□

The *Shannon capacity* of G is

$$\Theta(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}.$$

A *normal representation* of a graph G is a set $U \subset \mathbb{R}^n$ of vectors $u_1, \ldots, u_n$ with $\|u_i\| = 1$ with the property that u_i is orthogonal to u_j whenever i and j are nonadjacent in G. The *value* of a representation U is

$$v(U) = \min_{c\in S^{n-1}} \max_i \frac{1}{\langle c, u_i\rangle^2}.$$

The *Lovász Theta Function* of G is

$$\theta(G) = \min_U v(U).$$

The Lóvasz Theta function can be used to obtain a lower bound for the Shannon capacity of a graph.

Theorem 2.3 (Lovász bound)**.** *$\alpha(G^n) \le \theta(G^n)$. In particular,*

$$\Theta(G) \le \theta(G).$$

Proof.

Lemma 2.4. *If $u_1, \ldots, u_n$ and $v_1, ldots, v_m$ are orthonormnal representations of G and H repsectively then $u_i \otimes v_j$ is orthonormal representation of $G \otimes H$.*

Proof. We will use the fact that

$$(u \otimes v)(u' \otimes v')^T = (uu'^T)(vv'^T).$$

□

Lemma 2.5. *$\theta(G \otimes H) \le \theta(G)\theta(H)$.*

Proof. Let $u_1, \ldots, u_n$ and $uv_1, \ldots, v_m$ be optimal orthogonal representations of G and H respectively with handles c and d. Then $c \otimes d$ is a unit vector and

$$\begin{aligned}\theta(G \otimes H) &\le \max_{i,j} \frac{1}{\langle c \otimes d, u_i \otimes v_j\rangle^2} \\ &= \max_{i,j} \frac{1}{\langle c, u_i\rangle^2} \frac{1}{\langle d, v_j\rangle^2}\end{aligned}$$

□

We first show that $\alpha(G) \le \theta(G)$. Suppose that $1, \ldots, k$ is a maximal stable set and therefore $u_1, \ldots, u_k$ are mutually orthogonal. Hence

$$1 = \|c\|^2 \ge \sum_{i=1}^{k} \langle c, u_i\rangle^2 \ge \frac{\alpha(G)}{\theta(G)}.$$

Now, $\alpha(G^n) \le \theta(G^n) \le (\theta(G))^n$ so $\Theta(G) = \lim(\alpha(G^n)^{1/n}) \le \theta(G)$. □

A spectral bound can be derived for the Shannon capacity:

Proposition 2.6. *Let G be an r–regular graph and let λ be the smallest eigenvalue of its adjacency matrix. Then*

$$\Theta(G) \leq \frac{-n\lambda}{r-\lambda}.$$

Proof. We now use another proof. Let $M = A - \lambda I - ((r-\lambda)/n)J$. This is a positive semidefinite matrix (all eigenvalues are nonnnegative). Hence we can write $M = BB^T$ for some matrix B with rank $< n$. Let $x_1, \dots, x_n$ be the rows of B. We note that $x_i^T x_i = M_{ii} = -\lambda - (r-\lambda)/n$ while, if i, j are not adjacent, then $x_i^T x_j = M_{ij} = -(r-\lambda)/n$. Let c be a unit vector orthogonal to the rows of B and define

$$v_i = \frac{1}{\sqrt{-\lambda}} x_i + \frac{1}{\sqrt{-\lambda n((r-\lambda)}} c.$$

Hence, if ij are not adjacent, then $v_i^T v_j = (1/\lambda) x_i^T x_j + 1/(-\lambda n(r-\lambda) = 0$.

$$\frac{1}{\langle c, v_i \rangle^2} = \frac{-\lambda n}{r-\lambda}.$$

□

3. Exercices

(1) Let G be a connected graph. Show that

$$\tau(G) = \frac{1}{n} \mu_2 \mu_3 \cdots \mu_n,$$

where $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are the eigenvalues of the Laplacian of G. In particular, if G is r–regular, then

$$\tau(G) = \frac{1}{n} P'_G(r),$$

where P'_G is the derivative of the characteristic polynomial of G.

(2) Show that

$$\tau(K_{n,m}) = n^{m-1} m^{n-1}$$

(3) Let G be a directed graph. Prove a version of the matrix–tree theorem to count the number of rooted oriented spanning trees of G.

Proof. It is the same formula but now, if the root vertex is i then $\tau(G, i) = \det(L_i)$. □

(4) A multigraph G is a graph where multiple edges and loops are allowed. Let G be a loopless multigraph. Prove a version of teh Matrix–Tree theorem for multigraphs.

(5) Let G be the graph obtained from K_n by deleting $r < n/2$ independent edges. Show that

$$\tau(G) = n^{n-2}(1 - 2/n)^r.$$

Proof. Let $G = K_n - M$ where M is a matching. As it happens with the spectra of the adjacency matrix of the complement of a regular graph, the Laplacian spectra of a (nonecessarily regular) graph can be related to the one of its complement: From

$$L(G) + L(\overline{G}) = L(K_n) = (n-1)I_n - (J - I_n),$$

where I_n is the identity matrix and J is the all ones matrix, one sees that every eigenvector $\mathbf{x}$ of $L(G)$ orthogonal to $\mathbf{1}$ is also eigenvector of $L(\overline{G})$:

$$L(\overline{G})\mathbf{x} = nI_n\mathbf{x} - J\mathbf{x} - L(G)\mathbf{x} = n\mathbf{x} - \mu\mathbf{x} = (n-\mu)\mathbf{x}.$$

Therefore, if the Laplacian spectra of G is $0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n$ then the spectra of the Laplacian of $\overline{G}$ is

$$0 = \mu_1' \le (n - \mu_n) \le (n - \mu_{n-1} \le \cdots \le (n - \mu_2).$$

We therefore want to compute the spectra of a matching of i edges (together with $n - 2i$ isolated vertices. This graph has $n - i$ connected components, so the first $n - i$ eigenvalues are zero. For the remaining ones, recall that if a graph is thedisjoint union of two graphs then the spectra is simply the union of teh spectra. The spectra of an edge is $\{0, 2\}$ so the spectra of a matching of r edges is $\{0^r, 2^r\}$. It follows that the spectra of a matching of r edges on n points is

$$\{0^{n-r}, 2^r\}.$$

The spectra of the complement is

$$\{0, (n-2)^r, n^{n-r-1}\}.$$

By the matrix–tree Theorem,

$$\tau(G) = \frac{1}{n}(n - r - 1)^{n-r-1}(n-2)^r = n^{n-2}\left(1 - \frac{2}{n}\right)^r.$$

□

(6) Show that if G is an r–regular graph then

$$\tau(G) \le \frac{1}{n}\left(\frac{nr}{n-1}\right)^{n-1}$$

and characterize the case of equality.

(7) Let C be a Hamiltonian cycle in the complete graph K_n. Compute the number of spanning trees in K_n that have at least one edge in C.

(8) A wheel W_n is the graph obtained from the cycle C_n by adding one vertex adjacent to every vertex in the cycle. Find a formula for $\tau(W_n)$.

Proof. The number is

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

The cone of a graph G is the graph obtained from G by adding a new vertex adjacent to every vertex of G. If G is r–regular then the characteristic polynomial of the cone of G is $p(x) = (x^2 - rx - n)p_G(x)/(x - r)$. One can obtain a similar expression for the Laplacian (due to Kelman):

The Spectra of the Laplacian of the wheel W_n is

$$(0, n+1, 3 - 2\cos(2k\pi/n)), k = 1, \ldots, n-1)$$

□

(9) The bandwidth of a graph is $bw(G) = \max\{|i - j|, ij \in E(G)\}$ where the maximum is taken over all possible orderings of the vertices of G. Prove that

$$bw(G) \ge \lceil n\mu_2/\mu_n \rceil.$$

[Hint: Order the vertices according to the bandwidth of the graph. Then there are no vertices connecting the first $(n-bw(G))/2$ vertices and the last $(n-bw(G))/2$ vertices.]

(10) Let $S \subset V(G)$. Show that

$$\mu_2(G) \leq \mu_2(G \setminus S) + |S|.$$

Deduce that vertex connectivity $\kappa(G) \geq \mu_2(G)$.

[Hint: Consider a vector vanishing in S and whose coordinates in $V \setminus S$ form an eigenvector of the Laplacian of $G \setminus S$.]

(11) By checking C_5^2 give a lower bound on the Shannon capacity of the pentagon. Prove that $\Theta(C_5) = \sqrt{5}$.

(12) Compute the Shannon capacity of the Petersen graph.

(13) The Kneser graph $K(n,r)$, $n \geq 2r$, has the r–subsets of n as vertices and two vertices are adjacent if and only if they are disjoint. You may want to draw $K(5,2)$. Show that

$$\Theta(K(n,r)) = \binom{n-1}{r-1}.$$

[Hint: For the lower bound, find a stable set of $K(n,r)$ with cardinality $\binom{n-1}{r-1}$. For the upper bound, use that $K(n,r)$ has eigenvalues $(-1)^t\binom{n-r-t}{r-t}, t = 0,1,\ldots,r$. You may want to compute these values.]

(14) A graph is vertex–transitive if for every pair of vertices $x, y \in V(G)$ there is an automorphism of the graph ϕ such taht $\phi(x) = \phi(y)$. Show that, if G is vertex transitive, then $\theta(G)\theta(\bar{G}) = n$. In particular, if G is a vertex–transitive self–complementary graph then $\theta(G) = \sqrt{n}$.

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Random Graphs: The $\mathcal{G}_{n,p}$ model.

---

1. Models of Random graphs

The model $\mathcal{G}_{n,p}$ is the space of graphs of n vertices in which we choose each edge in $\binom{[n]}{2}$ independently with probability p. We denote by $N = \binom{n}{2}$.

Proposition 1.1. *Let H be a fixed (labelled) graph on n vertices and m edges. Let G be a random graph in $\mathcal{G}_{n,p}$. We have*

$$\Pr(H \subset G) = p^m.$$

Moreover, the probability that H is an induced graph of G is

$$\Pr(H \subset_i G) = p^m(1-p)^{N-m}.$$

In particular all graphs with n vertices have the same probability in $\mathcal{G}_{n,1/2}$

Proof. The probability that an edge ij in H is in G is p and they are independent. For the second part we should add that the nonedges of H are nonedges in G.

In particular, if $p = 1/2$ and H is a graph on n vertices, $\Pr(G = H) = (1/2)^{\binom{n}{2}}$, and there are $2^{\binom{n}{2}}$ (labelled) graphs on n vertices. □

The probability that a random graph contains an isomorphic copy of a given graph H is more involved but simple upper bounds can be obtained.

Proposition 1.2. *Let G be a random graph in $\mathcal{G}_{n,p}$. Then, for each fixed k,*

$$\Pr(\omega(G) \geq k) \leq \binom{n}{k} p^{\binom{k}{2}}.$$

Analogously

$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k} (1-p)^{\binom{k}{2}}.$$

Proof. For a fixed subset $K \in \binom{[n]}{k}$, we have

$$\Pr(G[K] = K_k) = p^{\binom{k}{2}}.$$

Since the event $\omega(G) \geq k$ means that $G[K] = K_k$ for some subset K,

$$\Pr(\omega(G) \geq k) \leq \Pr(\bigcup_{K \in \binom{[n]}{k}} \Pr(G[K] = K) \leq \sum_{K \in \binom{[n]}{k}} \Pr(G[K] = K) = \binom{n}{k} p^k.$$

The second part is analogous. □

The above inequalities can be derived by an argument based on expectation.

Proposition 1.3. *Let H be a fixed graph with k vertices and m edges. Let X_H be the random variable which counts the number of isomorphic copies of H in a random graph $G \in \mathcal{G}_{n,p}$. Then*

$$\mathbb{E}(X_H) = \binom{n}{k}\left(\frac{k!}{a}\right)p^m,$$

where a is the cardinality of the automorphism group of H.

In particular, the expected number of cycles of length k is

$$\mathbb{E}(X_{C_k}) = \frac{(n)_k}{2k}p^k.$$

where $(n)_k = n(n-1)\cdots(n-k+1)$.

Proof. For each $K \in \binom{[n]}{k}$ let Y_K be the number of isomorphic copies in $\mathbb{G}[K]$. For each labeled copy of H in K, let Z_H be the indicator function that this precise labeled copy is in $\mathbb{G}[K]$. We clearly have $\mathbb{E}(Z_H) = \Pr(H \subset \mathbb{G}) = p^m$. The number of labeled copies of H in K is $(k!/a)$ (every permutation of the elements in K which belongs to the automorphism group gives rise to the same labeled copy of H). Therefore

$$\mathbb{E}(X_H) = \sum_{K\in\binom{[n]}{k}} \mathbb{E}(Y_K) = \binom{n}{k}\sum_{H\in\mathbb{G}[K]} Z_H = \binom{n}{k}\left(\frac{k!}{a}\right)p^m.$$

For $H = C_k$ we have $Aut(C_k) = D_{2k}$ the dihedral group with cardinality $2k$. □

2. The probabilistic method

The versatility of the $\mathcal{G}_{n,p}$ model allows one to prove existence results by showing that the probability of an event is nonzero. This is the framework of the so–called probabilistic method. The classical (and most striking) examples of application are the following theorems of Erdős.

Theorem 2.1 (Erdős). *For every k there is a graph G with $n \geq 2^{k/2}$ such that*

$$\alpha(G) < k \text{ and } \omega(G) < k.$$

In other words, the k–th diagonal Ramsey number verifies $R(k) > 2^{k/2}$.

Proof. Consider a random graph $G \in \mathcal{G}_{n,p}$ with $p = 1/2$. By Proposition 1.2, we have

$$\Pr(\omega(G) \geq k, \alpha(G) \geq k) \leq \Pr(\alpha(G) \geq k)) + \Pr(\alpha(G) \geq k)) \leq \binom{n}{k}2^{-\binom{k}{2}+1}.$$

The conclusion follows if we show that the right–hand side is smaller than one for $n = 2^{k/2}$. Some rough estimations and the Stirling formula give

$$\binom{n}{k}2^{-\binom{k}{2}+1} \leq 2\frac{n^k}{(k/e)^k\sqrt{2\pi k}2^{k(k-1)/2}} \leq \left(\frac{ne}{2^{k/2}k}\right)^k \leq \left(\frac{e}{k}\right)^k,$$

which is less than one for $k \geq 3$. □

Explicit constructions of triangle–triangle free graphs with arbitrary larhe chromatic number are known. However, graphs with arbitrary large girth and chromatic number are more involved to construct. The probabilstic argument shows that such objects exist.

Theorem 2.2 (Erdős)**.** *For every fixed $k > 3$ there is a graph G with chromatic number $\chi(G) > k$ and girth $g(H) > k$.*

Proof. By Proposition 1.2 we know that a random graph $\mathbb{G} \in \mathcal{G}_{n,p}$ satisfies

$$\Pr(\alpha(G) \geq r) \leq \binom{n}{r}(1-p)^{\binom{n}{r}} \leq n^r e^{-pr(r-1)/2}.$$

If $p \geq 6k \ln n/n$ and $r \geq n/2k$ then the right hand side satisfies

$$ne^{-p(r-1)/2} \leq ne^{-3\ln n/2} e^{1/2} = (e/n)^{1/2} \to 0 \ (n \to \infty).$$

Therefore, with the appropriate range of p, for every sufficiently large n we have

$$\Pr(\alpha(G) \geq n/2k) < 1/2.$$

Let X be the number of cycles in $G \in \mathcal{G}_{n,p}$ with length at most k. By Proposition 1.3, the expected number of X satisfies

$$\sum_{\ell=3}^{k} \mathbb{E}(X_{C_l}) = \sum_{\ell=3}^{k} \frac{(n)_\ell}{2\ell} p^\ell \leq \frac{1}{2}\sum_{\ell=3}^{k}(np)^\ell \leq \frac{k}{2}(np)^k,$$

where the last inequality holds whenever $np \geq 1$. By Markov's inequality,

$$\Pr(X \geq n/2) \leq \frac{\mathbb{E}(X)}{n/2} \leq kn^{k-1}p^k \leq k(6k)^k \left(\frac{\ln n}{n}\right)^k \to 0 \ (n \to \infty).$$

Therefore, for $p \geq 6k \ln n/n$ so that both conditions for p are satisfied, the above probability is also less than $1/2$ for sufficiently large n. Hence there is a graph G with stability number $\alpha(G) \geq n/2k$ and at most $n/2$ cycles of length at most k. By deleting at most $n/2$ vertices in G we obtain a graph G' with girth at least k and chromatic number

$$\omega(G') \geq \frac{n/2}{\alpha(G')} \geq \frac{n/2}{n/2k} = k.$$

□

3. Properties of almost all graphs

Let P be a property that a graph may have. We identify with P the class of graphs which satsify the property P. Every class of graphs closed by isomorphisms is such a class. Let $p = p(n)$. We say that almost every graph in $\mathcal{G}_{n,p(n)}$ has property P if

$$P(G \in P) \to 1, \ (n \to \infty).$$

The following are examples of properties shared by almost all graphs in $\mathcal{G}_{n,p}$ for fixed $p \in (0, 1)$.

Proposition 3.1. *Let $0 < p < 1$ be fixed. For every graph H, almost every graph $G \in \mathcal{G}_{n,p}$ contains an induced copy of H.*

Proof. Let k be the number of vertices and m the number of edges of H. For every labeled copy $H \in \mathcal{G}_{n,p}$,

$$\Pr(H \subset_i G) = p^m (1-p)^{\binom{n}{2}-m} = \alpha.$$

Choose a partition of $[n]$ with $h = \lceil n/k \rceil$ sets of size k. The probability that none of them contains an induced copy of H is an upper bound for

$$\Pr(H \not\subset_i G) \leq (1-\alpha)^{n/k} \to 0, \ (n \to \infty).$$

□

For fixed positive integers i, j a graph G has the $\mathcal{P}(i,j)$ property if, for every choice of sets $U \in \binom{[n]}{i}$ and $W \in \binom{[n]}{j}$ there is $x \in V(G) \setminus (U \cup W)$ such that $U \subset N(x)$ and $W \cap N(x) = \emptyset$.

Theorem 3.2. *Fix $p \in (0,1)$ and positive integers i, j. Almost every graph in $\mathcal{G}_{n,p}$ has property $\mathcal{P}(i,j)$.*

Proof. Let U and W be fixed. The probability that some vertex $x \notin U \cup W$ has a neighborhood containing U and disjoint from W is $p^i(1-p)^j$. Therefore, the probability that such x does not exist is at most $(1-p^i(1-p)^j)^{n-i-j}$. There are at most n^{i+j} choices for the pairs $U, , W$. Therefore,

$$\Pr(\mathbb{G} \notin P(i,j)) \leq n^{i+j}(1-(1-p^i(1-p)^j))^{n-i-j} \to 0 \ (n \to \infty),$$

where the value of the limit follows from the fact that $1 - p^i(1-p)^j < 1$ for every $i, j \geq 1$. □

The property $\mathcal{P}(i,j)$ has a special relevance. It can be shown to prove almost sure properties for a wide class of statements.

Proposition 3.3. *Let $p \in (0,1)$ be fixed. For every positive integer k almost every graph in $\mathcal{G}_{n,p}$ is k–connected.*

Proof. Almost all graphs have property $\mathcal{P}(2, k-1)$. For every pair x, y of vertices and every subset $K \in \binom{[n]}{k-1}$, there is a vertex $z \notin K \cup \{x, y\}$ adjacent to x, y (and adjacent to no vertex in K). Therefore the removal of K does not disconnect the graph. □

More strikingly, property $\mathcal{P}(i,j)$ can be used to prove the following general $(0,1)$–law. Recall that a first order logic sentence is one which can be expressed by a finite number of connectives, negations, relations and quantifiers on elements of a relational system.

Theorem 3.4 ($(0,1)$–law)**.** *Let $p \in (0,1)$ be fixed. Let $\mathcal{P}$ be any property which can be expressed in first order logic. Then,*

$$\lim_{n\to\infty} \Pr(G \in \mathcal{P} \cap \mathcal{G}_{n,p}) \in \{0,1\}.$$

The proof is based on the *random graph* on a countable set of vertices which can be shown to verify the $\mathcal{P}(i,j)$ properties and its uniqueness up to isomorphisms (see the exercises). The proof uses some ingredients of model theory beyond the scope of this course.

As a last example we show that almost every graph in $\mathcal{G}_{n,p}$ has a considerably large chromatic number.

Theorem 3.5. *For every $\epsilon > 0$, almost every graph in $\mathcal{G}_{n,p}$ has chromatic number*

$$\chi(G) \geq \frac{\log 1/q}{2+\epsilon} \frac{n}{\log n},$$

where $q = 1 - p$.

Proof. By Proposition 1.2 we have

$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k} q^{\binom{k}{2}} \leq n^k q^{\binom{k}{2}} = e^{k \ln n + (k(k-1)/2) \ln q}.$$

The exponent can be written as

$$\frac{k}{2} \left(2 \ln n - (k-1) \ln(1/q)\right).$$

If $k = (2+\epsilon) \ln n / \ln(1/q)$ then $2 \ln n - (k-1) \ln(1/q) = -\epsilon \ln n + \ln(1/q)$ which is negative for every sufficiently large n. It follows that

$$\Pr(\alpha(G) \geq k) \leq e^{-\epsilon \ln n + \ln(1/q)} \to 0 \ (n \to \infty).$$

In particular for this value of k,

$$\Pr(\chi(G) > n/k) \leq \Pr(\alpha(G) < k) \to 1 \ (n \to \infty).$$

□

4. The second moment method

For a nonnegative random variable X and a positive real number c, the *Markov* inequality states

$$\Pr(X \geq c) \leq \mathbb{E}(X)/c.$$

This is the basis of the so–called *first moment method.*

Lemma 4.1 (First moment method)**.** *Let X_n be a random variable on $\mathcal{G}_{n,p}$ taking integer values. If*

$$\mathbb{E}(X_n) \to 0 \ (n \to \infty),$$

then

$$\Pr(X_n = 0) \to 1 \ (n \to \infty).$$

Proof. By the Markov inequality, for every positive $c \in \mathbb{R}$, we have $\Pr(X_n \geq c) \leq \mathbb{E}(X_n)/c \to 0$, $(n \to \infty)$. □

In many situations we would also like to show that X_n takes positive valuesfor almost all graphs. The so–called *second moment method* is devised to achieve this goal.

Application of Markov's inequality to $Y = \mathbb{E}(X - \mathbb{E}(X))^2$ the *Tchebyshev* inequality states

$$\Pr(|X - \mathbb{E}(X)| \geq c) \leq Var(X)/c^2.$$

Lemma 4.2 (Second moment method)**.** *Let X_n be a random variable on $\mathcal{G}_{n,p(n)}$ taking values in $\mathbb{N}$. Let $\mu_n = \mu(X_n)$ and $\sigma_n^2 = \sigma^2(X_n)$.*

(i) *If $\mu_n = o(1)(n \to \infty)$ then $X_n = 0$ for almost all graphs.*
(ii) *If $\sigma_n^2 = o(\mu_n^2)(n \to \infty)$ then $X_n > 0$ for almost all graphs.*

Proof. Part (i) is the first moment method. For part (ii) we can write

$$\Pr(X_n = 0) \leq \Pr(|X_n - \mathbb{E}(X_n)| \leq \mathbb{E}(X_n)) \leq \sigma_n^2/\mu_n^2.$$

If (ii) holds then $X_n > 0$ almost surely. □

The following is an application of the second moment method. Let X_r denote the number of copies of the complete graph K_r in a random graph. Let $\omega(G)$ denote the clique number of G.

Theorem 4.3. *Let $p \in (0,1)$ and denote by $r = 2\ln n/\ln(1/p)$. Almost all graphs in $\mathcal{G}_{n,p}$ verify*

$$r - 1 \leq \omega(G) \leq r.$$

Proof. Let $X_{n,r}$ be the number of copies of K_r in $G(n,p)$. We have

$$\mathbb{E}(X_{n,r}) = \binom{n}{r} p^{\binom{r}{2}} \leq (np^{(r-1)/2})^r = e^{(r/2)(2\ln n - (r-1)\ln(1/p))}.$$

By taking $r = 2\ln n/\ln(1/p)$ we see that the right–hand side is $e^{-r/2} \to 0\,(n \to \infty)$. By the first moment method, this shows that almost no graphs have a clique of size larger than r.

For the lower bound we need some more weapons. We can refine the estimation of $\mathbb{E}(X_{n,r}$ by using Stirling formula:

$$\mathbb{E}(X_{n,r}) = \binom{n}{r} p^{\binom{r}{2}} \geq \frac{(n-r)^r}{r!} p^{r(r-1)/2} \geq \left(\frac{e(n-r)}{r}\right)^r p^{r(r-1)/2}$$

The right–hand side can be written as

$$exp\{\frac{r}{2}\left(2 + 2\ln(\frac{n-r}{r}) - (r-1)\ln(1/p)\right)\}$$

Therefore, if $r - 1 < 2\ln n/\ln(1/p)$ the exponent is positive (one must lower bound $\ln(n/r - 1) \geq 2\ln r - O(\ln\ln n)$ whenever $r < 2\ln n$) and get $\mathbb{E}(X_n) \to \infty$, $(n \to \infty)$.

The above bounds can be used to show that $\mathbb{E}(X_{n,r}) \to \infty$ when $r = 2\log_{1/p} n - 1$, but this is not enough. By the second moment method, we are in good shape if we can prove

$$Var(X_{n,r})/(\mathbb{E}(X_{n,r})^2) \to 0 \ (n \to \infty).$$

We have

$$Var(X_{n,r}) = \sum_S Var(X_S) + \sum_{S,T} Cov(X_S, X_T)$$

where X_S is the indicator variable that $G[S]$ is an r–clique and the sum is extended to all r–subsets S, T of $[n]$. By using $X_S^2 = X_S$, we have

$$\begin{aligned} Var(X_{n,r}) &= \sum_S Var(X_S) + \sum_{S,T} Cov(X_S, X_T) \\ &\leq \sum_S \mathbb{E}(X_S^2) + \sum_{S,T} \mathbb{E}(X_S X_T) \\ &\leq \mathbb{E}(X_{n,r}) + \sum_{S,T:|S\cap T|\geq 2} \mathbb{E}(X_S X_T). \end{aligned}$$

Thus we have to show that $\sum_{S,T:|S\cap T|\geq 2}\mathbb{E}(X_S X_T)=o(\mathbb{E}(X_{n,r})^2)$.

$$\begin{aligned}\sum_{S,T:|S\cap T|\geq 2}\mathbb{E}(X_SX_T) &= \sum_{S,T:|S\cap T|\geq 2}\Pr(X_S=1)\Pr(X_T=1|X_S=1)\\ &= \sum_S \Pr(X_S=1)\sum_{T:|T\cap S|\geq 2}\Pr(X_T=1|X_S=1)\\ &= \mathbb{E}(X_{n,r})\sum_{T:|T\cap S_0|\geq 2}\Pr(X_T=1|X_S=1)\end{aligned}$$

where, by symmetry, all summands equal the one for a fixed $S=S_0$. Finally,

$$\sum_{T:|T\cap S_0|\geq 2}\Pr(X_T=1|X_S=1)=\sum_{i=2}^{r}\binom{r}{i}\binom{n-i}{r-i}p^{\binom{r}{2}-\binom{i}{2}}.$$

and the last expression, divided by $\mathbb{E}(X_{n,r})=\binom{n}{r}p^{\binom{r}{2}}$ equals

$$\sum_{i=2}^{r}\frac{\binom{r}{i}\binom{n-i}{r-i}}{\binom{n}{r}}p^{-\binom{i}{2}}\leq r^4/n^2\to 0,$$

as we wanted. □

5. Exercices

(1) A tournament is an oriented complete graph. Show that, for each n, there is a tournament with at least $n!/2^{n-1}$ direcetd Hamiltonian paths.

(2) Show that there is a constant $c=c(p)>0$ such that almost all graphs in $\mathcal{G}_{n,p}$ verify the following property: for each subset $X\in V(G)$ with cardinality $|X|\leq n/2$,

$$e(X,V\setminus X)\geq c|X|,$$

where $e(X,V\setminus X)$ denotes the number of edges connecting X with $V\setminus X$.

Proof. For a fixed $X\subset V$ with $|X|=x\leq n/2$, we have

$$\Pr(e(X)<cx)=\binom{x(n-x)}{cx}p^{cx}q^{n(n-x)-cx}\leq\binom{n^2}{cn}p^{cn}(1-p)^{n^2-cn}\leq$$

(should give an expression $e^{-n^2(p-c)}$ which works for $c<p$ and even absorvs 2^n sets. Check! □

(3) Show that for every graph H there is a function $p(n)$ with $\lim_{n\to\infty}p(n)=0$ such that almost all graphs in $\mathcal{G}_{n,p(n)}$ have an induced copy of H.

(4) Show that the greedy algorithm for coloring G (take an ordering of the vertices and give to vertex i the smallest available color) uses (with high probability) at most $\frac{n}{\log_{1/p}n}$ colors (Compare with Theorem 3.2(iii).)

Proof. Let X_j be the number of colours used to colour the first j vertices. We want to show that

$$\Pr(X_n>n/\log_{1/q}n)\to 0.$$

Let Y_j be the color of vertex j. Then

$$\Pr(Y_{j+1} = k+1) \leq \prod_{i=1}^{k}(1 - q^{U_i}),$$

where U_i is the number of vertices preceding $j+1$ which have colour i. This is so because $j+1$ must be adjacent to some vertex coloured i and the probability of this is $(1 - q^{U_i})$. Now, $\sum_i U_i \leq j$ so the right–hand side can be upper bounded by the value in the middle terms (because the function is convex):

$$\Pr(Y_{j+1} = k+1) \leq (1 - q^{j/k})^k \leq e^{-kq^{j/k}}.$$

Now,

$$\Pr(X \geq k+1) \leq \sum_{j=k+1}^{n} \Pr(Y_j = k+1) \leq ne^{-kq^{n/k}}.$$

By plugging in the value $k = n/\log_{1/q} n$ we see that

$$\Pr(X \geq k+1) \to 0, (n \to \infty).$$

□

(5) Show that, for p fixed, almost all graphs in $G_{n,p}$ have diameter two.
(6) Show that if almost all graphs have properties P and also property Q then almost all graphs have both properties.
(7) Show that, for p fixed, almost no graph in $G_{n,p}$ has a separating complete subgraph.
(8) Consider $\mathcal{G}_{\mathbb{N},p}$: the vertices are the natural numbers and each pair $\{i,j\}$ occurs with probability $p \in (0,1)$ independently.
    (a) Show that, for each r and s, the infinite random graph has the following property with probability 1: for each pair of disjoint subsets R, S of cardinalities r and s, there is a vertex x adjacent to all vertices in R and none in S.
    (b) Show that, for every finite set X, every automorphism of the subgraph $G[X]$ induced by X can be extended to an automorphism of the full graph. Conclude that every two graphs in $\mathcal{G}_{\mathbb{N},p}$ are isomorphic.
(9) Let $p(n) = (1-\epsilon)\ln n/n$ for some fixed $\epsilon > 0$. Let $\mathbb{G} \in \mathcal{G}_{n,p}$.
    (a) Show that almost all graphs in $\mathcal{G}_{n,p(n)}$ have an isolated vertex.
    (b) Show that $\mathbb{E}(\tau(\mathbb{G})) \to \infty$, $(n \to \infty)$, where $\tau(\mathbb{G})$ is the number of spanning trees in $\mathbb{G}$.
(10) Consider the following bipartite version $\mathcal{G}_{n,n,p}$ of random graphs. Let A, B be two sets of cardinality n and place each edge xy, $x \in A$, $y \in B$ with probability p independently. Is it true that, for fixed $0 < p < 1$, almost every graph in $\mathcal{G}_{n,n,p}$ has a perfect matching?

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Random Graphs:Threshold functions.

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1. Threshold fuctions

For fixed $p \in (0,1)$ the expected number of edges of a random graph with n vertices is $p\binom{n}{2}$. When instead one lets $p = p(n)$ be a function of n, Erdős and Rényi discovered that many natural properties of graphs appear suddenly according to the asymptotic behaviour of the function $p(n)$.

A property of graphs is *monotone* if it is preseved by addition of edges. A function $f : \mathbb{N} \to \mathbb{R}^+$ is a *threshold function* for a monotone graph property $\mathcal{P}$ if

$$Pr(G \in \mathcal{P}) = \begin{cases} 0, & \lim_{n\to\infty} \frac{p(n)}{f(n)} = 0 \\ 1, & \lim_{n\to\infty} \frac{p(n)}{f(n)} = \infty. \end{cases}$$

A graph G is *balanced* if the average degree of each subgraph H verifies $\bar{d}(H) \leq \bar{d}(G)$.

Theorem 1.1. *Let H be a balanced graph with k vertices and l edges. Let $p(n) = \gamma(n)n^{-k/l}$.*

(i) *If $\gamma = o(1)(n \to \infty)$ then almost no graph in $\mathcal{G}_{n,p(n)}$ contains H.*
(ii) *If $\gamma^{-1} = o(1)(n \to \infty)$ then almost all graphs in $\mathcal{G}_{n,p(n)}$ contain H.*

Proof. Let $\mathcal{H}$ be the family of graphs isomorphic to H in K_n. If h is the total number of graphs isomorphic to H in a set of cardinality $k = |V(H)|$ then

$$|\mathcal{H}| = \binom{n}{k} h \leq \binom{n}{k} k! \leq n^k.$$

Let X be the random variable which counts the number of copies of H in $\mathbb{G}$. For each subset $K \in \binom{[n]}{k}$ we denote the by X_K the number of copies of H in $\mathbb{G}[K]$. Each copy has probability p^l to appear in $\mathbb{G}$. Therefore,

$$\mathbb{E}(X) = \sum_{K \in \binom{[n]}{k}} \mathbb{E}(X_K) = \binom{n}{k} hp^l \leq n^k p^l = \gamma(n)^l.$$

It follows that

$$\gamma(n) \to 0 \;\Rightarrow\; \mathbb{E}(X) \to 0.$$

and, by Markov's inequality,

$$\Pr(H \in \mathbb{G}) = \Pr(X \geq 1) \leq \mathbb{E}(X) \to 0, \; (n \to \infty).$$

This proves the first part (the 0–statement).

For the second part (the 1–statement) we bound the variance of X. Let H, H' denote two labeled copies of H in K_n (some abuse of language here). We have

$$\Pr(H_1 \cup H_2 \in \mathbb{G}) = p^{2l-|E(H_1\cap H_2)|}.$$

Here we use the hypothesis that H is balanced to estimate the number of edges in $H_1 \cap H_2$ by its number of vertices: if $|V(H_1) \cap V(H_2)| = i$ then

$$|E(H_1 \cap H_2)| = i\bar{d}(H_1 \cap H_2)/2 \leq i\bar{d}(H)/2 = il/k.$$

For each labeled copy of H let 1_H denote the indicator random variable that $H \in \mathbb{G}$. We can write

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}(\sum_{H,H'\subset \mathbb{G}} 1_H)^2 \\
&= \sum_{H\in K_n} \mathbb{E}(1_H^2) + \sum_{i=0}^{k-1} \sum_{H\in K_n} \sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(1_H 1_{H'}) \\
&= \mathbb{E}(X) + \sum_{i=0}^{k-1} \sum_{H\in K_n} \sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(1_H 1_{H'}).
\end{aligned}$$

For $i = 0$, 1_H and $1'_H$ are independent and we get

$$\sum_{H\in K_n} \sum_{\substack{H'\subset K_n \\ H\cap H'=\emptyset}} \mathbb{E}(1_H 1_{H'}) = \sum_{H\in K_n} \mathbb{E}(1_H) \sum_{\substack{H'\subset K_n \\ H\cap H'=\emptyset}} \mathbb{E}(1_{H'}) \leq \mathbb{E}(X) \sum_{H\subset K_n} 1_H = (\mathbb{E}(X))^2.$$

If $|V(H) \cap V(H')| = i$ then, for fixed H,

$$\begin{aligned}
\sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(1_H 1_{H'}) &= \sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(\mathbb{E}(1_H 1_{H'} | 1_H = 1)) \\
&= \Pr(1_H = 1) \sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(1_{H'} | 1_H = 1) \\
&= \Pr(1_H = 1) \binom{k}{i} \binom{n-k}{k-i} hp^{l-il/k}.
\end{aligned}$$

For n large enough and every $1 \leq i \leq k-1$ we can write $\binom{k}{i}\binom{n-k}{k-i} \leq c_{k,i}\binom{n}{k} n^i$, where $c_{k,i}$ is some constant which only depends on k and i. Hence,

$$\begin{aligned}
\sum_{H\in K_n} \sum_{\substack{H'\subset K_n \\ |V(H)\cap V(H')|=i}} \mathbb{E}(1_H 1_{H'}) &\leq \sum_{H\in K_n} \Pr(1_H = 1) c_{k,i} n^i p^{-il/k} \binom{n}{k} hp^l \\
&= \mathbb{E}(X) c_{i,k} n^i p^{-il/k} \sum_{H\subset K_n} \Pr(1_H = 1) \\
&= (\mathbb{E}(X))^2 c_{k,i} \gamma^{-il/k}.
\end{aligned}$$

It follows that

$$\frac{Var(X)}{\mathbb{E}(X)^2} = \frac{\mathbb{E}(X^2) - \mathbb{E}(X)^2}{\mathbb{E}(X)^2} \leq c\gamma^{-l/k} \to 0, \; (n \to \infty).$$

By the second moment method, this implies that $X \geq 1$ for almost all $\mathbb{G} \in \mathcal{G}_{n,p}$ with this range of $p = \gamma n^{-k/l}$ and $\gamma^{-1} = o(1)$. $\square$

2. The evolution of random graphs

One way of looking at random graphs is viewing them as a random process in which we add edges one at a time. This process can be monitored with good approximation by setting the number m of edges at each time as the value of p in the $\mathcal{G}_{n,p}$ model for which $pn = m$. By using Theorem 1.1 we obtain the times when substructures appear.

Next we survey for which probabilities the most relevant properties appear. All the properties listed below have sharp thresholds and their are stated in an asymptotic sense: every property happens with high probability, meaning that the probability tends to one with $n \to \infty$.

- if $p \ll \frac{1}{n^2}$: Empty graph.
- if $p = \frac{1}{n^2}$: First edge appears.
- if $p = \frac{1}{n^{3/2}}$: First vertex of degree two (Birthday paradox).
- if $p = \frac{1}{n^{4/3}}$: First vertex of degree three.
- if $\frac{1}{n^{k+1/k}} \leq p \leq \frac{1}{n^{k+2/k+1}}$: Maximum degree k, all the trees on k vertices exists, no connected component of size $> k$.
- if $p = \frac{c}{n}$, $c < 1$: Largest component of size $\Theta(\log n)$. First triangle. All the cycles of constant size. Maximum degree of order $\Theta\left(\frac{\log n}{\log\log n}\right)$
- if $p = \frac{1}{n}$: Components of size $O(n^{2/3})$. Giant 3-core.
- if $p = \frac{c}{n}$, $c > 1$: Unique giant component of linear size. Other components of size $O(\log n)$.
- if $p = \frac{\log n - \omega(1)}{n}$, $c > 0$: One giant component of size $(1 - o(1))n$ plus some isolated vertices.
- if $p = \frac{\log n + \omega(1)}{n}$: Minimum degree one (Coupon collector problem), the graph is connected, there exists a perfect matching, diameter $\frac{\log n}{\log\log n}$.
- if $p = \frac{\log n + \log\log n}{n}$: Minimum degree two, the graph is 2-connected, there exists a hamiltonian cycle.
- if $p = \frac{\log n + (k-1)\log\log n}{n}$: Minimum degree k, the graph is k-connected, there exists $\lfloor k/2 \rfloor$ edge-disjoint hamiltonian cycles plus an edge-disjoint perfect matching if k is odd.
- if $p = \frac{6\log n}{n}$: All degrees are highly concentrated around its expected value, i.e. the graph is almost regular.
- if $p = \frac{\log n}{\sqrt{n}}$: Diameter 2.
- if $p = \frac{1}{n^{2/(k-1)}}$, : First complete graph of size k.
- if $p = 1/2$: Any graph has the same probability to appear. Independence number= clique number $= 2\log n$. Chromatic number $\frac{n}{2\log n}$

There are two specially interesting threshold to study, when the giant component appears which occurs when $p = \frac{1}{n}$ (percolation threshold) and when the graph becomes connected, $p \approx \frac{\log n}{n}$ (connectivity threshold).

3. Connectivity threshold

The following helpful result shows that a random variable counting objects which are almost independent approximates to a Poisson distribution (see e.g. [**?**, Thm 8.3.1]).

Theorem 3.1 (Brun's Sieve)**.** *Let X be an integer non-negative random variable. Suppose there is a constant λ so that $\mathbb{E}(X) \to \lambda$ and such that for every fixed t,*

$$\mathbb{E}\left(\binom{X}{t}\right) \to \frac{\lambda^t}{t!}.$$

Then $\Pr(X = k) \to \frac{\lambda^k}{k!}e^{-\lambda}$*, which implies that* $X \xrightarrow{d} Pois(\lambda)$ *in distribution.*

The following theorem unveils the connectivity threshold for $G(n, p)$.

Theorem 3.2. *Let $G \sim G(n, p)$ and let $c = np - \log n$. Then,*

(1) *If $c \to -\infty$, G contains w.h.p. isolated vertices.*
(2) *If $c \in \mathbb{R}$, $\Pr(G$ connected$) = e^{-e^{-c}}$.*
(3) *If $c \to +\infty$, G is w.h.p. connected.*

In particular we will show that the number of isolated vertices is a Poisson with parameter $\lambda = e^{-c}$.

4. Exercices

(1) Show that, for $p(n) = n^{\alpha}$, $\alpha < -3/2$, almost every graph in $\mathcal{G}_{n,p(n)}$ consists of independent edges.
(2) Show that for every graph H there is a function $p(n)$ with $\lim_{n\to\infty} p(n) = 0$ such that almost all graphs in $\mathcal{G}_{n,p(n)}$ have an induced copy of H.
(3) Show that, if $p(n) = cn^{-1-1/k}$ then almost every graph G has no vertices of degree $k+1$. Find a threshold function for the property that the maximum degree is at least k. Show that, if $p(n)n^{1+1/k} \to \infty$ and $p(n)n^{(k+2)/(k+1)} \to 0$ then almost every graph G has maximum degree k.
(4) Let H be a graph with four vertices obtained from K_4 by deleting two incident edges. Find the threshold function for the property that $\mathcal{G}_{n,p}$ contains H. Compare it with the threshold function of K_3.
   [Remark: The graph H in this problem is not balanced.]
(5) Let $G \sim G(n, c/n)$, $c < 1$. Let X_n count the number of triangles in G. What is the probability that $X_n = k$ for some $k \in \mathbb{N}$?
   [Hint: use Brun's sieve]
(6) Let H be a graph and let $\rho(H) = \max\{E(G[F])/|F| : F \subset V\}$. Show that the threshold function for the appearance of H as a subgraph is $n^{-1/\rho(H)}$.
(7) Let G be a random subgraph of $K_{n,n}$: every edge in $K_{n,n}$ is chosen independently with probability p. Let $p = (1+\epsilon)\ln n/n$.
   (a) Show that, if $\epsilon < 0$, then almost every graph G has no perfect matching.
   (b) Show that, if $\epsilon > 0$, then almost every graph G has a perfect matching.
   [Hint: If G has no perfect matching there is a set S such that $N(S) = |S| - 1$ and $G[S \cup N(S)]$ is connected.]

Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Random Walks

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1. Random walks in graphs

Let G be a connected graph (possibly with loops) with degree sequence $(d(1), \ldots, d(n))$. A simple random walk in G is the discrete time process X_k with

$$P(X_k = j | X_{k-1} = i) = \frac{a_{ij}}{d(i)},$$

where a_{ij} denotes the entry in the adjacency matrix $A = A(G)$ of G and $d(j)$ is the degree of j. In other words, at time $k-1$ we move from the current vertex j to a randomly chosen neighbor (with uniform distribution).

Denote by

$$\mathbf{p}_k^T = (p_k(1), \ldots, p_k(n))$$

the probability distribution of the random walk at time k, where where $p_k(i)$ denotes the probability of being at vertex i at time k.

Proposition 1.1. *Let $D = diag(d(1), \ldots, d(n)$. Then*

$$\mathbf{p}_k^T = \mathbf{p}_{k-1}^T (D^{-1}A) = \mathbf{p}_0^T (D^{-1}A)^k,$$

Proof. We have

$$p_k(j) = \sum_{i=1}^{n} \Pr(X_k = j | X_{k-1} = i) \Pr(X_{k-1} = i) = (p_{k-1}(D^{-1}A))_i.$$

□

The matrix $M = D^{-1}A$ is the *transition matrix* of the random walk. This means that the probability of visiting vertex j in k steps from vertex i is

$$p_k(i,j) = P(X_k = j | X_0 = i) = (M^k)_{i,j}.$$

Since M is not necessarily symmetric (unless G is r–regular), we consider instead the symmetric matrix

$$N = D^{-1/2} A D^{-1/2} = D^{1/2} M D^{-1/2},$$

with $N_{ij} = a_{ij}/\sqrt{d(i)d(j)}$.

Proposition 1.2. *The matrices N and M have the same spectra and $\mathbf{v}^T N = \lambda \mathbf{v}$ if and only if $\mathbf{w}^T M = \lambda \mathbf{w}^T$ with $\mathbf{w}^T = \mathbf{v}^T D^{1/2}$.*

The vector $\frac{1}{2m}\mathbf{1}^T D$ is an eigenvector of M with eigenvalue 1.

Proof. Let $\mathbf{v}$ be a left eigenvector of N with eigenvalue λ, and let $\mathbf{w}^T = \mathbf{v}^T D^{1/2}$. Then,

$$\mathbf{w}^T M = (\mathbf{v}^T D^{1/2})(D^{-1/2} N D^{1/2}) = \lambda(\mathbf{v}^T D^{1/2}) = \lambda \mathbf{w}^T.$$

We can check that

$$(\mathbf{1}^T D^{1/2})N = \mathbf{1}^T A D^{-1/2} = (d(1), \dots, d(n))D^{-1/2} = \mathbf{1}^T D^{1/2}$$

is an eigenvector with eigenvalue 1. Hence $\mathbf{1}^T D = (d(1), \dots, d(n))$ is an eigenvector of M with eigenvalue 1. The corresponding probabiltiy distribution is $\frac{1}{2m}(d(1), \dots, d(n))$. □

The eigenvector $\pi^T = \frac{1}{2m}\mathbf{1}^T D$ is called the *stationary distribution* because

$$\pi^T M = \pi^T.$$

Therefore, if the initial distribution $\mathbf{p}_0^T$ is set to π^T then $\mathbf{p}_k^T = \mathbf{p}_0^T$ for all $k \geq 0$. One of the main results is the fact that, for every initial distribution, $\mathbf{p}_k^T$ tends to π^T under reasonable conditions.

Theorem 1.3 (Limit distribution)**.** *Let G be a non bipartite connected graph. Then, for every initial distribution p_0 and for every i, j,*

$$p_k(i,j) \longrightarrow \frac{d(j)}{2m} \quad (k \to \infty)$$

and $\pi = (d(1)/2m, \dots, d(n)/2m)$ is the limit distribution.

Proof. We have already seen that $\mathbf{w}_1$ is a left eigenvalue of M with eigenvalue one.

If G is connected and nonbipartite then there is k such that $N^k = (D^{-1/2} A D^{-1/2})^k > 0$ (the ij–entry of N^k is the sum of weighted walks of length k between i and j, where edge ij has positive weight $1/\sqrt{d(i)d(j)}$ and the weight of a walk is the product of weights of their edges). By the Perron-Frobenius theorem, since $\mathbf{w}_1^T D^{-1/2} > 0$, the eigenvalue one has multiplicity one and is the spectral radius of N.

Therefore, the spectra of N satisfies

$$1 = \lambda_1 > \lambda_2 \cdots \geq \lambda_n > -1.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a corresponding orthonormal basis of eigenvectors of N, where

$$\mathbf{v}_1^T = \frac{1}{\sqrt{2m}} \mathbf{1}^T D^{1/2} = (\sqrt{d(1)/2m}, \dots, \sqrt{d(n)/2m}).$$

The above are also eigenvectors of every power N^k of N and, by using the spectral decomposition of N,

$$\begin{aligned} M^k &= D^{1/2} N^k D^{-1/2} \\ &= D^{1/2} \left(\sum_{\ell=1}^n \lambda_\ell^k \mathbf{v}_\ell \mathbf{v}_\ell^T \right) D^{-1/2} \\ &= D^{1/2} \mathbf{v}_1 \mathbf{v}_1^T D^{-1/2} + \sum_{\ell=2}^n \lambda_\ell^k (D^{-1/2} \mathbf{v}_\ell \mathbf{v}_\ell^T D^{1/2}). \end{aligned}$$

It follows that

(1) $$p_k(i,j) = \pi(j) + \sum_{\ell=2}^n \lambda_\ell^k (\mathbf{v}_\ell)_i (\mathbf{v}_\ell)_j \sqrt{\frac{d(i)}{d(j)}}$$

Since $|\lambda_\ell| < 1$ for $\ell = 2, \dots, n$, we have

$$p_k(i,j) \to \pi(j),\ (k \to \infty).$$

□

In particular, since $\lim_{k\to\infty} M^k$ has constant columns, $\mathbf{p}_0 M^k$ tends to the stationary distribution π for every initial distribution $\mathbf{p}_0$. We note that, if the graph is regular, then the limit distribution of a random walk is the uniform one.

In addition to having a probability limit, a key question in applications is to know the speed of convergence. One way of measuring this speed is the *mixing rate*:

$$\limsup_{k\to\infty} \max_{i,j} |p_k(i,j) - d(j)/2m|^{1/k}.$$

Theorem 1.4 (Mixing rate)**.** *Let G be a non bipartite connected graph. Let $\lambda = \min\{\lambda_2, \lambda_n\}$ denote the second largest eigenvalue (in absolute value) of N. Then the mixing rate of G is λ.*

Proof. It follows from (1) that

$$\begin{aligned}|p_k(i,j) - \pi(j)|^{1/k} &= \left|\sum_{\ell=2}^{n} \lambda_\ell^k (\mathbf{v}_\ell)_i (\mathbf{v}_\ell)_j \sqrt{\frac{d(i)}{d(j)}}\right|^{1/k} \\ &\le \lambda \left(\sum_{\ell=2}^{n} |(\mathbf{v}_\ell)_i (\mathbf{v}_\ell)_j| \sqrt{\frac{d(i)}{d(j)}}\right)^{1/k} \to \lambda\ (k \to \infty).\end{aligned}$$

On the other hand, if $\mathbf{w}$ is the eigenvector of λ,

$$|(\mathbf{w}^T + \pi)M^k - \pi|^{1/k} = |\lambda|,$$

therefore, by starting with the distribution corresponding to $\mathbf{w}^T + \pi$ one gets equality in the mixing rate. □

The following result gives an intuition on how to assess the speed of convergence.

Theorem 1.5. *Let G be an r–regular graph such that $|\lambda_i/r| \le 1/30$ for all eigenvalues of the adjacency matrix of G. Let $S \subset V(G)$ be a set of size at most $n/36$. Then the probability that a random walk starting in a random point is in S half of the time in k steps is at most $(2/\sqrt{5})^k$.*

Proof. By abuse of notation we denote by S and $\bar{S}$ the $n \times n$ matrices

$$S = diag(\mathbf{1}_S),\ \bar{S} = diag(\mathbf{1}_{\bar{S}}).$$

Let $\mathbf{p}_0^T = (1/n, \dots, 1/n)$ the initial uniform distribution. We can write

$$\Pr(X_0 \in S) = \sum_{i\in S} \mathbf{p}_0(i) = \|\mathbf{p}_0^T S\|_1.$$

where we use the 1–norm $\|\mathbf{x}\| = \sum_i |x_i|$ and M is the transition matrix of G. Similarly,

$$\Pr(X_0 \in S, X_1 \in S) = \Pr(X_1 \in S | X_0 \in S)\Pr(X_0 \in S) = \|\mathbf{p}_0^T SMS\|_1$$

and

$$\Pr(X_0 \in S, X_1 \in \bar{S}) = \|\mathbf{p}_0^T SM\bar{S}\|_1.$$

In general, if $A_i \in \{S, \bar{S}\}$,

(2)

$$\Pr(X_0 \in A_0, \dots, X_k \in A_k) = \|\mathbf{p}_0 A_0 M A_1 M A_2 \cdots M A_{k-1} M A_k\|_1 = \|\mathbf{p}_0 M A_0 M A_1 M A_2 \cdots M A_{k-1} M A_k\|_1,$$

where we again identify A_i with the matrix S or $\bar{S}$.

Let us show that, for every vector $\mathbf{v}$, under the conditions of the Theorem, we have

$$\|\mathbf{v}^T M S\|_2 \leq r\|\mathbf{v}\|_2/5.$$

We can write $\mathbf{v} = \alpha\mathbf{p}_0 + \mathbf{x}$ where $\mathbf{x}$ is orthogonal to $\mathbf{p}_0$. Since G is r–regular, $\mathbf{p}_0$ is an eigenvector of M with eigenvalue 1. It follows that

$$\|\mathbf{v}^T M S\|_2 \leq |\alpha| \cdot \|\mathbf{p}_0^T S\|_2 + \|\mathbf{x}^T M S\|_2.$$

By the hypothesis on $|S|$,

$$\|\mathbf{p}_0^T S\|_2 = \|\mathbf{p}_0\|_2 \|S\|_2 = \|\mathbf{p}_0\|_2 \sqrt{|S|/n} \leq \|\mathbf{p}_0\|_2/6,$$

and, since $\mathbf{x}$ is orthogonal to $\mathbf{p}_0$ and every eigenvalue of M distinct from 1 is at most $1/30$,

$$\|\mathbf{x}^T M S\|_2^2 \leq \|\mathbf{x}^T M\|_2^2 \leq (1/30)\|\mathbf{x}\|^2,$$

which implies,

$$\|\mathbf{v}^T M S\|_2 \leq \frac{|\alpha|\|\mathbf{p}_0\|}{6} + \frac{\|\mathbf{x}\|_2}{30} \leq \frac{\|v\|_2}{6} + \frac{\|v\|_2}{30} = \frac{\|\mathbf{v}\|_2}{5}.$$

By applying this to (2), if at least half of the A_i's are S,

$$\|\mathbf{p}_0^T A_0 M A_1 M A_2 \cdots M A_{k-1} M A_k\|_2 \leq \frac{\|\mathbf{p}_0\|_2}{5^{k/2}} = \frac{1}{\sqrt{n}5^{k/2}}.$$

It follows that, by using $\|\mathbf{v}\|_1 \leq \sqrt{n}\|\mathbf{v}\|_2$,

$$\|\mathbf{p}_0^T A_0 M A_1 M A_2 \cdots M A_{k-1} M A_k\|_1 \leq 5^{-k/2}.$$

By adding up all the sequences with this property (at most 2^k) we obtain $(2/\sqrt{5})^k$. □

2. Exercices

(1) Show that the mixing rate of the n–cube with a loop at every vertex is $(n-1)/n$. What is the number of steps of a random walk in Q_4 such that the probability of being at a vertex is $1/16 \pm 10^{-3}$?

(2) Show that the mixing rate of the torus $C_{2m+1}^l = \underbrace{C_{2m+1} \square \cdots \square C_{2m+1}}_{l}$ can be approximated by

$$1 - \frac{2\pi^2}{l(2m+1)^2}.$$

[Hint: Use that $1 - \cos x \approx x^2/2$.]

(3) The access time $H(i,j)$ is the expected number of steps to reach j from i. Compute the access time for complete graph K_n and for the path P_n (here depending on i and j.

(4) Let G be a vertex transitive graph. Show that, for every integer k and every pair i, j of vertices,

$$p_{i,i}(2k) \geq p_{i,j}(2k).$$

Hint: Use the fact that, for every permutation π *of* $[n]$ *and every sequence* $(a_1, \ldots, a_n)$ *of* n *real numbers,*

$$\sum_{i=1}^{n} a_i a_{\pi(i)} \leq \sum_{i=1}^{n} a_i^2.$$

(5) Consider a random walk in the complete graph K_n. What is the expected number of steps to visit each node at least once?

(6) Let G be a nonbipartite graph. Show that for every $\epsilon > 0$ there is k such that, if $j - i > k$ then $|P(X(i) = x, X(j) = y) - P(X(i) = x)P(X(j) = y)| < \epsilon$.

Graph Theory. Fall 2018

Extremal Graph Theory

The main problem in this area is, given a fixed graph H, determine the maximum number of edges in a (simple) graph with n vertices not containing H as a subgraph.
The first result is the following.

Theorem 1. *(Mantel) A triangle-free graph on n vertices has at most $n^2/4$ edges.*

This generalizes to

Theorem 2. *(Turán) A graph on n vertices containing no K_{r+1} as a subgraph has at most*

$$\left(1-\frac{1}{r}\right)\frac{n^2}{2}$$

edges.

Definition 3. $ex(n,H)$ is the maximum number of edges of a graph with n vertices which contains no copy of H.

Hence, if G is a graph with n vertices and more than $ex(n,H)$ edges then G contains a copy of H as a subgraph.

The following is the key result in this area. There is no easy proof of it.

Theorem 4. *(Erdős-Stone) For all $\epsilon > 0$ there exists an $n_0 \geq n$ such that if $n \geq n_0$ then*

$$\left(1-\frac{1}{\chi-1}-\epsilon\right)\frac{n^2}{2} \leq ex(n,H) \leq \left(1-\frac{1}{\chi-1}+\epsilon\right)\frac{n^2}{2},$$

where χ is the chromatic number of H.

When H is bipartite, the theorem says in particular that $ex(n,H)$ is subquadratic.

Theorem 5. *(Kovari-Sos-Turán) For every pair of integers $s \geq t \geq 2$, there exists a constant (depending only on s and t) such that*

$$ex(n,K_{s,t}) \leq cn^{2-1/t}.$$

The following lower bound comes from a random construction.

Theorem 6. *(Erdős-Spencer) $ex(n,K_{s,t}) \geq cn^{2-(s+t-2)/(st-1)}$*

The following lower bound comes from the incidence graph of a projective plane (and the density of primes).

Theorem 7. For all $\epsilon > 0$ there exists an $n_0 \geq n$ such that

$$ex(n,K_{2,2}) \geq \frac{1}{2\sqrt{2}}(1-\epsilon)n^{3/2}.$$

The following lower bound comes from the polarity graph of a projective plane (and the density of primes).

Theorem 8. For all $\epsilon > 0$ there exists an $n_0 \geq n$ such that

$$\tfrac{1}{2}(1-\epsilon)n^{3/2} \leq ex(n,K_{2,2}) \leq \tfrac{1}{2}(1+\epsilon)n^{3/2}$$

Exercises

(1) Let G be a graph with n vertices in which every vertex has degree d and suppose G contains no C_4 and no C_3. Prove that $n \geqslant d^2+1$ and that the pentagon and the Petersen graph attain the bound.

(2) Consider the following graph H. Take five pentagons P_h and five pentagrams Q_i, so that vertex j of P_h is adjacent to vertices $j-1, j+1$ of P_h, and vertex j of Q_i is adjacent to vertices $j-2, j+2$ of Q_i. Now join vertex j of P_h to vertex $hi+j$ of Q_i. (All indices mod 5.)

Show that H is 7-regular and has no C_3 and no C_4. [It is known as the Hoffman-Singleton graph.]

(3) Determine the maximum number of edges on a disconnected graph on n vertices and find the extremal examples.

(4) Let G be a graph with n vertices in which every vertex has degree at least $n/2$. Prove that G contains a cycle of length n.

[Hint: Consider a path $x_1, \ldots, x_k$ of maximal length. Prove that there is an i for which $x_1 x_{i+1}$ is an edge and $x_i x_k$ is an edge.]

(5) A graph without C_3 and C_4 contains at most

$$\frac{n\sqrt{n-1}}{2}$$

edges.

(6) (a) Let T be a tree on k vertices. If G is a graph for which $\delta(G) \geq k-1$, then G contains T as a subgraph.

(b) If G is a graph with n vertices ($n \geq k+1$) and m edges, such that

$$m \geq (k-1)n - \binom{k}{2} + 1,$$

then G contains a subgraph with minimum degree $\geq k$. [Hint: induction on n.]

Deduce that such a graph contains every tree on $k+1$ vertices.

(7) Let G be a graph with $\delta \geq (2n+1)/3$. Show that every edge is contained in a subgraph isomorphic to K_4.

(8) A graph with more than $2n-3$ edges contains a subdivision of K_4.

(9) Let S be a set of n points in the real plane. Prove that there are at most $cn^{3/2}$ points that at pairwise distance one from each other, for some constant c that does not depend on n. Prove that if the points are in $\mathbb{R}^3$, then there are at most $c'n^{5/3}$ pairs.

(10) Let (P, L) be a projective plane of order q. Let G be its incidence graph, that is: $V(G) = P \cup L$, and the edges are of the form $\{p, \ell\}$, where $p \in P, \ell \in L$ and p lies in ℓ. Show that the number of edges is $(q^2+q+1)(q+1)$. What does this say with respect to Theorem 5.

(11) Let V be the set points in the projective plane over $\mathbb{F}_q$, with homogeneous coordinates $x = (x_0, x_1, x_2)$. Let two pointsx x and y be adjacent if $\sum x_i y_i = 0$. Show that the associated graph has no C_4 and the number of edges is $(q+1)^2 q$. What does this say with respect to Theorem 5.

(12) Show that there are $K_{s,s}$-free graphs with $\Omega(n^{2-2/(s+1)})$ edges. [Compute the expected number of copies of $K_{s,s}$ in $G(n, n^{-2/(s+1)})$ and remove one edge from each copy.]