Master on Applied Mathematics and Mathematical Engineering

Graph Theory

Final exam. January 2018

Choose three problems, not including one you presented in the course.

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1. Call a graph G apex if there exists a vertex v such that $G - v$ is planar. Show that apex graphs form a minor-closed class. Find at least two excluded minors for the class of apex graphs.

2. (a) Let T be a tree on k vertices. If G is a graph for which $\delta(G) \geq k-1$, then G contains T as a subgraph.

   (b) If G is a graph with n vertices ($n \geq k+1$) and m edges, such that

   $$m \geq (k-1)n - \binom{k}{2} + 1,$$

   then G contains a subgraph with minimum degree $\geq k$. [Hint: induction on n.]

   Deduce that such a graph contains every tree on $k+1$ vertices.

3. Let G be an r–regular graph and let $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its adjacency matrix.

   (a) Show that the stability number $\alpha(G)$ satisfies

   $$\alpha(G) \leq \frac{-n\lambda_n}{r - \lambda_n}.$$

   (b) Find an analogous inequality for $\omega(G)$ the cardinality of the largest clique of G.

4. Let G be an r–regular graph with n vertices and denote by $\tau(G)$ the number of spanning trees of G. Show that

   $$\tau(G) \leq \frac{1}{n}\left(\frac{nr}{n-1}\right)^{n-1}$$

   and characterize the case of equality.

   [Hint: You may want to use the arithmetic–geometric mean inequality.]

5. The k–dimensional torus $T_{k,n}$ is the cartesian product of k cycles C_n. By using $1 - \cos x \approx x^2/2$, show that the mixing rate of $T_{k,2m+1}$ can be approximated by

   $$1 - \frac{2\pi^2}{k(2m+1)^2}.$$

   For k large, would you consider a random walk on $T_{k,2m+1}$ rapidly mixing (getting fast to the uniform distribution)?

6. Let $p_\lambda(n) = n^{-\lambda}$ and let H be a fixed graph. Show that there is $\lambda > 0$ such that almost all graphs in $\mathcal{G}_{n,p_\lambda(n)}$ have an induced copy of H.