Facultat de Matemàtiques i Estadística
UNIVERSITAT POLITÈCNICA DE CATALUNYA

Assignatura:

Estudiant/a: [redacted]

Data:

QUALIFICACIÓ
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2- a) We prove it by induction on K.

$K=1$ is the case ~~of one vertex so it~~ where T is a single vertex so it trivially holds.

For $K=2$, T must be an edge with its end vertices (it is the only tree on 2 vertices). So if $\delta(G) \geq K-1 = 1$ it means that there is an edge in G and hence there is a copy of T in G.

Now, suppose the result holds for less than K vertices, and let T be a tree on K vertices and G a graph for which $\delta(G) \geq K-1$.

Let $u \in T$ be a leaf of T and consider the tree $T \setminus \{u\}$. $T\setminus\{u\}$ has $K-1$ vertices and $\delta(G) \geq K-1 > (K-1)-1$, so by induction hypothesis G contains $T\setminus\{u\}$ as a subgraph.

Let $\varphi: T\setminus\{u\} \longrightarrow G$ be an embedding of $T\setminus\{u\}$ in G. We want to extend such embedding to the whole T.

Now, let $v \in T$ be the only neighbour of u in T (unique because u is a leaf). Then the degree of $\varphi(v) \in G$ satisfies $d(\varphi(v)) \underset{\substack{\uparrow \\ \text{min degree of } G}}{\geq} \delta(G) \geq K-1$ and $|\operatorname{Im} \varphi(T\setminus\{u,v\})| = K-2 \Longrightarrow$

$\Longrightarrow \exists w \in V(G)$ st. $w \sim v$, $w \notin \operatorname{Im}\varphi(T\setminus\{u\})$ so we can extend φ to u by defining $\varphi(u) := w$. And this implies that G contains the whole T as a subgraph.

b) We have G a graph with $|V(G)| = n$, $n \geq K+1$ and $|E(G)| = m$ such that $m \geq (K-1)\cdot n - \binom{K}{2} + 1$.

We fix $K \geq 2$ and apply induction on n. (for $K=1$ $\binom{K}{2}$ is not well defined)

The base case corresponds to $n = K+1$. In this case the inequality for the number of edges is:

$$m \geq (K-1)\cdot n - \binom{K}{2} + 1 \underset{\substack{\downarrow \\ n=K+1}}{=} (K-1)\cdot(K+1) - \binom{K}{2} + 1 = K^2 - \frac{K\cdot(K-1)}{2} = \frac{K^2}{2} + \frac{K}{2} =$$

$$= \frac{K\cdot(K+1)}{2} = \binom{K+1}{2} = \binom{n}{2}$$

which implies $m = \binom{n}{2}$ as the number of edges is always less or equal than $\binom{n}{2}$.

As $m = \binom{n}{2} \Longrightarrow G = K_n$ and $\delta(G) = n-1 = K$. So in particular G itself is a graph with minimum degree $K \rightarrow \delta(G) \geq K$ holds.

Suppose the result holds until $n-1$ vertices and let G be a graph with n vertices.

If $\delta(G) \geq K$, the result holds (the subgraph with minimum degree $\geq K$ is itself).

If $\delta(G) < K$, there exists $v \in V(G)$ such that $d(v) \leq K-1$. So consider the graph $G \setminus v$.

$G \setminus v$ has $n-1$ vertices and $\cancel{\delta(G\setminus v)}$ the number of edges satisfies

$$|E(G\setminus v)| \geq (K-1)\cdot n - \binom{K}{2} + 1 - (K-1) = (K-1)\cdot(n-1) - \binom{K}{2} + 1$$

due to the fact that $|E(G\setminus v)| = |E(G)| - d(v) \geq |E(G)| - (K-1)$.

We can apply the induction hypothesis to $G \setminus v$ so $G \setminus v$ contains a subgraph with minimum degree $\geq K$.

In particular (by monotonicity) G contains such subgraph, so the result holds.

Due to result ~~(a) and~~ (b) a graph G with n vertices, $n \geq K+1$ and $m \geq (K-1)\cdot n - \binom{K}{2} + 1$ contains a subgraph H with minimum degree $\geq K$, $\delta(H) \geq K$. Due to a), H contains every tree on $K+1$ vertices, and as H is a subgraph of G $\Longrightarrow$ G contains every tree on $K+1$ vertices

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4- Let G be an r-regular graph with $|V(G)| = n$.

And let $0 = \mu_1 \le \mu_2 \le \dots \le \mu_n$ be the eigenvalues of $L(G)$ the Laplacian matrix of G.

Then it is satisfied the following inequality of means:

$$\sqrt[n-1]{\mu_2 \cdots \mu_n} \le \frac{\mu_2 + \dots + \mu_n}{n-1} \quad \text{(Arithmetic-geometric mean inequality)}$$

Now, as G is r-regular $\mu_1 = 0$ so:

$$\frac{\mu_2 + \dots + \mu_n}{n-1} = \frac{\mu_1 + \mu_2 + \dots + \mu_n}{n-1} = \frac{\sum_{i=1}^{n} \mu_i}{n-1} = \frac{Tr(L)}{n-1} = \frac{nr}{n-1}$$

where $L(G) = r \cdot I - A(G)$ because G is r-regular so $Tr(L)$ is the sum of the elements in the diagonal, $Tr(L) = n \cdot r$.

Now:

$$\sqrt[n-1]{\mu_2 \cdots \mu_n} \le \frac{\mu_2 + \dots + \mu_n}{n-1} = \frac{n \cdot r}{n-1} \implies \mu_2 \cdots \mu_n \le \left(\frac{n \cdot r}{n-1}\right)^{n-1};$$

$$\frac{1}{n} \cdot (\mu_2 \cdots \mu_n) \le \frac{1}{n} \cdot \left(\frac{rn}{n-1}\right)^{n-1}$$

And from the matrix tree theorem $\tau(G) = \frac{1}{n} \cdot (\mu_2 \cdots \mu_n)$

$$\implies \boxed{\tau(G) \le \frac{1}{n} \cdot \left(\frac{rn}{n-1}\right)^{n-1}}$$

The equality holds if and only if $\sqrt[n-1]{\mu_2 \cdots \mu_n} = \frac{\mu_2 + \dots + \mu_n}{n-1}$ which is ~~satisfi~~ satisfied if and only if $\boxed{\mu_2 = \mu_3 = \dots = \mu_n.}$

This happens for instance when $G = K_n$. In this case $\lambda_1 = n-1$ and $\lambda_i = -1$ for $i = 2, \dots, n$ where $(\lambda_1, \dots, \lambda_n)$ are the eigenvalues of $A(G)$.

Hence as G is $(n-1)$-regular $\mu_i = (n-1) - \lambda_i \implies \mu_i = n \quad \forall i = 2, \dots, n$

OK

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6- Let $G \in \mathcal{G}_{n,p(n)}$ a random graph on n vertices and let H be a graph on K vertices and l edges. For fixed n, let us call p the probability of having an edge.

Let $K \subset V(G)$ with $|K| = K$. Then:

$$\Pr[G[K] = H] = c \cdot p^l \cdot (1-p)^{\binom{K}{2}-l} \underset{:=\alpha_n}{} \rightarrow \text{Define this probability as } \alpha_n$$

where c stands for the number of isomorphic copies of H that can be embedded in $G[K]$.

Now, let S be a partition of the vertices of G in K-subsets. S contains $\lceil \frac{n}{K} \rceil$ subsets and $\lfloor \frac{n}{K} \rfloor$ of which contain K elements. Then:

$$\Pr[H \nsubseteq_{ind} G] = \Pr[G[K] \neq H \quad \forall K \subset \binom{[n]}{K}] \leq \Pr[G[K] \neq H \quad \forall K \in S] =$$

$$= (1-\alpha_n)^{\lfloor \frac{n}{K} \rfloor}$$

due to independence of events $G[K] \neq H$

✓ ✓

Let us now use the exponential and logarithm function:

$$\Pr[H \nsubseteq_{ind} G] \leq (1-\alpha_n)^{\lfloor \frac{n}{K} \rfloor} = \exp\left\{\lfloor \tfrac{n}{K} \rfloor \cdot \log(1-\alpha_n)\right\} \leq \exp\left\{-\lfloor \tfrac{n}{K} \rfloor \cdot \alpha_n\right\}$$

$\log(1-\alpha_n) \leq -\alpha_n$ if $\alpha_n < 1$ (and $\alpha_n < 1$ because it is a probability) ✓

We want to see that $\exists \lambda > 0$ s.t. $p = p_\lambda(n) = n^{-\lambda}$ satisfies that this probability tends to 0 as n tends to ∞:

$$\Pr[H \nsubseteq_{ind} G] \xrightarrow[n\to\infty]{} 0 \iff \exp\left\{-\lfloor \tfrac{n}{K} \rfloor \cdot \alpha_n\right\} \xrightarrow[n\to\infty]{} 0 \iff \lfloor \tfrac{n}{K} \rfloor \cdot \alpha_n \xrightarrow[n\to\infty]{} \infty$$ ✓

Substituting α_n and erasing the constant terms, the condition is fulfilled if and only if:

$$n \cdot p^l \cdot (1-p)^{\binom{K}{2}-l} \xrightarrow[n\to\infty]{} \infty \quad \text{and} \quad (1-p)^{\binom{K}{2}-l} \xrightarrow[n\to\infty]{} 1$$ imposing $p = p_\lambda(n) = n^{-\lambda}$ with $\lambda > 0$.

Hence, it is needed that:

$$\lim_{n\to\infty} n \cdot p^l = \infty \iff \lim_{n\to\infty} \frac{p^l}{1/n} = \infty \iff \lim_{n\to\infty} \frac{n^{-\lambda l}}{n^{-1}} = \lim_{n\to\infty} n^{-\lambda l + 1} = \infty$$

$$\iff -\lambda l + 1 > 0 \iff 1 > \lambda l \iff \frac{1}{l} > \lambda.$$ ✓ OK

So for $p_\lambda(n) = n^{-\lambda}$ with $0 < \lambda < \frac{1}{\ell}$ and $G \in \mathcal{G}_{n, p_\lambda(n)}$ we have that $\lim_{n \to \infty} \Pr[H \notin_{ind} G] = 0$ so almost all graphs in $\mathcal{G}_{n, p_\lambda(n)}$ have an induced copy of H.