(4) Let G be a vertex transitive graph. Show that, for every integer k and every pair i, j of vertices,

$$p_{i,i}(2k) \geq p_{i,j}(2k)$$

Hint: Use the fact that, for every permutation π of $[n]$ and every sequence $(a_1, ..., a_n)$ of n real numbers,

$$\sum_{i=1}^{n} a_i a_{\pi(i)} \leq \sum_{i=1}^{n} a_i^2$$

Recall the definition of a vertex transitive graph. A graph G is vertex transitive if its automorphism group acts transitively; i.e. $\forall v_1, v_2 \in V(G)$, $\exists f$ automorphism $f : V(G) \longrightarrow V(G)$ such that $f(v_1) = v_2$. As f is an automorphism it preserves vertex adjacencies.

Let M be the transation matrix of G.
The first observation is that G is regular and the powers of M, M^k are symmetric $\forall k \in \mathbb{N}$. Since G is vertex transitive, the degree of all its vertices must be equal, so G is regular. We prove that M^k is symmetric by induction on k:
If $k = 1$, $M_{i,j} = p_{i,j} = \frac{a_{i,j}}{d(i)} = \frac{a_{j,i}}{d(j)} = p_{j,i} = M_{i,j}$.
Using the induction hypothesis we prove it for k: $(M^k)^t = (M^{k-1} * M)^t = M^t(M^{k-1})^t = MM^{k-1} = M^k$

The second observation is that all the rows of M^k have the same entries. First, notice that a graph automorphism preserves the probability of a t-walk between two points, i.e: $\forall i, j \in V(G)$, and ϕ automorphism, $p_{i,j}(k) = p_{\sigma(i),\sigma(j)}(k)$. This is due to the fact that σ preserves adjacencies, so for any step s in an arbitrary path from i to j, there exists an equivalent step r under the transformation σ (with same probability). And no edges are created or deleted.
Let's prove that all the rows of M^k have the same entries. Let $M_{i,j}^k$ be an entry of M^k. Then choose another row of M^k, the z-th one, with z$\neq$ i. As G is vertex transitive, let σ be the automorphism sending i to z, $\sigma(i) = z$, and let t be the image of j by σ, $t := \sigma(j)$. Then, by the previous claim, $M_{i,j}^k = p_{i,j}^k = p_{\sigma(i),\sigma(j)}^k = M_{z,t}^k$. So basically, $\forall i, z$ rows and $M_{i,j}$ entry, $\exists t$ such that $M_{z,t}^k = M_{i,j}^k$ and all rows have the same entries. We will denote such entries as $\{a_1, ..., a_n\}$.

Finally, we have that:

$$p_{i,j}^{2k} = M_{i,j}^{2k} = (M^k M^k)_{i,j} = \sum_{s=1}^{n} M_{i,s}^k M_{s,j}^k = \sum_{t=1}^{n} a_t a_{\pi(t)} \leq \sum_{t=1}^{n} a_t^2 = \sum_{s=1}^{n} M_{i,s}^k M_{s,i}^k = M_{i,i}^{2k} = p_{i,i}^{2k}$$

where we have used that all the rows have the same entries up to a permutation, the hint and the fact that M^k is symmetric.