UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Full 1 de 5

QUALIFICACIÓ
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Assignatura: Graph Theory

Estudiant/a: [redacted]

Data: 09/01/19

Problem 3

G r-regular graph.

$\text{Spec}(G) = \{\lambda_1, \dots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\lambda_1 = r$.

Call the eigenvalues of the corresponding Laplacian matrix $\mu_1, \dots, \mu_n$ such that $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$.

Observation: the cardinality of the largest clique in G or G^c is the size of the largest independent set of G^c or G, respectively.

(a) We look for the stability number $\alpha(G)$, which is the cardinality of the largest clique in G^c.

Define S as the largest independent set in G. Then, by the previous observation, it is clear that $\alpha(G) = |S|$.

$$\mu_n = \min_{x \in \mathbb{R}^n} \frac{x^T L x}{x^T x} \implies \mu_n \leq \frac{x^T L x}{x^T x} \quad \text{for any } x \in \mathbb{R}^n$$

Let us choose a value for x_i in the following way:

$$x_i = \begin{cases} 1 - c & \text{if } v_i \in S \\ -c & \text{if } v_i \notin S \end{cases}$$

✓

Clearly, we can compute

$$x^T L x = \sum_{v_i \sim v_j} (1-c-(-c))^2 = r \cdot |S|$$

$$x^T x = \sum_{v_i \in S} (1-c)^2 \sum_{v_j \notin S} (-c)^2 = |S|(1-2c) + nc^2$$

Therefore, defining $c = \frac{|S|}{n}$, we have

$$\frac{x^T L x}{x^T x} = \frac{r \cdot |S|}{|S|(1-2c)+nc^2} = \frac{r \cdot |S|}{|S|\left(1-\frac{2|S|}{n}\right) + n\frac{|S|^2}{n^2}} =$$

$$= \frac{r \cdot |S|}{|S|\left(1-\frac{2|S|}{n} + \frac{|S|}{n}\right)} = \frac{nr}{n-|S|}$$

So finally we have that

$$\mu_n \leq \frac{nr}{n-|S|} \iff n\mu_n - \mu_n |S| \leq nr \iff |S| \leq -\frac{n(r-\mu_n)}{\mu_n}$$

Recall that G is r-regular. As $L(G) = D(G) - A(G)$ (where $D(G)$ is the diagonal matrix with entries the degrees of vertices), $\mu_n = r - \lambda_n$.

Hence,

$$|S| \leq -\frac{n(r-r+\lambda_n)}{r-\lambda_n} = \frac{-n\lambda_n}{r-\lambda_n}$$ ✓ OK

Finally, we proved that

$$\alpha(G) \leq \frac{-n\lambda_n}{r-\lambda_n}$$

[Part (b) in the next sheet]

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Estudiant/a: XAVIER GOMBAU PASCUAL

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[continuation of exercise 3]

(b) Now, we are looking for the cardinality of the largest clique in G, which, by the observation, corresponds to the size of the largest independent set of G^c.

Define $A(G^c) = A^c$ and $L(G^c) = L^c$.

$$\mathrm{Spec}(A^c) = \{\lambda_1^c, \dots, \lambda_n^c\}$$
$$\mathrm{Spec}(L^c) = \{\mu_1^c, \dots, \mu_n^c\}$$

$$L + L^c = L(K_n) = nI - J \quad \Rightarrow \quad L^c = nI - J - L$$

Let v_i be an eigenvector of L^c.

$L^c v_i = (nI - J - L) v_i = (n - \mu_i) v_i$, since the eigenvector of nI is n, for J is 0 and for L is μ_i.

Therefore, if μ_i is eigenvalue of $L(G)$, $\mu_i^c = n - \mu_i$

Reordering these eigenvalues, we get that

$$\begin{cases} \mu_1^c = \mu_1 = 0 \\ \mu_i^c = n - \mu_{n+2-i}, \quad \text{for } i = 2, \dots, n \end{cases}$$

Notice that the graph G^c is $(n-r-1)$-regular, so

$$\lambda_i^c = (n-r-1) - \mu_i^c$$

$$\lambda_n^c = n-r-1-\mu_n^c \underset{\substack{\uparrow \\ (\mu_n^c = n-\mu_2)}}{=} n-r-1-n+\mu_2 \underset{\substack{\uparrow \\ \left(\begin{array}{l} G \text{ r-regular} \\ \mu_2 = r-\lambda_2 \end{array}\right)}}{=} -r-1+r-\lambda_2 =$$

$$= -1-\lambda_2$$

By using the formula in (a):

$$\omega(G) \leq \frac{-n\lambda_n^c}{(n-r-1)-\lambda_n^c} = \frac{-n(-1-\lambda_2)}{n-r-1+1+\lambda_2} = \frac{n(1+\lambda_2)}{n-r+\lambda_2}$$

which gives the following bound:

$$\omega(G) \leq \frac{n(1+\lambda_2)}{n-r+\lambda_2}$$ ✓ ok

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Problem 4

Let G be an r-regular graph with n vertices.

We define $\text{Spec}(L(G)) = \{\mu_1, \mu_2, \dots, \mu_n\}$, where $\mu_1 = 0$ and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$.

By the geometric-arithmetic mean inequality, we have that

$$\sqrt[n-1]{\mu_2 \cdots \mu_n} \leq \frac{\mu_2 + \dots + \mu_n}{n-1} \underset{(\mu_1 = 0)}{=} \frac{\mu_1 + \dots + \mu_n}{n-1}$$

The sum of eigenvalues of the Laplacian matrix corresponds to the trace of this matrix. As G is r-regular, the Laplacian matrix of G will have 'r' in the diagonal, since $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with entries the degree of vertices.

$$\mu_1 + \dots + \mu_n = \text{Tr}(L) = nr \quad ✓$$

Therefore,

$$\sqrt[n-1]{\mu_2 \cdots \mu_n} \leq \frac{rn}{n-1} \iff \mu_2 \cdots \mu_n \leq \left(\frac{rn}{n-1}\right)^{n-1} \iff$$

$$\frac{1}{n} \cdot \mu_2 \cdots \mu_n \leqslant \frac{1}{n} \left(\frac{nr}{n-1} \right)^{n-1}$$

By the matrix tree Theorem, $\tau(G) = \frac{1}{n} \mu_2 \cdots \mu_n$, hence

$$\tau(G) \leqslant \frac{1}{n} \left(\frac{nr}{n-1} \right)^{n-1}$$ ✓

Equality will occur when $\mu_2 = \mu_3 = \ldots = \mu_n$, since the previous arithmetic-geometric mean becomes an equality.

This happens in the case where $G = K_n$, since

$\mathrm{Spec}(K_n) = \{ n-1, -1, \ldots, -1 \}$ and $\mu_i = r - \lambda_i$

In this case, as K_n is $(n-1)$-regular, we have the following:

$$\tau(K_n) = \frac{1}{n} \left(\frac{n \cancel{(n-1)}}{\cancel{n-1}} \right)^{n-1} = \frac{1}{n} \cdot n^{n-1} = n^{n-2}$$ ✓

So in the case of K_n, the number of spanning trees is n^{n-2}.

only one case

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Problem 5

$T_{k,n}$ k-dimensional torus, cycles C_n

$T_{k,2m+1}$ k-dimensional torus, cycles C_{2m+1}

<u>Notation</u>

- Eigenvalues of A : $\{\lambda_1, \dots, \lambda_n\}$
- Eigenvalues of L : $\{\mu_1, \dots, \mu_n\}$
- Eigenvalues of N: $\{\theta_1, \dots, \theta_n\}$

As $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with entries the degrees of vertices, if G is r-regular, then

$$\lambda_i = r - \mu_i$$

Since $N = \frac{A}{r}$, $\theta = \frac{\lambda}{r} = 1 - \frac{\mu}{r}$

The mixing rate of a nonbipartite connected graph is the second largest eigenvalue (in absolute value) of N.

The largest θ is $\theta_1 = 1$ and this happens when $\mu = \mu_1 = 0$. Therefore, we need to look for the smallest μ different from 0, that is μ_2, since $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. As we look for the second largest θ in absolute value, it could exist a μ_n which made θ_n greater than θ_2 (in absolute value), but we will see that there is not.

The spectra of the cycle graph C_{2m+1} is

$$\lambda_j = 2\cos\left(\frac{2\pi(j-1)}{2m+1}\right) \quad \text{for} \quad j = 1, \dots, n$$

Any cycle graph is 2-regular. Then $\mu_j = 2 - \lambda_j$.

$$\mu_j = 2 - 2\cos\left(\frac{2\pi(j-1)}{2m+1}\right) \checkmark$$

Notice that the previous formula confirms that $\mu_0 = 0$. Indeed, we can clearly see that it is maximized when $j = 2$, since we cannot choose $j = 1$.

$$\mu_2 = 2 - 2\cos\left(\frac{2\pi}{2m+1}\right)$$

Notice that, as we do the cartesian product of k 2-regular cycles, $T_{k,2m+1}$ is $2k$-regular.

Applying $\theta = 1 - \frac{\mu}{r}$, ? You should use the spectra of the cartesian product.

$$\theta_2 = 1 - \frac{1}{2k}\left(2 - 2\cos\left(\frac{2\pi}{2m+1}\right)\right) = 1 - \frac{1}{k}\left(1 - \cos\left(\frac{2\pi}{2m+1}\right)\right)$$

Using $1 - \cos x \approx x^2/2$,

$$\theta_2 \approx 1 - \frac{1}{2k}\left(\frac{2\pi}{2m+1}\right)^2 = 1 - \frac{2\pi^2}{k(2m+1)^2}$$

Therefore, the mixing rate of $T_{k,2m+1}$ can be approximated by

$$1 - \frac{2\pi^2}{k(2m+1)^2} \checkmark \text{ ok}$$

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[Continuation of problem 5]

The mixing rate is used to measure the speed of convergence.

For a large k, we easily see that the mixing rate of $T_{k,2m+1}$ tends to 1, which is the maximum value it can take.

A random walk on $T_{k,2m+1}$ would tend to get fast to the uniform distribution, since its mixing rate is tending to the maximum.

✓ OK.