Problem 7 (list 2) - Spectral Graph Theory

[redacted]

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Problem: Let G be a graph with n vertices such that every pair of vertices has a unique common neighbour. Prove that there is a vertex with degree $n-1$.

This is a popular problem known as the *Friendship theorem* which states that the finite graphs with the property that every two vertices have exactly **one** neighbour in common are exactly the friendship graphs. Informally, suppose in a group of people we have the situation that any pair of persons has precisely **one** common friend. Then there is always a person (the "politician") who is everybody's friend.

The **friendship graph** F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex. The **windmill graph** $Wd(k,n)$, however, is an undirected graph constructed by joining n copies of the complete graph K_k at a shared universal vertex. As we can see, the friendship graphs are included in the windmill graphs and we can say that by construction, the friendship graph F_n is isomorphic to the windmill graph $Wd(3,n)$.

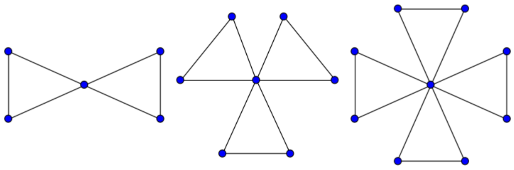

Figure 1: The friendship graphs F_2, F_3, F_4

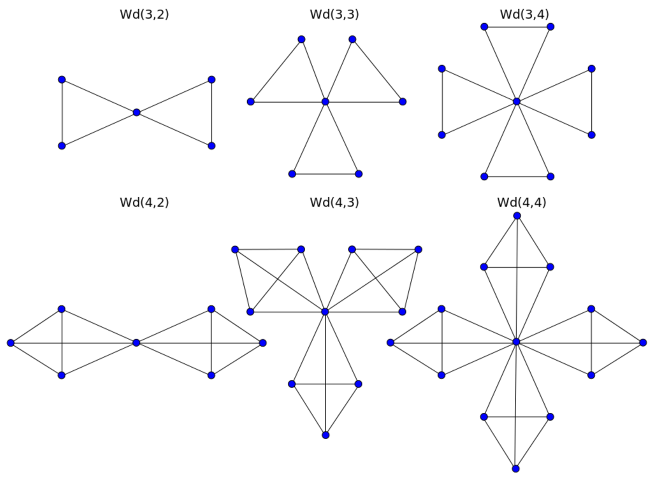

Figure 2: Examples of windmill graphs

The friendship theorem is equivalent to say that for any pair of nodes, there is exactly **one** path of length 2 between them.

One first observation is the following. If G contains C_4 as a subgraph, then there will exist two vertices which will have 2 common neighbours, and that would be a contradiction of the friendship theorem. Then, we can write that if a graph G is friendship, then it won't contain any C_4 as a subgraph.

Let's consider u and v two non-adjacent vertices ($u \nsim v$) with $d(u) = k$ and $W = \{w_1, w_2, ..., w_k\}$ the set of vertices adjacent to u ($\forall i \; w_i \sim u$).

One (and only one) vertex $w_i \in W$ must be adjacent to v, as u and v must have **one** common neighbour. Say w_2. Also, w_2 must be adjacent to one (and only one) other w_i to have a common neighbour with u. Say w_1. Note that w_2 and w_1 cannot be connected with any more vertex w_i, as they would have two common neighbours with u.

Now, to have a common neighbour with v, w_2 must be connected to an external vertex $x \notin W$ and also $x \sim v$.

$\forall w_i$ except w_1, v must have a common neighbour and this neighbour must be distinct in order to avoid a cycle of length 4. Note that v cannot be connected to any other w_i in order to keep having a unique neighbour with u.

If we do that, $d(v) \geq k = d(u)$. By symmetry, $d(u) \geq d(v)$ so $d(u) = d(v) = k$. Therefore, **any two non-adjacent vertices must have the same degree**.

Now we have constructed a graph such that w_2 is the only shared neighbour of u and v, so any other vertex is adjacent to at most **one** of u and v, but not both. Suppose G is a k-regular graph $\Leftrightarrow d(v) = k \;\; \forall v \in V(G)$.

Now we know the degree of all vertices. Let's count the total number of vertices. We count the degree of each w_i adjacent to u to obtain the total number of vertices, because there are no external vertex $x \notin W$ connected to u but all of them must have a common neighbour with u, that is, one w_i. All vertices have to be connected to some w_i. Thus, the total number of vertices should be k^2. But we counted u a total of k times. So we subtract $k-1$. Therefore the number of vertices is $n = k^2 - k + 1$.

It is time to write the adjacency matrix A. It will be a matrix $n \times n$ with k one's in each row (as the degree of each vertex is k). By the friendship condition, any two rows have **one** column where they both have a one (a common neighbour).

Therefore, A^2 will have k's down the diagonal, because A is symmetric and we will be adding 1's to the diagonal k times, and 1's everywhere else because there cannot be more than one common neighbour. For example, if $k = 2$, we get, among others, the triangle:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

To find the eigenvalues of A^2, we write $A^2 = J + (k-1)I$ where J is the unit matrix (all 1's) and I is the identity matrix.

- The eigenvalues of J are n and 0.
- The eigenvalue of I is 1. The eigenvalue of $(k-1)I$ is $k-1$.

From this we see that the eigenvalues of A^2 are $n + k - 1 = k^2 - k + 1 + k - 1 = k^2$ and $k - 1$. Therefore, the eigenvalues of A are k, $+\sqrt{k-1}$ and $-\sqrt{k-1}$.

We know the multiplicity of k is 1 (*Perron-Frobenius Theorem*). Consider the other eigenvalues have multiplicity r and s such that $r + s = n - 1$. The trace of the adjacency matrix must be 0:

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0$$

This implies that $r \neq 0$ because if $r = 0$ then $s = n-1$ and $k = 0$. Working on the previous equation we get

$$
\begin{gathered}
k + (r - s)\sqrt{k-1} = 0 \\
(r - s)^2(k - 1) = k^2 \\
(r - s)^2 = \frac{k^2}{k-1}
\end{gathered}
$$

$k - 1$ divides k^2 which is only true for $k = 2$. Therefore, a graph satisfying the friendship condition must be a 2-regular graph. The only compatible one is the triangle, which is indeed a solution:

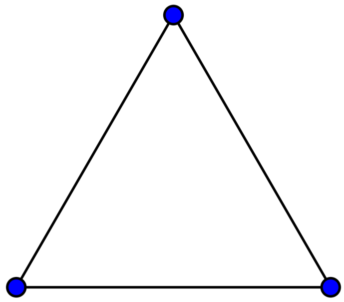

Figure 3: The unique 2-regular graph satisfying the friendship condition

Now suppose G is not regular anymore. We previously proved that two non-adjacent vertices have the same degree. As G is not regular, there must be a vertex which is adjacent to all the others — let's call it p — because otherwise, it would have the same degree as the other ones (regular).

Another argument previously written is that G cannot contain any cycle of length 4. As all vertices must be connected to p and there can't exist C_4 as a subgraph, the only possible construction is to glue cycles of length 3 (i.e. triangles) such that all of them share the vertex p.

We obtained that G must be a friendship graph with **one vertex of degree** $n-1$ in order to satisfy the friendship condition.