

Spectral Graph Theory

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Problem 10. Let $G \square H$ denote the cartesian product of G and H . The vertex set is $V(G) \times V(H)$ and $(x, y) \sim (z, t)$ if one of the coordinates agree and the other one is a pair of adjacent vertices.

1. Show that the Laplace eigenvalues of $G \square H$ are precisely $\mu_i(G) + \mu_j(H)$ for all i, j .
2. The n -cube Q_n is defined as $Q_1 = K_2$ and $Q_n = K_2 \square Q_{n-1}$ for $n \geq 2$. Determine $\mu_2(Q_n)$
3. Show that $i(Q_n) = 1$

Proof. (1) Using the properties of Kronecker product operation, we have the followings:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

Now, if

$$L(G)X_i = \mu_i X_i \text{ and } L(H)Y_j = \mu_j Y_j$$

$$L(G \times H)(X_i \otimes Y_j) = (L(G) \otimes I_m + I_n \otimes L(H))(X_i \otimes Y_j) = L(G)X_i \otimes I_m Y_j + I_n X_i \otimes L(H)Y_j = \mu_i X_i \otimes Y_j + \mu_j X_i \otimes Y_j = (\mu_i + \mu_j)X_i \otimes Y_j \quad \square$$

Proof. (2) The μ_2 states for algebraic connectivity of the graph. We know that for Q_n n -cube, having the formula $Q_n = K_2 \square Q_{n-1}$ for $n \geq 2$ we have the unique μ_2 , that being said, μ_2 does not change with number of dimensions of a vector space. To prove that, let us remember, that μ_2 is the second eigenvalue of the Laplace matrix. Since the n -cube Q_n we have is always regular, we have

$$\theta_1 \geq \theta_2 \geq \dots \geq \theta_m$$

for adjacency matrix, and

$$0, k - \theta_2, \dots, k - \theta_m$$

for Laplacian eigenvalues. Since the spectrum of n -cube is

$$\{(n - 2k) \binom{n}{k} \text{ for } k = 0, \dots, n\}$$

and we have proved the case for $k = 2 = n$ having eigenvalues of

$$\{0, 2, \dots, 2n\}$$

having that $\mu_2(Q_n) = 2$. □

Proof. (3) The $i(Q_n)$ is the isoperimetric number states for the surface of a graph G (in our case, Q_n) on n vertices and defined as:

$$i(G) = \min\left\{\frac{e(A)}{|A|}, A \subset V, |A| \leq \frac{n}{2}\right\}$$

Knowing the fact that each vertex of n -cube Q_n is a Boolean vector of length d , we have a set of $\{0, 1\}^d$. Two vectors are connected if their difference is of 1 bit, that is, their Hamming distance is 1. Let us take the Q_n and subdivide it by 2 $(n-1)$ dimensional subcubes C_1 and C_2 . Let us state, that the last digit of the coordinate for each vertex in the C_1 subcube is 1, while - 0 in the C_2 . Every vertex in C_1 has only one edge connecting it with it's complement in C_2 . Thus,

$$\frac{e(A)}{|A|} = 1$$

where A states for C_1 or C_2 . And since

$$\frac{\mu_2}{2} \leq \frac{e(A)}{|A|}$$

there is no other set that would have a smaller quotient. □