COMPUTATIONAL MECHANICS

15:00-18:00 hours

Final Exam

4th June 2019

Surname: **Name:**

Open class notes. Open book. The highest possible score is 100 marks.

1. (20 marks) The constitutive law of an elastic material is given by,

$$\boldsymbol{\sigma} = \eta \boldsymbol{\varepsilon}^{dev}$$

with η a material parameter, and $\boldsymbol{\varepsilon}^{dev}$ the deviatoric strain.

(a) Write the expression of the elemental stiffness $\mathbf{K}^e_{ij}$ coupling contributions from node i and j. You can make use of the deformation matrix $\mathbf{B}_i$ such that $\{\boldsymbol{\varepsilon}^e\} = \mathbf{B}_i \boldsymbol{u}_i$, with $\{\boldsymbol{\varepsilon}^e\}$ the strain tensor in Voigt notation, and $\boldsymbol{u}_i$ the nodal displacements of the element.

(b) Which is the dimension of the elemental matrix $\mathbf{K}^e$ for a three-dimensional eight noded hexahedral element? What do you think is the maximum rank of the elemental stiffness matrix in this case? Justify your answers.

Solution:

(a) The elemental matrix is such that the weak form is given by,

$$\int_{\Omega^e} \boldsymbol{\varepsilon}(\delta \boldsymbol{v}) : \boldsymbol{\sigma} dV = \int_{\Omega^e} \{\boldsymbol{\varepsilon}(\delta \boldsymbol{v})\}^T \{\boldsymbol{\sigma}\} dV = \delta \boldsymbol{v}_i \cdot \mathbf{K}^e_{ij} \boldsymbol{u}_j$$

In our case,

$$\begin{aligned}\{\boldsymbol{\varepsilon}(\delta \boldsymbol{v})\}^T \{\boldsymbol{\sigma}\} &= \eta \delta \boldsymbol{v}_i^T \mathbf{B}_i^T \{\boldsymbol{\varepsilon} - trace(\boldsymbol{\varepsilon}) \mathbf{I})\} \\ &= \eta \delta \boldsymbol{v}_i^T \mathbf{B}_i^T \left(\mathbf{B}_j \boldsymbol{u}_j - \frac{1}{3} [\partial_x N_j \ \partial_y N_j \ \partial_z N_j] \, \boldsymbol{u}_j \bar{\mathbf{1}} \right)\end{aligned}$$

with $\bar{\mathbf{1}} = \{1\ 1\ 1\ 0\ 0\ 0\}^T$. Therefore matrix $\mathbf{K}^e_{ij}$ may be expressed as

$$\mathbf{K}^e_{ij} = \int_{\Omega^e} \eta \mathbf{B}_i^T \left(\mathbf{B}_j - \frac{1}{3} \bar{\mathbf{1}} \otimes \nabla N_j \right) dV.$$

(b) In total the elemental matrix has $nodes \times dim = 8 \times 3 = 24$ rows or columns. Since there no energy associated to the volumetric deformation , there are at least 6 rigid body motion and one volumetric deformation ($\boldsymbol{\varepsilon} = \alpha \mathbf{I}$) that produce no nodal forces, and thus yield 0 eigenvalues for $\mathbf{K}^e$. Therefore, the rank is equal or lower than $24 - 7 = 17$.

2. (20 marks) Consider the following two noded element with dimension h and with the following stiffness and mass matrices:

$$\mathbf{K} = \frac{E}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \mathbf{M} = \frac{h\rho}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

with E and ρ the Young modulus and the density of the material.

(a) Compute the eigen-frequencies w_i for the free oscillation of the element, with the consistent mass matrix $\mathbf{M}$ given above.

(b) Compute the eigen-frequencies w_i when using the lumped mas matrix.

(c) If the critical time-step in central differences scheme is given by $\Delta t_{crit} = \frac{2}{w_{max}}$, which mass matrix seems better from the stability point of view?

(d) How is going to affect to the stability of the time-integration a reduction of h, E or ρ?

Solution:

(a) The eigen-values of ω of the element are given by

$$\omega^2 = eig\left(\mathbf{M}^{-1}\mathbf{K}\right) = eig\left(\frac{2}{h\rho}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\frac{E}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\Big) = eig\left(\frac{6E}{\rho h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$
$$= \frac{12E}{\rho h^2}(0,1)$$

Therefore, $\omega_{min} = 0$ and $\omega_{max} = \frac{2\sqrt{3}}{h}\sqrt{\frac{E}{\rho}}$.

(b) The lumped mass matrix is equal to $\mathbf{M}_L = \frac{h\rho}{2}\mathbf{I}$. In this case,

$$\omega^2 = eig\left(\mathbf{M}_L^{-1}\mathbf{K}\right) = eig\left(\frac{2E}{\rho h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \frac{4E}{\rho h^2}(0,1)$$

That is, $\omega_{min} = 0$ and $\omega_{max} = \frac{2}{h}\sqrt{\frac{E}{\rho}}$.

(c) From the results above, and setting $c = \sqrt{\frac{E}{\rho}}$, we have that,

$$\Delta t_{crit} = \frac{2}{\omega_{max}} = \frac{h}{\sqrt{3}c}$$
$$\Delta t_{crit}^L = \frac{2}{\omega_{max}} = \frac{h}{c}$$

Since $\Delta t_{crit}^L > \Delta t_{crit}$, the lumped matrix is preferable form the stability point-view since the critical time-step is larger for an arbitrary mesh-size.

(d) Reducing h, E or ρ is going to decrease, increase and decrease the critical time-step, respectively.

3. (20 marks) A specimen is subjected to an homogeneous shear stress,

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If the material follows a Von Mises plasticity criteria with yield stress σ_y, determine how the maximum shear stress τ will depend on the material parameter σ_Y.

Solution:

The material will plasticity when $\sqrt{\frac{3}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}} = \sigma_Y$. In this case, since $\sigma_m = trace(\boldsymbol{\sigma}) = 0$, we have that $\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma}$, and thus $\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev} = 2\tau^2$. Therefore, the material will plastify when

$$\tau = \frac{\sigma_Y}{\sqrt{3}}. \quad (1)$$

4. (20 marks) The steady flow past an obstacle, with characteristic size L_1 and inflow velocity V_1 has been computed with a finite element solution of the incompressible Navier-Stokes equations. The computation took 10 hours.

   a) The same problem is computed next with exactly the same parameters and conditions, but with inflow velocity $V_2 = 100V_1$. Unfortunately, the program stopped with an error message"Newton-Raphson iterations failed to converge". What to you think the problem may be? What would you do to get a solution?

   b) The problem is to be solved now with a much smaller obstacle, with characteristic size $L_2 = L_1/100$, and inflow velocity V_2. Do we have to do the computation or can we obtain the solution from the previous ones? How would you compute the velocity field in the domain?

Solution:

   a) Given that the Reynolds number is much larger, the solution may develop smaller features, that require a refined mesh, in particular along boundary layers. Thus, the initial mesh may not be fine enough, precluding convergence of the non-linear solver. On other hand, the initial guess for the non-linear solver may be too far from the solution. In order to get a better initial guess, intermediate solutions with smaller increases of the Reynolds number can be computed, using each solution as initial guess for the next Reynolds number computation.

   b) The behaviour of the solution depends on the Reynolds number $Re = VL/\nu$. Thus we do not need to do a new computation, because $L_1V_1 = L_2V_2$, and therefore the velocity for the obstable with size L_2 and inflow velocity V_2, v_2, is just a multiple of the solution for the obstacle with size L_1 and velocity V_1, v_1. More precisely, $\boldsymbol{v}_2(\boldsymbol{x}) = 100\boldsymbol{v}_1(100\boldsymbol{x})$ in the domain.

5. (20 marks) The 1D wave equation for acoustics is $\partial^2 p/\partial t^2 = c^2\partial^2 p/\partial x^2$, where p is the acoustic pressure and $c > 0$ is the speed of sound. This equation can be written in conservative form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0} \quad \text{with} \quad \mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} := \begin{bmatrix} p \\ q \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = -c\begin{bmatrix} U_2 \\ U_1 \end{bmatrix} \quad (2)$$

with an additional unknown q. The system (2) is to be solved with a Finite Volumes (FV) scheme with a flux-splitting (upwind) numerical flux.

   a) Write the expression of the upwind flux splitting for normal vector $n = -1$ and for normal vector $n = 1$. Note that,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

   Check that the proposed formulas are conservative. That is, $\widetilde{\mathbf{F}}_{-n}(\mathbf{U}^{out}, \mathbf{U}) = -\widetilde{\mathbf{F}}_n(\mathbf{U}, \mathbf{U}^{out})$.

Consider a 1D domain (interval) Ω, divided in n volumes, $\{V_e\}_{i=1}^n$, with centers $\{x_e\}_{i=1}^n$, that is $V_e = (x_{e-1/2}, x_{e+1/2})$ with $x_{e\pm1/2} = x_e \pm \frac{h}{2}$, $h = |V_e|$.

b) Derive the FV scheme and write the formula for the computation of the solution at time t^{n+1} from the solution at time t^n using explicit Euler time integration. Recall that in 1D $\int_a^b \partial f/\partial x\, dx = (-1)f(a) + (1)f(b)$.

Solution:

a) The normal flux is $\mathbf{F}_n(\mathbf{U}) = -cn \begin{bmatrix} U_2 \\ U_1 \end{bmatrix} = \mathbf{A}_n\mathbf{U}$ with $n = \pm 1$ and

$$\mathbf{A}_n = -cn \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -cn & 0 \\ 0 & cn \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Thus, for $n = 1$

$$\mathbf{A}_1^+ = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{c}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{A}_1^- = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = -\frac{c}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Note that, $\mathbf{A}_1^+ + \mathbf{A}_1^- = \mathbf{A}_1$. Now, the numerical normal flux for $n = 1$ is

$$\widetilde{\mathbf{F}}_1(\mathbf{U}, \mathbf{U}^{out}) = \mathbf{A}_1^+\mathbf{U} + \mathbf{A}_1^-\mathbf{U}^{out} = \frac{c}{2}(U_1 - U_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{c}{2}(U_1^{out} + U_2^{out}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note that $\widetilde{\mathbf{F}}_1(\mathbf{U}, \mathbf{U}) = \mathbf{F}_1(\mathbf{U})$. Analogously,

$$\widetilde{\mathbf{F}}_{-1}(\mathbf{U}, \mathbf{U}^{out}) = \mathbf{A}_{-1}^+\mathbf{U} + \mathbf{A}_{-1}^-\mathbf{U}^{out} = \frac{c}{2}(U_1 + U_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{c}{2}(U_1^{out} - U_2^{out}) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The numerical normal flux is conservative:

$$\widetilde{\mathbf{F}}_{-1}(\mathbf{U}^{out}, \mathbf{U}) = \frac{c}{2}(U_1^{out} + U_2^{out}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{c}{2}(U_1 - U_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\widetilde{\mathbf{F}}_1(\mathbf{U}, \mathbf{U}^{out}).$$

b) Integrating the conservation equation in a volume (interval), V_e, and applying integration by parts, leads to

$$\int_{Ve} \frac{d\mathbf{U}}{dt}\, dx = -\int_{Ve} \frac{d\mathbf{F}(\mathbf{U})}{dx}\, dx = -\left(\mathbf{F}_{-1}(\mathbf{U}(x_{e-1/2})) + \mathbf{F}_1(\mathbf{U}(x_{e+1/2}))\right).$$

Now, considering constant approximation $\mathbf{U}^e$ in each volume V_e and replacing the normal flux by the numerical normal flux, we get

$$h\frac{d\mathbf{U}^e}{dt} \simeq= -\left(\widetilde{\mathbf{F}}_{-1}(\mathbf{U}^e, \mathbf{U}^{e-1}) + \widetilde{\mathbf{F}}_1(\mathbf{U}^e, \mathbf{U}^{e+1})\right).$$

Discretizing in time, the scheme is

$$[\mathbf{U}^e]^{n+1} = [\mathbf{U}^e]^n - \frac{\Delta t}{h}\left(\widetilde{\mathbf{F}}_{-1}([\mathbf{U}^e]^n, [\mathbf{U}^{e-1}]^n) + \widetilde{\mathbf{F}}_1([\mathbf{U}^e]^n, [\mathbf{U}^{e+1}]^n)\right).$$