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Assignment 1: Continuum Mechanics and

Elasticity

1 In small strains, the constitutive law can be deduced from the strain energy function $W(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ as $\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}$.

a) Deduce the expression of the strain energy function $W(\boldsymbol{\varepsilon})$ for a linear isotropic material. Write the expression in terms of the invariants $I_1(\boldsymbol{\varepsilon}) = \text{trace}(\boldsymbol{\varepsilon})$ and $I_2(\boldsymbol{\varepsilon}) = \text{tr}(\boldsymbol{\varepsilon}^2)$, and the Lamé parameters λ and μ .

We know that $\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}$, so:

$$\boldsymbol{\sigma} = \begin{pmatrix} \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{xx} & 2\mu \varepsilon_{xy} & 2\mu \varepsilon_{xz} \\ 2\mu \varepsilon_{yx} & \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{yy} & 2\mu \varepsilon_{yz} \\ 2\mu \varepsilon_{zx} & 2\mu \varepsilon_{zy} & \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{zz} \end{pmatrix}$$

Using the Voigt notation, we have that:

$$\{\boldsymbol{\sigma}\} = \begin{pmatrix} \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{xx} \\ \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{yy} \\ \lambda \text{tr}(\boldsymbol{\varepsilon}) + 2\mu \varepsilon_{zz} \\ 2\mu \varepsilon_{xy} \\ 2\mu \varepsilon_{xz} \\ 2\mu \varepsilon_{yz} \end{pmatrix}, \quad \{\boldsymbol{\varepsilon}\} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{pmatrix}$$

$$\text{So, } W(\boldsymbol{\varepsilon}) = \frac{1}{2} \{\boldsymbol{\sigma}\}^t \cdot \{\boldsymbol{\varepsilon}\} =$$

$$= \frac{1}{2} \left[\lambda \text{tr}(\varepsilon)^2 + 2\mu \underbrace{(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2 + 2(\varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2))}_{\text{tr}(\varepsilon^2)} \right] =$$

$$= \frac{\lambda}{2} I_1(\varepsilon)^2 + \mu I_2(\varepsilon) \quad \square$$

b) St. Venant's constitutive model is obtained by replacing in the previous expression of $W|_{\varepsilon}$ the invariants $I_i(\varepsilon)$ by $I_i(E)$, with E the Green-Lagrange strain tensor. The resulting strain energy function $W|_E$ allows to compute the (2nd Piola) stress tensor S as $S = \frac{\partial W|_E}{\partial E}$. Compare the stresses σ and S for a displacement $u^t = \{X, 0, 0\}$, using the same Lamé parameters in both models. Are

σ and S equal? Why?

$$\text{We have } W|_E = \frac{\lambda}{2} I_1(E)^2 + \mu I_2(E)$$

$$\bullet \frac{\partial}{\partial E} \text{tr}(E) = \begin{pmatrix} \frac{\partial \text{tr}(E)}{\partial E_{11}} & \frac{\partial \text{tr}(E)}{\partial E_{31}} \\ \frac{\partial \text{tr}(E)}{\partial E_{13}} & \frac{\partial \text{tr}(E)}{\partial E_{33}} \end{pmatrix} = \text{Id}$$

$$\bullet \text{tr}(E^2) = E_{11}^2 + E_{22}^2 + E_{33}^2 + 2E_{12}E_{21} + 2E_{13}E_{31} + 2E_{23}E_{32}$$

$$\bullet \frac{\partial}{\partial E} \text{tr}(E^2) = \begin{pmatrix} \frac{\partial \text{tr}(E^2)}{\partial E_{11}} & \frac{\partial \text{tr}(E^2)}{\partial E_{31}} \\ \frac{\partial \text{tr}(E^2)}{\partial E_{13}} & \frac{\partial \text{tr}(E^2)}{\partial E_{33}} \end{pmatrix} = 2E$$

$$\text{So, } S = \frac{\partial}{\partial E} W|_E = \lambda I_1(E) \cdot \frac{\partial \text{tr}(E)}{\partial E} + \mu \frac{\partial \text{tr}(E^2)}{\partial E} = \lambda I_1(E) \text{Id} + 2\mu E$$

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If we consider $u^t = \{X, 0, 0\}$, then:

$$F = \frac{\partial x}{\partial X} = \text{Id} + \frac{\partial u}{\partial X} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \frac{1}{2} (F^t F - \text{Id}) = \begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\nabla u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon = \frac{1}{2} (\nabla u + \nabla u^t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{So, } S = \lambda I_1(E) \cdot \text{Id} + 2\mu E = \begin{pmatrix} \frac{3\lambda}{2} + 3\mu & 0 & 0 \\ 0 & 3\lambda/2 & 0 \\ 0 & 0 & 3\lambda/2 \end{pmatrix}$$

$$\sigma = \lambda I_1(\varepsilon) \cdot \text{Id} + 2\mu \varepsilon = \begin{pmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$S \neq \sigma$ because we aren't under the assumption of small deformations and so, $E \neq \varepsilon$. 

c) Do you think that St. Venant's model for large strains is invariant under a rigid body rotation R ? Answer this question by applying a rigid body rotation to a displacement field u , and checking whether $I_1 E$ and S vary under R .

Let's consider a general rotation R applied to a movement x :

$$\tilde{x} = Rx$$

$$\text{Then: } \tilde{F} = \frac{\partial \tilde{x}}{\partial X} = \frac{\partial \tilde{x}}{\partial x} \cdot \frac{\partial x}{\partial X} = RF$$

$$\tilde{E} = \frac{1}{2} (\tilde{F}^t \tilde{F} - \text{Id}) = \frac{1}{2} (F^t \underbrace{R^t R}_{\text{Id}} F - \text{Id}) = E$$

Since $\tilde{E} = E$, we have that $W_E = W_{\tilde{E}}$ and $S_E = S_{\tilde{E}}$,

so we can state that St. Venant's model for large strains is invariant under a rigid body rotation R . 



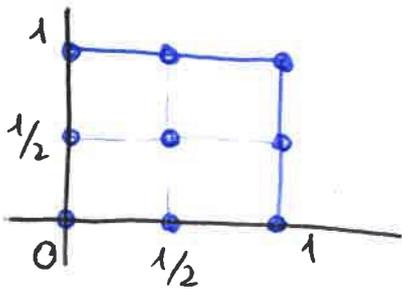
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2 Consider a motion defined by the displacements $u_x(X, Y)$ and $u_y(X, Y)$, in the directions X and Y respectively, of the square domain $\Omega = [0, 1] \times [0, 1] \in \mathbb{R}^2$, with:

$$u_x = X^2 \quad u_y = -\beta XY$$

We are assuming plane strain and small deformations. The material is isotropic elastic with Young modulus $E > 0$ and $\nu = 0$. Then,

a) Draw the deformed shape of a square domain formed by a mesh of 2×2 quadrilaterals, indicating the position of the interior nodes.



We have $u = (X^2, -\beta XY)$
and $x = u + X$.

So, $A_{\text{beg}} = (0, 0) \rightsquigarrow A_{\text{end}} = (0, 0)$

$B_{\text{beg}} = (1/2, 0) \rightsquigarrow B_{\text{end}} = (3/4, 0)$

$C_{\text{beg}} = (1, 0) \rightsquigarrow C_{\text{end}} = (2, 0)$

$D_{\text{beg}} = (1, 1/2) \rightsquigarrow D_{\text{end}} = (2, 1/2(1-\beta))$

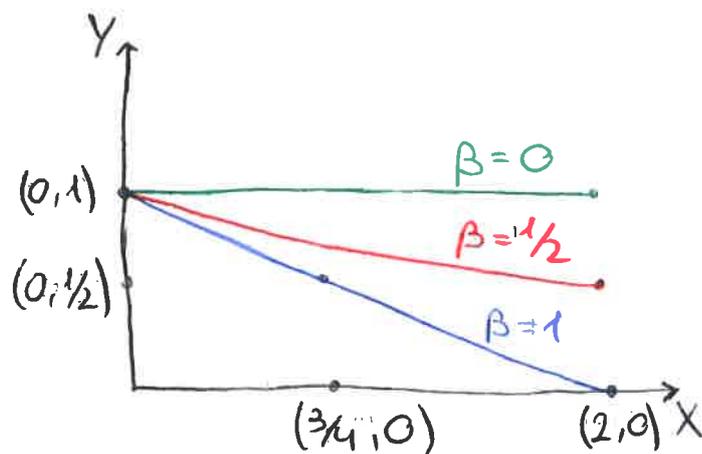
$E_{\text{beg}} = (1, 1) \rightsquigarrow E_{\text{end}} = (2, 1-\beta)$

$$F_{\text{beg}} = (1/2, 1) \rightsquigarrow F_{\text{end}} = (3/4, 1 - \beta/2)$$

$$G_{\text{beg}} = (0, 1) \rightsquigarrow G_{\text{end}} = (0, 1)$$

$$H_{\text{beg}} = (0, 1/2) \rightsquigarrow H_{\text{end}} = (0, 1/2)$$

$$I_{\text{beg}} = (1/2, 1/2) \rightsquigarrow I_{\text{end}} = (3/4, 1/2 - \beta/4)$$



Above we can see the result of apply the deformation to Ω for certain values of β .

b) Write the expression of the Cauchy stress tensor σ as a function of X and Y .

$$\nabla u = \begin{pmatrix} 2X & 0 \\ -\beta Y & -\beta X \end{pmatrix}$$

$$\varepsilon = \frac{1}{2} (\nabla u + \nabla u^t) = \begin{pmatrix} 2X & -\beta/2 Y \\ -\beta/2 Y & -\beta X \end{pmatrix}$$

Since $\nu = 0$, we have that $\lambda = 0$ and, using this, we also get that $\mu = E/2$.

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So, $\sigma = \lambda(\epsilon) \cdot \text{Id} + 2\mu\epsilon = E \cdot \epsilon = \begin{pmatrix} 2EX & -\frac{E}{2}\beta Y \\ -\frac{E}{2}\beta Y & -E\beta X \end{pmatrix}$

c) Evaluate the stress tensor on the four edges of the boundary of Ω .



(1) $Y=0$: $\sigma_1 = \begin{pmatrix} 2EX & 0 \\ 0 & -E\beta X \end{pmatrix}$

(2) $X=1$: $\sigma_2 = \begin{pmatrix} 2E & -\frac{E}{2}\beta Y \\ -\frac{E}{2}\beta Y & -E\beta \end{pmatrix}$

(3) $Y=1$: $\sigma_3 = \begin{pmatrix} 2EX & -\frac{E}{2}\beta \\ -\frac{E}{2}\beta & -E\beta X \end{pmatrix}$

(4) $X=0$: $\sigma_4 = \begin{pmatrix} 0 & -\frac{E}{2}\beta Y \\ -\frac{E}{2}\beta Y & 0 \end{pmatrix}$

d) Represent graphically the traction vectors on the four edges forming the boundary of Ω .

Let's know that $t = \sigma \cdot n$. Following the same nomenclature of the previous point, we have:

$$n_1^t = (0, -1), \quad n_2^t = (1, 0)$$

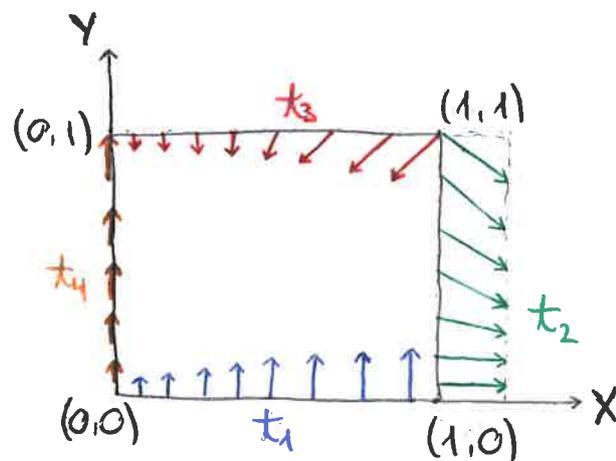
$$n_3^t = (0, 1), \quad n_4^t = (-1, 0)$$

So, computing the product we get:

$$t_1^t = (0, E\beta X), \quad t_2^t = (2E, -\frac{E}{2}\beta Y), \quad t_3^t = (-\frac{E}{2}\beta, -E\beta X),$$

$$t_4^t = (0, \frac{E}{2}\beta Y)$$

A graphical representation of this traction vectors can be found below:



e) Verify that the sum of the tractions on the boundary of Ω is different from zero.

Using t_1, t_2, t_3, t_4 from the previous point, we get that the sum of the tractions on the boundary of Ω is:

$$\sum_{i=1}^4 t_i = \begin{pmatrix} (2 - \beta/2)E \\ 0 \end{pmatrix}$$



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We can also compute the loads at the boundary:

$$F_i = \int_0^1 t_i$$

$$\text{So: } F_1 = \int_0^1 t_1 dx = \begin{pmatrix} 0]_0^1 \\ \frac{E}{2} \beta x^2]_0^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{E}{2} \beta \end{pmatrix}$$

$$F_2 = \int_0^1 t_2 dy = \begin{pmatrix} 2EY]_0^1 \\ -\frac{E}{4} \beta Y^2]_0^1 \end{pmatrix} = \begin{pmatrix} 2E \\ -\frac{E}{4} \beta \end{pmatrix}$$

$$F_3 = \int_0^1 t_3 dx = \begin{pmatrix} -\frac{E}{2} \beta x]_0^1 \\ -\frac{E}{2} \beta x^2]_0^1 \end{pmatrix} = \begin{pmatrix} -\frac{E}{2} \beta \\ -\frac{E}{2} \beta \end{pmatrix}$$

$$F_4 = \int_0^1 t_4 dy = \begin{pmatrix} 0]_0^1 \\ \frac{E}{4} \beta y^2]_0^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{E}{4} \beta \end{pmatrix}$$

The sum of the loads is:

$$\sum_{i=1}^4 F_i = \begin{pmatrix} (2 - \beta/2)E \\ 0 \end{pmatrix}$$

f) If we know that the body is in static equilibrium, why is the sum of the tensions on the boundary of Ω not zero?

The equation of a static equilibrium body is :

$$\nabla \cdot \sigma + \rho b = 0$$

We know that $\sigma = \begin{pmatrix} 2EX & -\frac{E}{2}\beta Y \\ -\frac{E}{2}\beta Y & -E\beta X \end{pmatrix}$, so we can compute

$$\rho b = -\nabla \cdot \sigma = -\begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \end{pmatrix} = -\begin{pmatrix} 2E - \frac{E}{2}\beta \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} (\beta/2 - 2)E \\ 0 \end{pmatrix}$$

With this we can calculate the internal body forces :

$$f_v = \int_{\Omega} \rho b \, dV = \int_0^1 dx \int_0^1 dy \begin{pmatrix} (\beta/2 - 2)E \\ 0 \end{pmatrix} = \begin{pmatrix} (\beta/2 - 2)E \\ 0 \end{pmatrix}$$

As we can see, these internal body forces are not zero, so it's

reasonable to think that these forces along with the ones in the boundary make the whole body be in a static equilibrium.



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3] A specimen with an undeformed configuration given by the square domain $[-1, 1]^2$ and made of an isotropic elastic material with Young modulus E and Poisson ratio ν is being compressed according to the following boundary conditions:

- $y = -1$: $u = 0$
- $x = 1$: $t = 0$
- $y = 1$: $u = \{0, -\bar{u}\}$
- $x = -1$: $t = 0$

with \bar{u} a small positive constant. The initial and deformed configurations are represented in Figure 1. Assuming plane strain and small deformations, answer the following questions:

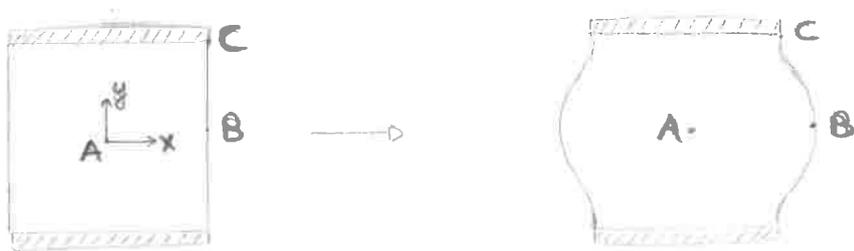


Figure 1: Initial and deformed configuration of the compressed square.

- a) If $E > 0$ and $0 < \nu < 0.5$, indicate which components of the small strain tensor ϵ and the stress tensor σ at the points $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 1)$ are zero and why.

Point A



We have a vertical symmetry, so $\nabla_x u = 0$ and so $\epsilon_{xx} = 0$.

Since there is deformation on the y-axis we have $\epsilon_{yy} \neq 0$ and,

because there is no angular distortion, $\epsilon_{xy} = 0$.

$$\text{So, } \epsilon = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{yy} \end{pmatrix}$$



Using the plane strain formulas we have that:

$$\sigma_{xx} = (\lambda + 2\mu) \epsilon_{xx} + \lambda \epsilon_{yy} \Rightarrow \sigma_{xx} \neq 0$$

$$\sigma_{yy} = \lambda \epsilon_{xx} + (\lambda + 2\mu) \epsilon_{yy} \Rightarrow \sigma_{yy} \neq 0$$

$$\sigma_{xy} = 2G \epsilon_{xy} = 0$$

$$\text{So, } \sigma = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix}$$

Point B

Using that $t = \sigma \cdot n$ and that $t = 0$ on B:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \end{pmatrix} \Rightarrow \begin{cases} \sigma_{xx} = 0 \\ \sigma_{xy} = 0 \end{cases}$$

Since the body is compressed, $\sigma_{yy} \neq 0$

$$\text{So, } \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{yy} \end{pmatrix}$$



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$$\epsilon_{xy} = \frac{1}{2G} \tau_{xy} \stackrel{=0}{=} \Rightarrow \epsilon_{xy} = 0$$

Since there is a horizontal deformation, $\epsilon_{xx} \neq 0$.

Using that $0 = \tau_{xx} = \underbrace{(\lambda + 2\mu)}_{\neq 0} \underbrace{\epsilon_{xx}}_{\neq 0} + \underbrace{\lambda}_{\neq 0} \epsilon_{yy} \rightarrow \epsilon_{yy} \neq 0$.

So $\epsilon = \begin{pmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{yy} \end{pmatrix}$

Point C

We have $u^t = (0, -\bar{u}) \rightarrow \mathcal{F} = \text{Id} \rightarrow \epsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$
 $\Rightarrow \tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 

b) If $E > 0$ and $\nu = 0$, which components of the small strain tensor ϵ and stress tensor τ at points A, B and C are zero? Whenever it is possible, predict also the sign of the non-zero components

Point A

Since we have a vertical symmetry, $\nabla_x u = 0 \Rightarrow \epsilon_{xx} = 0$, so

$$\tau_{xx} = E \cdot \cancel{\epsilon_{xx}}^0 = 0$$

Since there is a "negative" deformation on the y-axis: $\epsilon_{yy} < 0$,

$$\text{so } \tau_{yy} = \hat{E} \cdot \hat{\epsilon}_{yy} < 0$$

And since there is no angular distortion $\epsilon_{xy} = 0$ and so $\tau_{xy} = 0$.

$$\text{So } \epsilon = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{yy} \end{pmatrix} \quad , \quad \tau = \begin{pmatrix} 0 & 0 \\ 0 & \tau_{yy} \end{pmatrix}$$

Point B

Using the same arguments of the point (a), since $0 = t = \sigma \cdot n$,

$$\tau_{xx} = \tau_{xy} = 0, \text{ and from this we get } \epsilon_{xx} = \epsilon_{xy} = 0$$

Since the body is being compressed $\tau_{yy} < 0$ and so $\epsilon_{yy} < 0$.

$$\text{So } \epsilon = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{yy} \end{pmatrix} \quad , \quad \tau = \begin{pmatrix} 0 & 0 \\ 0 & \tau_{yy} \end{pmatrix}$$

Point C

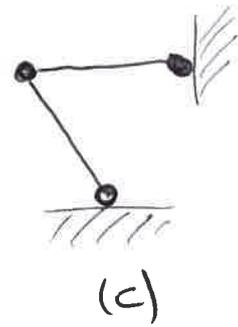
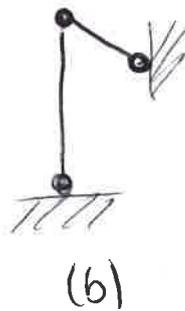
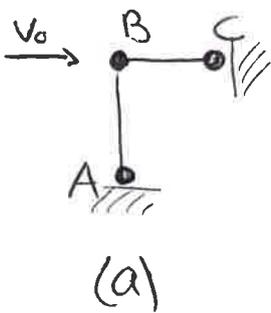
Using exactly the same arguments of (a), we get $\epsilon = \tau = 0$.



Assignment 2: Dynamics

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1 We aim to analyse the dynamic response of the undamped 2 bar system depicted in Figure 1a. Since parts A and C are fixed ($u_A = u_C = 0$), the motion is fully described by the displacement of point B, and therefore there are only two degrees of freedom. The two normalised eigen modes a_1 and a_2 , with eigen frequencies ω_1 and ω_2 , are respectively indicated in Figures 1b and 1c.



We aim to compute the solution $u_B(t)$ using modal analysis. The system is unloaded, initially at rest ($u(0) = 0$), and subjected to the initial velocity v_0 shown in figure 1a.

a) Write the system of uncoupled COEs that need to be solved in order to find $u_B(t)$.

As we are using modal analysis, we have that $M\ddot{u}_B(t) + k\bar{u}_B(t) = 0$

We are going to approximate \bar{u}_B as: $\bar{u}_B(t) = \sum_{i=1}^2 y_i(t) \cdot a_i$

Substituting in the equation:

$$\sum_{i=1}^2 [M a_i \ddot{y}_i(t) + K a_i y_i(t)] = 0$$

Multiplying by a_j^T :

$$\sum_{i=1}^2 [a_j^T M a_i \ddot{y}_i(t) + a_j^T K a_i y_i(t)] = 0$$

δ_{ij} $\omega_i^2 \delta_{ij}$
 $\bar{a}_i a_i$ $\omega_i^2 \delta_{ij}$
 M- and K-orthogonal

$$\rightarrow \ddot{y}_j + \omega_j^2 y_j = 0 \quad \text{OK}$$

So the system of uncoupled ODEs is:

$$\begin{cases} \ddot{y}_1 + \omega_1^2 y_1 = 0 \\ \ddot{y}_2 + \omega_2^2 y_2 = 0 \end{cases}$$

b) Indicate the initial conditions of the previous ODEs as a function of v_0 , $a_{1,2}$ and the diagonal mass matrix M .

We know that $u_0 = \sum_i \phi_i(0) a_i$

Multiplying by a_j and M : $a_j^T M u_0 = \sum_i \phi_i(0) a_j^T M a_i$

δ_{ij}

$$\rightarrow \phi_j(0) = a_j^T M u_0$$

Using $u_0 = \sum_i \phi_i(0) a_i$ and doing the same we get

$$\phi_j(0) = a_j^T M u_0$$

So :

$$\begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \\ \dot{y}_1(0) = 0 \\ y_2(0) = a_2 M v_0 \end{cases} \quad \text{OK}$$

c) Deduce the solution $u_B(t)$ as a function of v_0 , the mass matrix M , $a_{1,2}$ and $\omega_{1,2}$. Which will be the frequency of the oscillations at point B?

The first ODE to solve is :

$$\begin{cases} \ddot{y}_1(t) + \omega_1^2 y_1(t) = 0 \quad (*) \\ y_1(0) = 0 \\ \dot{y}_1(0) = 0 \end{cases}$$

We know that the solution of (*) is : $y_1(t) = A \cos(\omega_1 t) + i B \sin(\omega_1 t)$

Since $y_1(0) = 0$ we have that $A = 0$, so $y_1(t) = i B \sin(\omega_1 t)$

Using now that $\dot{y}_1(0) = 0$ we have $B = 0$.

So $y_1(t) = 0$. OK

The second ODE to solve is :

$$\begin{cases} \ddot{y}_2(t) + \omega_2^2 y_2(t) = 0 \quad (*) \\ y_2(0) = 0 \\ \dot{y}_2(0) = a_2 M v_0 \end{cases}$$

The solution of (*) is also $y_2(t) = \tilde{A} \cos(\omega_2 t) + i \tilde{B} \sin(\omega_2 t)$

Since $y_2(0) = 0$ we get that $\tilde{A} = 0$ and so $y_2(t) = i \tilde{B} \sin(\omega_2 t)$

Using now that $y_2(0) = a_2 M v_0$, we get that $\tilde{B} = \frac{a_2 M v_0}{i \omega_2}$

So, $y_2(t) = \frac{1}{\omega_2} a_2 M v_0 \sin(\omega_2 t)$

Finally, the solution is $u_B(t) = \sum_{i=1}^2 y_i(t) a_i =$

$$= \left(\frac{1}{\omega_2} a_2 M v_0 \sin(\omega_2 t) \right) a_2$$

→ The frequency of oscillation at point B will be ω_2 .

OK

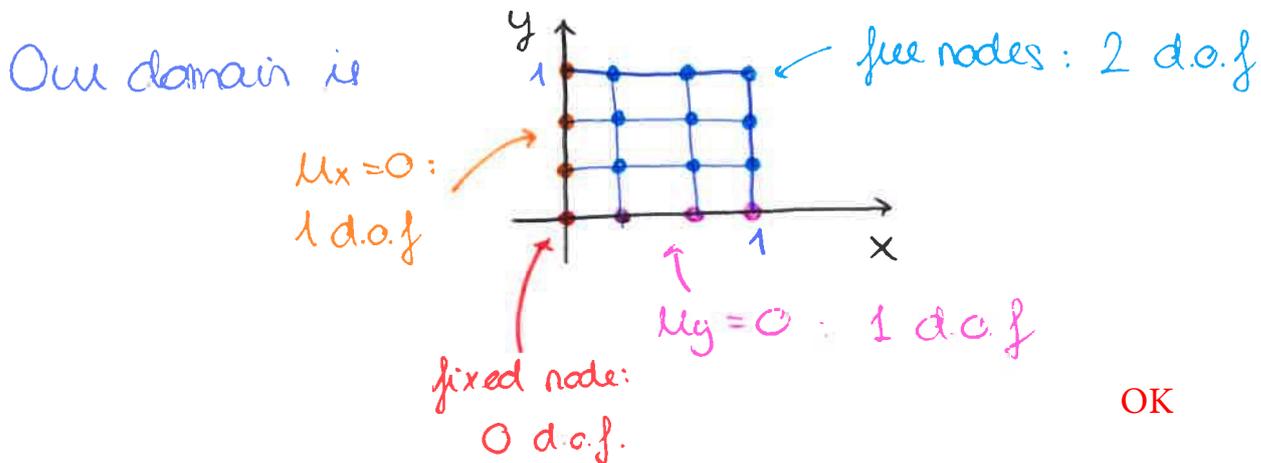
2 We aim to find the eigen-frequencies of an elastic two-dimensional square $[0, 1] \times [0, 1]$. The Dirichlet boundary conditions of the problem are,

$$u_x = 0, \text{ on } x=0$$

$$u_y = 0, \text{ on } y=0$$

The domain is discretised with bilinear quadrilateral finite elements, using 3 divisions along each direction (total 9 elements)

a) How many eigen-frequencies should we expect?



So we have $9 + 3 + 3 = 24$ d.o.f, so 24 eigen-frequencies.

b) If we double the Young modulus E , how are the eigen-frequencies going to be affected? And if we double the density ρ ? Justify your answer.

After the discretization we have that the system matrices are:

$$m_{ij} = \int_V \rho N_i(x) N_j(x) \cdot I_d \cdot dV, \quad K_{ij} = \int_V B_i(x)^T \cdot C \cdot B_j(x) dV,$$

where B is a matrix depending on N_i and N_j , and $C = E \cdot C^*$,
 where C^* is a matrix depending on the Poisson ratio ν .

We know that the eigen-frequencies ω_i can be calculated as $\omega_i^2 = \lambda_i$,
 where $\lambda_i = \text{eigenval.}(M^{-1}K)$.

• If we double the Young modulus E , we have that $\tilde{C} = 2C$ and
 so $\tilde{K} = 2K$.

$$\text{Thus, } \tilde{\lambda}_i = \text{eigenval.}(\tilde{M}^{-1}\tilde{K}) = \text{eigenval.}(2M^{-1}K) = 2 \text{ eigenval.}(M^{-1}K) = 2\lambda_i$$

$$\text{So, } \tilde{\omega}_i^2 = \tilde{\lambda}_i = 2\lambda_i = 2\omega_i^2 \Rightarrow \boxed{\tilde{\omega}_i = \sqrt{2} \omega_i}$$

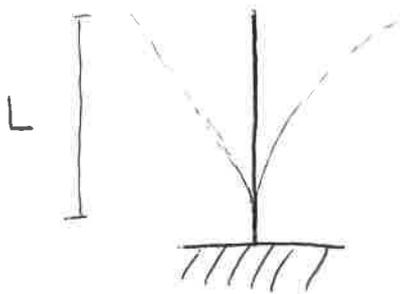
• If we double the density ρ , $\bar{M} = 2M$ and so $\bar{M}^{-1} = \frac{1}{2} M^{-1}$

$$\text{Thus, } \bar{\lambda}_i = \text{eigenval.}(\bar{M}^{-1}\bar{K}) = \text{eigenval.}\left(\frac{1}{2} M^{-1}K\right) = \frac{1}{2} \text{ eigenval.}(M^{-1}K) = \frac{1}{2} \lambda_i$$

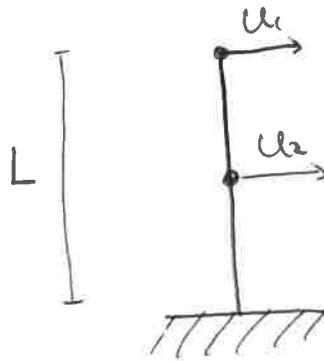
$$\text{So, } \bar{\omega}_i^2 = \bar{\lambda}_i = \frac{1}{2} \lambda_i = \frac{1}{2} \omega_i^2 \Rightarrow \boxed{\bar{\omega}_i = \frac{1}{\sqrt{2}} \omega_i}$$

OK

3] In order to know whether a thin rod with length $L = 1\text{ m}$ is made of copper or steel, we perform modal analysis. The rod is clamped at the bottom and subjected to free oscillations, as shown in Figure 2(a). We experimentally measure the lowest vibration frequency, which is approximately $f = 810\text{ Hz}$ (Remark: if ω is the pulse in rad/s, then $f = \omega/2\pi$).



(a)



(b)

Determine the material of the rod using the simplified model in Figure 2(b). Use two finite elements, and consider only the horizontal displacements (u_1, u_2). The global stiffness and mass matrices of the model, K and M respectively, are given by,

$$K = \frac{2GA}{L} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad M = \frac{\rho AL}{12} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

where A , ρ and G are the cross-section area, the density and the shear modulus, respectively. The values of ρ and G for the steel and copper are given in the following table:

	$\rho [\text{kg} \cdot \text{m}^{-3}]$	$G [\text{N} \cdot \text{m}^{-2}]$
Steel	7850	$82.0 \cdot 10^9$
Copper	8900	$48.5 \cdot 10^9$

Let's start calculating the eigenvalues of $M^{-1}K$. Using Maple and $L=1$, we get:

$$\lambda_i = \text{eigenval}(M^{-1}K) = \left(\begin{array}{c} \frac{24(5/7 + 3\sqrt{2}/7)}{\rho} \cdot G \\ \frac{24(5/7 - 3\sqrt{2}/7)}{\rho} \cdot G \end{array} \right)$$

The lowest eigenvalue is the second one.

• For steel: $f_{\text{steel}} = \frac{\omega_{\text{steel}}}{2\pi} = \frac{\sqrt{\lambda_{\text{steel}}}}{2\pi} = 828,89 \text{ Hz}$ $G = 82 \cdot 10^9, \rho = 7850$

• For copper: $f_{\text{cop}} = \frac{\omega_{\text{cop}}}{2\pi} = \frac{\sqrt{\lambda_{\text{cop}}}}{2\pi} = 598,89 \text{ Hz}$ $G = 48.5 \cdot 10^9, \rho = 8900$

Since $f_{\text{steel}} = 828,89 \text{ Hz} \approx 810 \text{ Hz} = f$, we can assume that the rod is made of steel. OK

COMPUTATIONAL MECHANICS

ASSIGNMENT 3: PLASTICITY

EXERCISE 1

Imagine that a small steel sphere is dropped underwater at depth H . If it plastifies according to a Von-Mises criteria, do you think it may plastify for a given value of H ? Justify your answer.

Answer

The Von-Mises stress is computed as $\sigma_{eq} = \sqrt{3J_2}$, where J_2 is defined as

$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

Since we are considering a small sphere underwater, we have that $\sigma = p_h Id$, where p_h is the hydrostatic pressure, depending on the depth H . So, $\sigma_1 = \sigma_2 = \sigma_3$ and thus $J_2 = 0$.

Since the Von-Mises stress $\sigma_{eq} = \sqrt{3J_2} = 0$, we can say that the sphere won't plastify for any value of H under Von-Mises criteria.

ok. 3.0

EXERCISE 2

A cubic domain $[0, 1]^3$ made of a perfect plastic a following a Von-Mises criteria is subjected to two different sets of boundary conditions, (A) and (B), given in the next table:

	$x = 0$	$x = 1$	$y = 0$	$y = 1$	$z = 0$	$z = 1$
(A)	$u_x = 0$	$\sigma \mathbf{n} = \{t, 0, 0\}^T$	$u_y = 0$	$\sigma \mathbf{n} = \mathbf{0}$	$u_z = 0$	$\sigma \mathbf{n} = \mathbf{0}$
(B)	$u_x = 0$	$\sigma \mathbf{n} = \{t, 0, 0\}^T$	$u_y = 0$	$u_y = 0$	$u_z = 0$	$u_z = 0$

with $t > 0$ a loading parameter.

- (a) Compute the displacement u_x at $x = 1$ in the two cases as a function of the Young modulus E and Poisson ratio ν in the elastic range. For a given value of t , in which case will u_x at $x = 1$ be larger? NOTE: In the two cases, the solution of the elastic problem has a constant deformation and stress field.
- (b) For a given material parameter σ_Y , in which case the material is going to plastify with a smaller value of t ?
- (c) In which case the loading parameter t will reach a larger magnitude?
- (d) If we now use $t < 0$, is the answer in question (c) going to change?

Answer (a)

For the whole section (a), we will use that in the two cases, the solution of the elastic problem has a constant deformation and stress field.

For case (A), since we know that $u_{\{x=1\}} = (1, 0, 0)$, $u_{\{y=1\}} = (0, 1, 0)$, $u_{\{z=1\}} = (0, 0, 1)$, and we have that $\sigma \mathbf{n} = (t, 0, 0)^T$ in $x = 1$ and $\sigma \mathbf{n} = 0$ in $y = 1$, $z = 1$, we can state that

$$\sigma = \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By using the constitutive equation $\varepsilon = -\frac{\nu}{E} \text{tr}(\sigma)I + \frac{1+\nu}{E}\sigma$, we obtain that

$$\varepsilon = \begin{bmatrix} t/E & 0 & 0 \\ 0 & -\nu t/E & 0 \\ 0 & 0 & -\nu t/E \end{bmatrix}$$

Using that $\varepsilon = \nabla^s u$, we get that, at point $x = 1$, $\frac{\partial u_x}{\partial x} = \varepsilon_{xx} = t/E$. So, integrating, $u_x(x) = \frac{t}{E}x + h(y, z)$, where h is a function depending only on components y and z . Using now the boundary condition at point $x = 0$, we get that $0 = u_x(0) = f(y, z)$.

So the displacement u_x at point $x = 1$ in case (A) is

$$u_x^A = \frac{t}{E}$$

For case (B) we have that $\sigma n = (t, 0, 0)^T$ at $x = 1$ and $u_y = u_z = 0$ at $y = 1$ and $z = 1$, respectively. So,

$$\sigma = \begin{bmatrix} t & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the constitutive equations

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \end{aligned}$$

and knowing that $\varepsilon_{yy} = \varepsilon_{zz} = 0$ and $\sigma_{xx} = t$, we get that

$$\begin{aligned} \sigma_{yy} = \sigma_{zz} &= \frac{\nu}{1-\nu} t \\ \varepsilon_{xx} &= \frac{(\nu+1)(1-2\nu)}{1-\nu} \frac{t}{E} \end{aligned}$$

Using the same argument we used in the previous case, from ε_{xx} we can state that the displacement u_x at point $x = 1$ in case (B) is

$$u_x^B = \frac{(\nu+1)(1-2\nu)}{1-\nu} \frac{t}{E}$$

Since we have that

$$\frac{(\nu+1)(1-2\nu)}{1-\nu} \leq 1, \quad \forall \nu \in (-1, 0.5)$$

we can state that, for a given value of t , $u_x^B \leq u_x^A$ in $x = 1$.

Answer (b)

We know that Von-Mises yield function is defined as $f(\sigma) = \sqrt{3J_2} - \sigma_Y$, where

$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

For case (A), we have that $\sigma_1 = t$ and $\sigma_2 = \sigma_3 = 0$, so $J_2 = \frac{1}{3}t^2$. Thus, $f(\sigma) = t - \sigma_Y$, and the material is going to plastify when $f(\sigma) = 0$, so when $t = \sigma_Y$.

For case (B), we have that $\sigma_1 = t$ and $\sigma_2 = \sigma_3 = \frac{\nu}{1-\nu}t$, so $J_2 = \frac{1}{3}t^2 \left(\frac{1-2\nu}{1-\nu}\right)^2$. Thus, $f(\sigma) = t \left(\frac{1-2\nu}{1-\nu}\right) - \sigma_Y$, and the material is going to plastify when $f(\sigma) = 0$, so when $t = \sigma_Y \left(\frac{1-\nu}{1-2\nu}\right)$.

Absolute value but OK. 1.0

Since $\frac{1-\nu}{1-2\nu} < 1, \forall \nu \in (-1, 0)$, we can state that for $\nu \in (-1, 0)$ the material will plastify with a smaller value of t in case (B), and for $\nu \in (0, 0.5)$ the material will plastify with a smaller value of t in case (A). For $\nu = 0$, the material will plastify with a smaller value of t in both cases.

Answer (c)

Based on what we have said in section (b), for $\nu \in (-1, 0)$, t will reach the maximum value for case (A), and for $\nu \in (0, 0.5)$, t will reach the maximum value for case (B). For $\nu = 0$, t will reach the maximum value for both cases.

True, but because stress is homogeneous. 0.5

Answer (d)

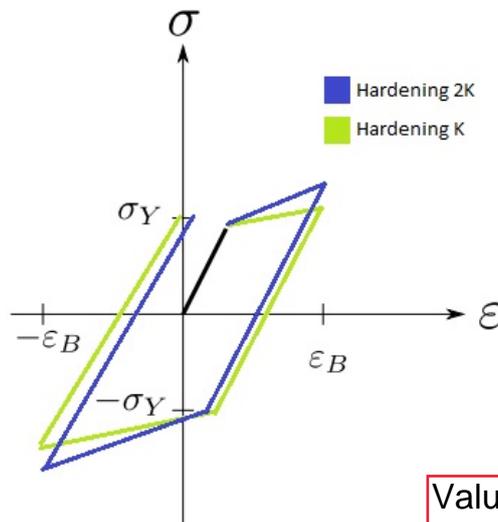
The answer is not going to change because in both cases t is squared when calculating J_2 , and when we calculate the square root, we take the positive solution.

~OK. 1.0

EXERCISE 3

Two specimens made of different materials have been subjected to the strain loading cycle $0 \rightarrow \varepsilon^B \rightarrow -\varepsilon^B \rightarrow 0$. Complete in the following figure the approximated plot of the stress-strain curves of the complete loading path for (a) a material with hardening K and (b) a material with hardening $2K$.

Answer



Value of plastification should change. 2.0

COMPUTATIONAL MECHANICS

ASSIGNMENT 4: FLUIDS

EXERCISE 1

We are modelling an incompressible Stokes flow, and we use a P_2^+/P_{-1} interpolation for velocities and pressures. This triangle element consists on a continuous quadratic interpolation of velocities (6 nodes per element) and a discontinuous bubble pressure with 3 nodes per element.

Consider a two-dimensional square domain $[0, 1] \times [0, 1]$, discretized with n divisions per side and with $2n^2$ triangles (n^2 squares with two triangles each).

(a) Compute the limit

$$\lim_{n \rightarrow \infty} \frac{n_v}{n_p} = \lim_{n \rightarrow \infty} \frac{\dim(Q^h)}{\dim(P^h)}$$

with n_v and n_p the number of degrees of freedom for the velocity and pressure fields, respectively.

(b) Does this element satisfy the necessary stability requirements regarding the dimensions of Q^h and P^h ? Justify your answer.

Answer (a)

For velocities we do a quadratic interpolation, so we have $(2n + 1)^2$ nodes for velocities. Since for pressure we do a discontinuous bubble with 3 nodes per element, we have $(n + 1)^2$. So

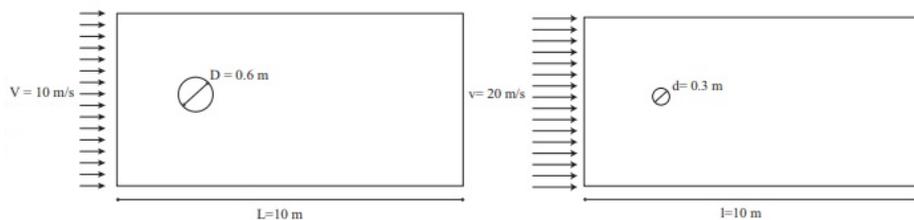
$$\lim_{n \rightarrow \infty} \frac{n_v}{n_p} = \lim_{n \rightarrow \infty} \frac{2 \cdot (2n + 1)^2}{3 \cdot 1 \cdot 2n^2} = \frac{4}{3}$$

Answer (b)

Yes, because in our case the necessary condition $\dim(P^h) \leq \dim(Q^h)$ is satisfied.

EXERCISE 2

We aim to study the flow around cylinders by analyzing the behaviour of different characteristic lengths of the problem. we propose studying the two problems in the figure, considering in both cases the same value for the viscosity ν . Is it necessary to solve both problems? Why?



Answer

For the first case we have that

$$Re_1 = \frac{VD}{\nu} = \frac{10 \cdot 0.6}{\nu} = \frac{6}{\nu}$$

For the second one,

$$Re_2 = \frac{vd}{\nu} = \frac{20 \cdot 0.3}{\nu} = \frac{6}{\nu}$$

We can see that $Re_1 = Re_2$, so both problems are equivalent, because the flow will behave in the same way. Thus, we don't need to solve both of them.

EXERCISE 3

Explain the main differences between the Stokes and Navier-Stokes equations. That is, comment the differences in the physical assumptions and in the numerical solution. Must the LBB condition be taken into account in both cases?

Answer

The Navier-Stokes equation is used for incompressible and viscous newtonian fluids. In order to solve it we need to be careful with some aspects, since it's a PDE with a coupled velocity-pressure, so usual FEM won't work in cases with a low Reynolds number. Plus, it is usually convection-dominated. To sort out the first problem we need to approximate velocity and pressure with spaces that satisfy the LBB condition. For the non-linear case, we can solve it using a non-linear method, such as Picard or Newton-Raphson. If we use the last one, we need to be careful with the magnitude of the Reynolds number, because the method may fail to converge for very low values of it. In order to avoid it, it's recommended to use finer meshes, compute the solution for lower Reynolds numbers and using this solutions as initial solution for the bigger ones.

If we consider a highly viscous isotropic incompressible flow (i.e., a very low Reynolds number in the nondimensionalized Navier-Stokes equation) we can neglect the convection term and get the Stokes equation. As well as with the Navier-Stokes equation with a low Reynolds number, we need to take into account the LBB condition in order to solve it.

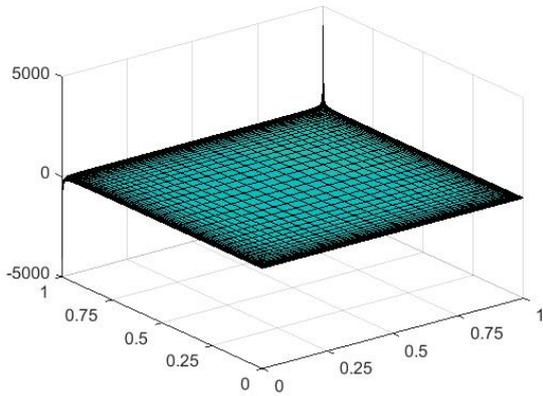
EXERCISE 4

The code at http://ww2.lacan.upc.edu/huerta/exercises/Incompressible/Incompressible_Ex2.htm solves the steady incompressible Navier-Stokes equations for the cavity flow problem. Show the streamlines and the pressure for $Re = 1$ and the biggest Re you can compute, and comment on the result in each case. Explain how did you compute the solution in each case and how many iterations of the non-linear solver were necessary.

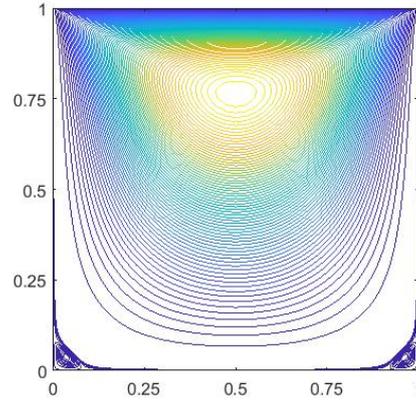
Answer

For both Reynolds numbers, we used an adapted mesh of quadrilaterals Q_2Q_1 , with 101 elements in both X and Y axis. To compute the solution we used the Newton-Raphson method in each case.

For $Re = 1$, 2 iterations of the Newton-Raphson method were necessary to get the solution, which can be seen below:



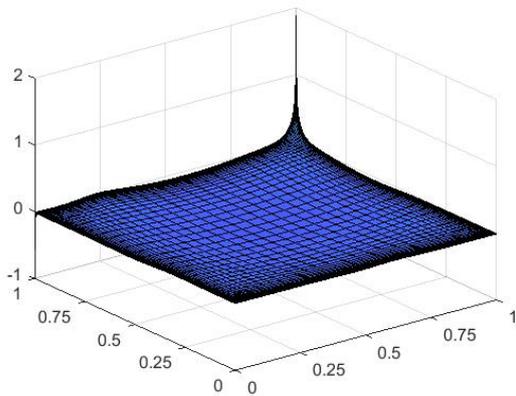
(a) Pressure Field.



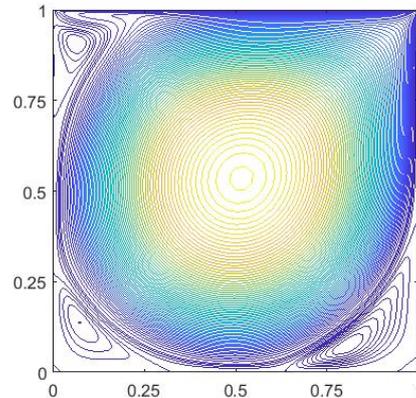
(b) Streamlines.

In figure (a) we can see that at the point $(1, 1)$ we have a very high pressure while in the remaining surface it is very low, basically due to the boundary conditions the upper side. This behaviour of the pressure and boundary conditions explain the streamlines of figure (b), where we can see elliptical trajectories, with two small vortexes on the bottom corners.

The biggest Reynolds we could compute was $Re = 5000$. To do that, we went through 4 intermediate Reynolds, and used their solution as initial solution for the next one. First we computed $Re = 1000$, which took 16 iterations of Newton-Raphson. Then, we went through $Re = 2000$, $Re = 3500$ and $Re = 4500$, that took 20, 46 and 77 iterations, respectively. Finally, we get $Re = 5000$ with 95 more iterations. The result is the following one:



(a) Pressure Field.



(b) Streamlines.

In figure (a) we can see a similar shape of the pressure field as the one with $Re = 1$, but in this case the maximum pressure is much more lower than in the previous case. The most notorious difference between both problems can be seen in figure (b). In this case, we can see how the streamlines have a more circular shape than in the previous example, due to the bigger Reynolds number (as Re is bigger, the diffusion term is less important). We can also see in this example that, due to the highest Reynolds number, the two vortexes from the bottom corners are bigger and a new vortex is appearing at the top left corner.