Total: 7.5/10

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Assignment 1: Continuum Mechanics & Elasticity

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(1) In small strains, the constitutive law can be deduced from the strain energy function $W_\epsilon = \frac{1}{2}\sigma : \epsilon$ as $\sigma = \frac{\partial W_\epsilon}{\epsilon}$.

(a) Deduce the expression of the strain energy function W for a linear isotropic material. Write the expression in terms of the invariants $I_1(\epsilon) = tr(\epsilon)$ and $I_2(\epsilon) = tr(\epsilon^2)$, and the Lamé parameters λ and μ.

Solution Using the direct Lamé equation:

$$\sigma = \lambda tr(\epsilon)I + 2\mu\epsilon$$

$$\sigma_{ijk|i=j=k} = \lambda tr(\epsilon) + 2\mu\epsilon_{ijk}$$

$$\sigma_{ijk|non-diagonal} = 2\mu\epsilon_{ijk}$$

And substituting in the strain energy definition:

$$W_\epsilon = \frac{1}{2}\sum_{i,j,k}\sigma_{ijk}\epsilon_{ijk}$$

First the diagonal terms:

$$\sum_{i,j,k|i=j=k}(\lambda tr(\epsilon) + 2\mu\epsilon_{ijk})\epsilon_{ijk}$$

$$\sum_{i,j,k|i=j=k}\lambda tr(\epsilon)\epsilon_{ijk} + \sum_{i,j,k|i=j=k}2\mu\epsilon_{ijk}\epsilon_{ijk}$$

$$\lambda tr(\epsilon)^2 + 2\mu tr(\epsilon^2)$$

Now the terms of the other terms.

$$\sum_{i,j,k|non-diagonal}2\mu\epsilon_{ijk}^2 = 2\mu\sum_{i,j,k|non-diagonal}\epsilon_{ijk}^2 = 2\mu(\sum_{i,j,k}\epsilon_{ijk}^2 - tr(\epsilon^2))$$

Adding all terms we get:

$$W_\epsilon = \frac{1}{2}(\lambda tr(\epsilon)^2 + 2\mu tr(\epsilon^2) + 2\mu(\sum_{i,j,k}\epsilon_{ijk}^2 - tr(\epsilon^2))) = \frac{1}{2}(\lambda tr(\epsilon)^2 + 2\mu\sum_{i,j,k}\epsilon_{ijk}^2) = \frac{1}{2}(\lambda tr(\epsilon)^2 + 2\mu tr(\epsilon^2))$$

$$\boxed{W_\epsilon = \frac{1}{2}\lambda tr(\epsilon)^2 + \mu tr(\epsilon^2)}$$

(b) St Venants constitutive model is obtained by replacing in the previous expression of W_ϵ the invariants $I_i(\epsilon)$ by $I_i(E)$, with E the Green-Lagrange strain tensor. The resulting strain energy function W_E allows to compute the (2nd Piola) stress tensor S as $S = \frac{\partial W_E}{\partial E}$. Compare the stresses σ and S for a displacement $u^T = \{X, 0, 0\}$, using the same Lamé parameters in both models. Are σ and S equal? Why?

Solution

$$W_E = \frac{1}{2}\lambda tr(E)^2 + \mu tr(E^2)$$

By undoing previous exercise computations we get:

$$S = \frac{\partial W_E}{\partial E} = \lambda tr(E)I + 2\mu E$$

Under the small deformation hypothesis:

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{X_j} + \frac{\partial u_j}{X_i})$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon = \frac{1}{2}(\nabla_X u^T + \nabla_X u)$$

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As ϵ and E are equal σ and S must be equal. This is because we are assuming the small displacement hypothesis.

(c) Do you think that St. Venants model for large strains is it invariant under a rigid body rotation R? Answer this question by applying a rigid body rotation to a displacement field u, and checking whether W_E and S vary under R.

Solution

$$F = RU$$

$$E = \frac{1}{2}(F^T F - I) = \frac{1}{2}(U^T R^T RU - I) = \frac{1}{2}(U^T U - I)$$

As we have just seen E does not depend on R which implies S does not depend on R.

(2) Consider a motion defined by the displacements $u_X(X, Y)$ and $u_Y(X, Y)$, in the directions X and Y respectively, of the square domain $\Omega = [0, 1][0, 1] \in R^2$, with: $u_X = X^2$ and $u_Y = -\beta XY$.

We are assuming plane strain and small deformations. The material is isotropic elastic with Young modulus $E > 0$ and $v = 0$. Then,

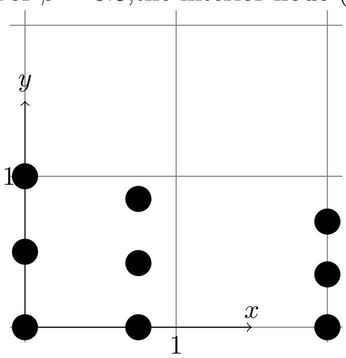
(a) Draw the deformed shape of a square domain formed by a mesh of 2 x 2 quadrilaterals, indicating the position of the interior nodes.

Solution

$$X \to X + X^2$$

$$Y \to Y - \beta XY$$

For $\beta = 0.3$,the interior node (0.5,0.5) goes to (0.75,0.425).



(b) Write the expression of the Cauchy stress tensor σ as a function of X and Y.

Solution As we are assuming small deformations:

$$\epsilon = \frac{1}{2}(\nabla_X u^T + \nabla_X u)$$

We compute $\nabla_X u = \frac{\partial u_i}{\partial X_j}$:

$$\nabla_X u = \begin{bmatrix} 2X & 0 \\ -\beta Y & -\beta X \end{bmatrix}$$

And substituting in the ϵ formula:

$$\epsilon = \begin{bmatrix} 2X & -\beta Y/2 \\ -\beta Y/2 & -\beta X \end{bmatrix}$$

We can compute the parameters λ and μ from the Young Modul and the poison:

$$\lambda = \frac{Ev}{(1+v)(1-2v)} = 0$$

$$\mu = \frac{E}{2(1+v)} = \frac{E}{2}$$

Using Cauchy stress definition:

$$\sigma = \lambda tr(\epsilon) I + 2\mu\epsilon$$

$$\sigma = E \begin{bmatrix} 2X & -\beta Y/2 \\ -\beta Y/2 & -\beta X \end{bmatrix}$$

(c) Evaluate the stress tensor on the four edges of the boundary of Ω.
Solution

- X=0:

$$\sigma = E \begin{bmatrix} 0 & -\beta Y/2 \\ -\beta Y/2 & 0 \end{bmatrix}$$

- X=1:

$$\sigma = E \begin{bmatrix} 2 & -\beta Y/2 \\ -\beta Y/2 & -\beta \end{bmatrix}$$

- Y=0:

$$\sigma = E \begin{bmatrix} 2X & 0 \\ 0 & -\beta X \end{bmatrix}$$

- Y=1:

$$\sigma = E \begin{bmatrix} 2X & -\beta/2 \\ -\beta/2 & -\beta X \end{bmatrix}$$

(d) Represent graphically the traction vectors on the four edges forming the boundary of Ω
Solution

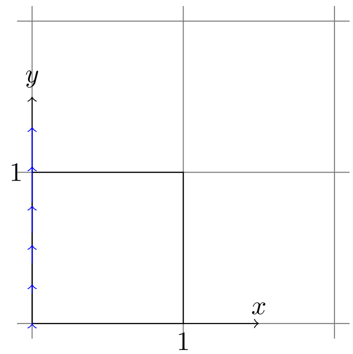
$$\vec{t} = \sigma \vec{n}$$

- X=0, $\vec{n} = \{-1, 0\}$:

$$\sigma = E \begin{bmatrix} 0 & -\beta Y/2 \\ -\beta Y/2 & 0 \end{bmatrix}$$

$$\vec{t} = E \begin{bmatrix} 0 \\ \beta Y/2 \end{bmatrix}$$

Plot for $E = 2$ and $\beta = 0.3$:

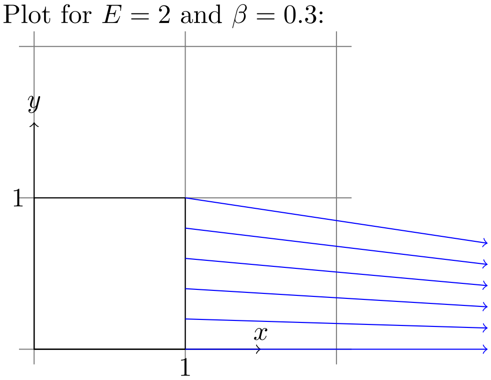


- X=1, $\vec{n} = \{1, 0\}$:

$$\sigma = E \begin{bmatrix} 2 & -\beta Y/2 \\ -\beta Y/2 & -\beta \end{bmatrix}$$

$$\vec{t} = E \begin{bmatrix} 2 \\ -\beta Y/2 \end{bmatrix}$$

Plot for $E = 2$ and $\beta = 0.3$:

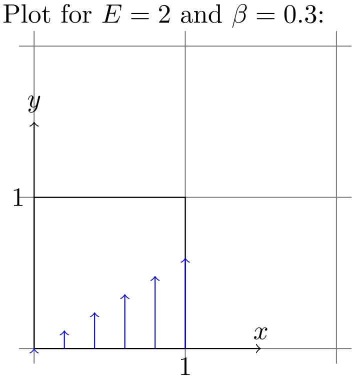


- Y=0, $\vec{n} = \{0, -1\}$:

$$\sigma = E \begin{bmatrix} 2X & 0 \\ 0 & -\beta X \end{bmatrix}$$

$$\vec{t} = E \begin{bmatrix} 0 \\ +\beta X \end{bmatrix}$$

Plot for $E = 2$ and $\beta = 0.3$:

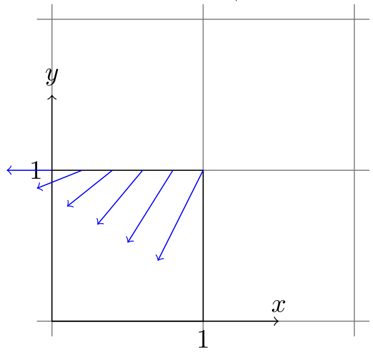


- Y=1, $\vec{n} = \{0, 1\}$:

$$\sigma = E \begin{bmatrix} 2X & -\beta/2 \\ -\beta/2 & -\beta X \end{bmatrix}$$

$$\vec{t} = E \begin{bmatrix} -\beta/2 \\ -\beta X \end{bmatrix}$$

Plot for $E = 2$ and $\beta = 0.3$:



(e) Verify that the sum of the tractions on the boundary of Ω is different from zero.

Solution

- X=0, $\vec{n} = \{-1, 0\}$:

$$\vec{t} = E \begin{bmatrix} 0 \\ \beta Y/2 \end{bmatrix}$$

$$\int_{Y=0}^{Y=1} \vec{t} dY = E \begin{bmatrix} 0 \\ \beta/4 \end{bmatrix}$$

- X=1, $\vec{n} = \{1, 0\}$:

$$\vec{t} = E \begin{bmatrix} 2 \\ -\beta Y/2 \end{bmatrix}$$

$$\int_{Y=0}^{Y=1} \vec{t} dY = E \begin{bmatrix} 2 \\ -\beta/4 \end{bmatrix}$$

- Y=0, $\vec{n} = \{0, -1\}$:

$$\vec{t} = E \begin{bmatrix} 0 \\ +\beta X \end{bmatrix}$$

$$\int_{X=0}^{X=1} \vec{t} dX = E \begin{bmatrix} 0 \\ \beta/2 \end{bmatrix}$$

- Y=1, $\vec{n} = \{0, 1\}$:

$$\vec{t} = E \begin{bmatrix} -\beta/2 \\ -\beta X \end{bmatrix}$$

$$\int_{X=0}^{X=1} \vec{t} dX = E \begin{bmatrix} -\beta/2 \\ -\beta/2 \end{bmatrix}$$

$$\Sigma = E \begin{bmatrix} 2 - \beta/2 \\ 0 \end{bmatrix}$$

(f) If we know that the body is in static equilibrium, why is the sum of the tensions on the boundary of Ω not zero?

Solution There are two ways to achieve this result:

- The first one is due to errors produced by assuming small strains. As seen in the deformed shape figure, the strains in the X axis are not small.
- Using the balance of linier momentum equation:

$$\int \rho \frac{dv}{dt} dV = \int \rho b dV + \int \nabla \sigma dV$$

The body can be in static equilibrium if the volume loads are minus the surface loads.

(3) A specimen with an undeformed configuration given by the square domain $[1,1]^2$ and made of an isotropic elastic material with young modulus E and Poisson ratio v is being compressed according to the following boundary conditions:

- y = -1: u = 0
- x = 1: t = 0
- y = 1: u = $\{0, -\vec{u}\}$
- x = -1: t = 0

with $\vec{u}$ a small positive constant. The initial and deformed configurations are represented in Figure 1. Assuming plane strain and small deformations, answer the following questions:

(a) If $E > 0$ and $0 < v < 0.5$, indicate which components of the small strain tensor ϵ and the stress tensor σ at points A = (0, 0), B = (1, 0) and C = (1, 1) are zero and why.

Solution

$$\lambda = \frac{Ev}{(1+v)(1-2v)} > 0$$

$$\mu = \frac{E}{2(1+v)} > 0$$

- A(0,0): The axis angle is not deforming:

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{yy} \end{bmatrix}$$

Using Cauchy stress definition:

$$\sigma = \lambda tr(\epsilon) I + 2\mu\epsilon$$

$$\sigma = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{bmatrix}$$

- B(1,0): Due to boundary conditions:

$$\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{yy} \end{bmatrix}$$

Using Lamé constitutive equation:

$$\epsilon = -\frac{v}{E} tr(\epsilon) I + \frac{1+v}{E}\sigma$$

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{yy} \end{bmatrix}$$

- C(1,1): Due to boundary conditions:

$$\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon_{yy} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{yy} \end{bmatrix}$$

(b) If $E > 0$ and $v = 0$, which components of the small strain tensor ϵ and stress tensor σ at points A, B and C are zero? Whenever it is possible, predict also the sign of the non-zero components.

Solution

$$\lambda = \frac{Ev}{(1+v)(1-2v)} = 0$$

$$\mu = \frac{E}{2(1+v)} = \frac{E}{2}$$

- A(0,0): The axis angle is not deforming:

$$\epsilon = \begin{bmatrix} +\epsilon_{xx} & 0 \\ 0 & -\epsilon_{yy} \end{bmatrix}$$

  Using Cauchy stress definition:

$$\sigma = 2\mu\epsilon$$

$$\sigma = \begin{bmatrix} +\sigma_{xx} & 0 \\ 0 & -\sigma_{yy} \end{bmatrix}$$

- B(1,0): Due to boundary conditions:

$$\sigma = \begin{bmatrix} 0 & 0 \\ 0 & -\sigma_{yy} \end{bmatrix}$$

  Using Lamé constitutive equation:

$$\epsilon = \frac{1}{E}\sigma$$

$$\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & -\epsilon_{yy} \end{bmatrix}$$

- C(1,1): Due to boundary conditions:

$$\epsilon = \begin{bmatrix} 0 & 0 \\ 0 & -\epsilon_{yy} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 0 & 0 \\ 0 & -\sigma_{yy} \end{bmatrix}$$

Total: 9/10

[redacted]

Assignment 2: Dynamics

3.5/4

(1) Two bar system

(a) Uncoupled ODE

$$M \cdot \ddot{u} + K \cdot u = 0$$

$$u(t) = \Sigma \phi_i(t) \cdot a_i$$

$$\Sigma(M \cdot \ddot{\phi}_i(t) \cdot a_i + K \cdot \phi_i(t) \cdot a_i) = 0$$

Pre-multiplying by a_j and using the orthogonalities yields the following uncoupled system of equations:

$$\ddot{\phi} + \omega_i^2 \phi_i = 0$$

With ω_i^2 the eigenvalues of $M^{-1} \cdot K$ and $\vec{a}_i$ the normalized eigenvectors.

(b) Initial conditions

The initial conditions are:

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{u}(0) = \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$$

Which must be equal to:

$$\phi_i(0) = a_i \cdot M \cdot u(0)$$

$$\dot{\phi}_i(0) = a_i \cdot M \cdot \dot{u}(0)$$

As $\phi_i(t) = C1_i \cdot sin(\omega_i \cdot t) + C2_i \cdot cos(\omega_i \cdot t)$

$$\phi_i(0) = C2_i = a_i \cdot M \cdot u(0)$$

$$\dot{\phi}_i(0) = C1_i \cdot \omega_i = a_i \cdot M \cdot \dot{u}(0)$$

An as seen in the figures 1b and 1c

$$a_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Direction ok, modulus may not be necessarily 1.

$$a_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and

$$M = \begin{bmatrix} m \end{bmatrix}$$

Diagonal, not scalar.

So $C2_i = 0$, $C1_1 = 0$ and $C1_2 = a_2 \frac{M \cdot v_0}{\omega_2}$,

(c) $u_B(t)$

$$u_B(t) = \Sigma \phi_i(t) \cdot a_i$$

As $\phi_1(t) = 0$, and $a_2 \cdot M \cdot a_2 = \delta_{22}$ According to a_2 given, should be m, not \delta_{22}.

$$u_B(t) = \phi_2(t) \cdot a_2 = a_2 \frac{M \cdot v_0}{\omega_2} \cdot a_2 \cdot sin(\omega_2 \cdot t) = \delta_{22} \frac{v_0}{\omega_2} \cdot sin(\omega_2 \cdot t) = \begin{bmatrix} \frac{v_0}{\omega_2} \cdot sin(\omega_2 \cdot t) \\ 0 \end{bmatrix}$$

The frequency of oscillation will be $\frac{\omega_2}{2\pi}$. OK

1.5/2

(2) Square $[0, 1][0, 1]$

(a) Number of eigen-frequencies.
The number of eigen-frequencies is at most the number of degrees of freedom of the system. There is 1 node with 0 degrees of freedom, 6 with 1 degree of freedom and 9 with 2 degrees of freedom. This makes a total of 24 DOF and therefore 24 eigen-frequencies. OK

(b) Double Young Module E, Double density ρ

$$M^{-1}K\vec{a} = \omega^2\vec{a}$$
$$\frac{1}{|M|}|K| = |\omega^2|$$

Determinant at both sides? Also on vector?

If we double the density ρ we will be dividing by two the determinant of M so $\omega' = \omega\sqrt{\frac{1}{2}}$. çdeterminant of M^{-1}
If we Young Module E we will be multiplying by two the determinant of K so $\omega' = \omega\sqrt{2}$.

OK

(3) Rod

4/4

Firstly we need to compute $M^{-1}K$:

$$M^{-1}K = \frac{24 \cdot G}{7 \cdot L^2 \cdot \rho}\begin{bmatrix} 5 & -6 \\ -3 & 5 \end{bmatrix}$$

Then we compute the eigenvalues of

$$eig(\begin{bmatrix} 5 & -6 \\ -3 & 5 \end{bmatrix}) = 5 \pm 3\sqrt{2}$$

The smallest one is $5 - 3\sqrt{2}$. So the smallest eigenvalue ω^2 of $M^{-1}K$ is $\frac{24 \cdot G}{7 \cdot L^2 \cdot \rho} \cdot (5 - 3\sqrt{2})$
We then compute the frequency $f = \frac{\omega}{2\pi}$

$$f = \frac{\sqrt{\frac{24 \cdot G}{7 \cdot L^2 \cdot \rho} \cdot (5 - 3\sqrt{2})}}{2\pi}$$

Substituting the L, ρ and G for both materials we get:

$$f_{steel} \approx 828Hz$$
$$f_{copper} \approx 598Hz$$

So between steel and copper we would choose steel because it's closer to the given frequency. OK

Total: 8

[redacted]

Assignment 3: Plasticity

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(1) Imagine that a small steel sphere is dropped under water at depth H. If it plastifies according to a Von Mises criteria, do you think it may plastify for a given value of H? Justify your answer

Solution: The Von Mises plasticity yield function is defined as:

$$f(\sigma) := \sqrt{\frac{3}{2}} \cdot ||\sigma'|| - \sigma_y = \sqrt{3 \cdot J_2} - \sigma_y$$

And J_2 will always be 0 because is defined as the angular distance to the spherical (hydrostatic) stress state ($\sigma_1 = \sigma_2 = \sigma_3$)

So the yield function will never become 0 and therefore it will never plastify.

OK. 3.0

(2) A cubic domain...

(a) Compute the displacement ...

$$\epsilon = -\frac{v}{E} tr(\sigma) I + \frac{1+v}{E}\sigma$$

In both cases we have 3 unknowns and 3 equations:

case (A)

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\epsilon_{xx} = -\frac{v}{E} \cdot t + \frac{1+v}{E} \cdot t = \frac{t}{E} = u_x$$

case (B)

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} t & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

Due tue symmetry:

$$\sigma_{yy} = \sigma_{zz}$$

Imposing that the y displacement is 0:

$$\epsilon_{yy} = 0 = -\frac{v}{E} \cdot (t + 2\sigma_{yy}) + \frac{1+v}{E} \cdot \sigma_{yy}$$

Simplifying we obtain:

$$\sigma_{yy} = \frac{v \cdot t}{1-v}$$

Substituting in the constitutive equations:

$$\epsilon_{xx} = -\frac{v}{E} \cdot (t + 2\frac{v \cdot t}{1-v}) + \frac{1+v}{E} \cdot t$$

$$\epsilon_{xx} = -\frac{t}{E} \cdot (\frac{-2 \cdot v^2}{1-v} + 1) = u_x$$

OK, but which one is larger? 0.5

(b) For a given material parameter ...

$$f(\sigma) := \sqrt{\frac{3}{2}} \cdot ||\sigma'|| - \sigma_y = \sqrt{3 \cdot J_2} - \sigma_y$$

$$J_2 := \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

case (A)

$$J_{2A} := \frac{t^2}{3}$$

case (B)

$$J_{2B} := \frac{1}{3}(t - \frac{v \cdot t}{1 - v})^2 = \frac{1}{3}(t \cdot (1 - \frac{v}{1 - v}))^2 = \frac{t^2}{3} \cdot (1 - \frac{v}{1 - v})^2$$

The bigger J_2 for the same t will plastify before:

(i) If $v > 0$ then $J_{2A} > J_{2B}$ so A plastify before

(ii) If $v = 0$ then $J_{2A} = J_{2B}$ so they plastify at the same time.

(iii) If $v < 0$ then $J_{2A} < J_{2B}$ so B plastify before

OK. 1.0

(c) In which case the loading parameter t will reach a larger magnitude?

The smaller J_2 for a given t will reach a larger t magnitude before plastifying:

(i) If $v > 0$ then $J_{2A} > J_{2B}$ so B will reach a larger t ma[illegible]

(ii) If $v = 0$ then $J_{2A} = J_{2B}$ so they will reach the same t[illegible]

(iii) f $v < 0$ then $J_{2A} < J_{2B}$ so A will reach a larger t mag[illegible]

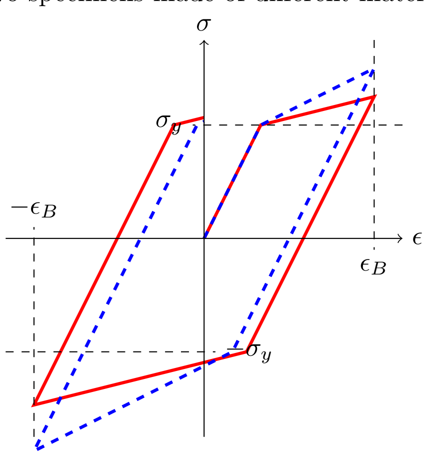
Yes, but because sigma is homogeneous. 0.5

(d) If we now use $t < 0$, is the response in question (c) going to [illegible]

Results do not change because J_2 only depend on the absolute value of t and not on the sign

OK. 1.0

(3) Two specimens made of different materials have been subjected to the strain loading cycle



Values of platification should be different from sigma_y in blue and red curves, after first plstification point. 2

I have use the following parameters in order to make the drawing:

E	2
K_{red}	0.25
K_{blue}	0.5
σ_y	2
ϵ_B	3

Process	Red	Blue
Start point	(0,0)	(0,0)
Elastic until σ_y	(1,2)	(1,2)
Plastifying until ϵ_B	(3,2.5)	(3,3)
Elastic until $-\sigma_y$	(0.75,-2)	(0.5,-2)
Plastifying until $-\epsilon_B$	(-3,-2.9375)	(-3,-3.75)
Elastic until σ_y	(-0.53125,2)	(-0.125,2)
Plastifying until 0	(0,2.1328125)	(0,2.0625)

[redacted]

Assignment 4: Fluids

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(1) We are modelling an incompressible Stokes flow, and we use a P_2^+/P_1^- interpolation for velocities and pressures. This triangle element consists on a continuous quadratic interpolation of velocities (6 nodes per element) and a discontinuous bubble pressure with 3 nodes per element.

Consider a two-dimensional square domain $[0,1]$x$[0,1]$, discretised with n divisions per side and with $2n^2$ triangles (n^2 squares with two triangles each).

(a) Compute the limit,

$$\lim_{n\to\infty} \frac{n_v}{n_p} = \lim_{n\to\infty} \frac{dim(Q^h)}{dim(P^h)}$$

$$n_p = 3 \cdot 2 \cdot n^2 = 6n^2$$

Neglecting the linear terms of the borders, each node in a vertex is shared between 6 elements and each node in a side is shared by 2 elements. We multiply by two because there are 2 degrees of freedom by node.

$$n_v \approx (\frac{3}{6} + \frac{3}{2}) * 2 \cdot 2 \cdot n^2 = 8n^2$$

$$\lim_{n\to\infty} \frac{n_v}{n_p} = \lim_{n\to\infty} \frac{8n^2}{6n^2} = \frac{4}{3}$$

(b) Does this element satisfy the necessary stability requirements regarding the dimensions of Q_h and P_h? Justify your answer.

$$\lim_{n\to\infty} \frac{dim(Q^h)}{dim(P^h)} > 1$$

So when we make the smallest mesh we can ($n \to \infty$) it's unstable:

$$dim(Q^h) > dim(P^h)$$

So we may say that it won't be stable in most of the meshes.

(2) We aim to study the flow around cylinders by analyzing the behavior of different characteristic lengths of the problem. We propose studying the two problems in the figure, considering in both cases the same value for the viscosity v. Is it necessary to solve both problems? Why?

Both flow share the same Reynolds Re number.

$$Re_1 = \frac{LV}{v} = \frac{0.6 * 10}{v} = \frac{6}{v}$$

$$Re_2 = \frac{LV}{v} = \frac{0.3 * 20}{v} = \frac{6}{v}$$

So the flow will have the same non-dimension solution.

$$v^* = \frac{v}{V}$$

We need to take in to account that the velocities of second case solution are twice the ones in the first case.

$$p^* = \frac{p_1}{\rho V^2} = \frac{p_1}{\rho \cdot 100}$$

$$p^* = \frac{p_2}{\rho V^2} = \frac{p_2}{\rho \cdot 400}$$

And the pressures of second case solution are 4 times the ones in the first case.

(3) Explain the main differences between the Stokes and Navier-Stokes equations. That is, comment the differences in the physical assumptions and in the numerical solution. Must the LBB condition be taken into account in both cases?

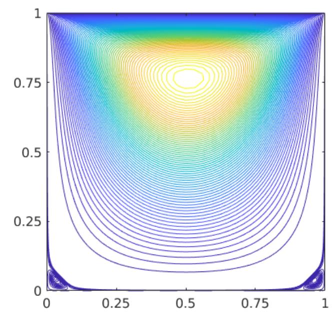
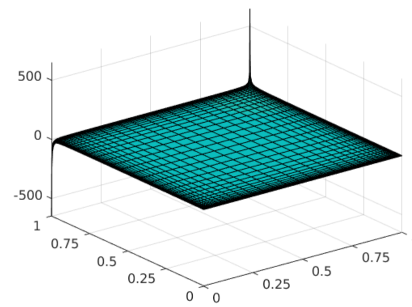
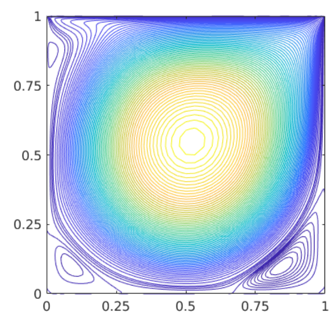
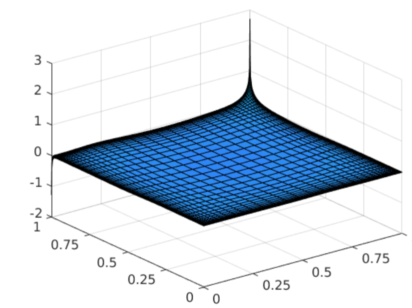
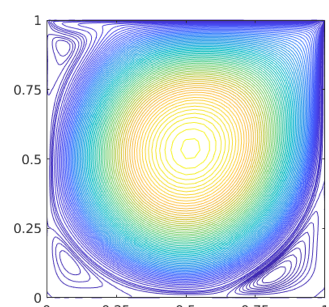
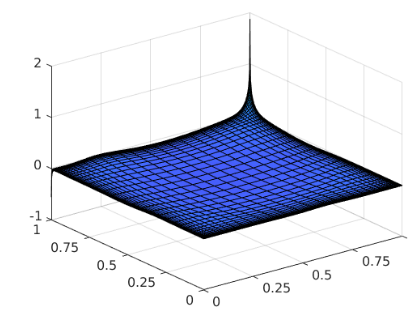
Stokes flow is a type of fluid flow where advective inertial forces are small compared with viscous forces. The Reynolds number is low. This is a typical situation in flows where the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small

Navier-Stokes adds the non-liner convective term to take in to account advective inertial forces.

The LBB condition must be betaken into account in the Stokes case. When solving Navier-Stokes we will need to use an iterative algorithm. In Navier-Stokes it's also recommended to fullfil the LBB condition.

(4) The code at http://ww2.lacan.upc.edu/huerta/exercises/Incompressible/Incompressible Ex2.htm solves the steady incompressible Navier-Stokes equations for the cavity flow problem. Show the streamlines and the pressure for Re = 1 and for Re = 2000, and comment on the result in each case. Explain how did you compute the solution in each case and how many iterations of the non-linear solver were necessary

For all cases I used element type Q2Q1, adapted mesh of 81x81.

Re	It	Streamlines	Pressures
5	3		
2000	22		
4000	59		

The maximum Reynolds we could compute with this parameters with the given script was 4837. We can see that new vortices appear when we increase the Reynolds number.

In order to reach greater Reynolds I did take an interactive approx:

We start with a Reynolds(1000) that we know we can compute and a step (1024). We compute the solution and save the velocities. Then we increase the Reynolds by the step. Then compute the solution with the previous velocities as initial guess. If the algorithm did converge we repeat the process. Otherwise we divide the step by two and we repeat until we reach a minimum step.

I was able compute a Reynolds of up to 6464 in about 30 minutes.