Assignment 1:
Continuum Mechanics and Elasticity
Computational Mechanics

[redacted]

March 2019

1. In small strains, the constitutive law can be deduced from the strain energy function $W_\epsilon = \frac{1}{2}\sigma : \epsilon$ as $\sigma = \frac{\partial W_\epsilon}{\epsilon}$.

   (a) Deduce the expression of the strain energy function W_ϵ for a linear isotropic material. Write the expression in terms of the invariants $I_1(\epsilon) = trace(\epsilon)$ and $I_2(\epsilon) = tr(\epsilon)^2$, and the Lamé parameters λ and μ.

   We have $\frac{\partial W(\epsilon)}{\partial \epsilon_{ij}} = \sigma_{ij}$ with $i, j \in \{1, 2, 3\}$.

   W is the potential energy function of a conservative system. In other words, we know that the stress tensor is derived from this scalar function: We use stress-strain equation ($\sigma_{ij} = \sum_{ij} \lambda tr(\epsilon_{ij}) + 2\mu\epsilon_{ij}$):

$$\frac{\partial W(\epsilon)}{\partial \epsilon_{ij}} = \sum_{ij}(\lambda tr(\epsilon_{ij}) + 2\mu tr(\epsilon_{ij})).$$

   Integrating both sides with respect to ϵ_{ij}:

$$W(\epsilon) = \frac{1}{2}\lambda tr(\epsilon)^2 + \mu tr(\epsilon^2) = \frac{1}{2}\lambda I_1^2 + \mu I_2.$$

   (b) St Venant's constitutive model is obtained by replacing in the previous expression of W_ϵ the invariants $I_i(\epsilon)$ by $I_i(E)$, with E the Green-Lagrange strain tensor. The resulting strain energy function W(E) allows to compute the (2nd Piola) stress tensor S as $S = \frac{\partial W_E}{\partial E}$. Compare the stresses σ and S for a displacement $u^T = \{X, 0, 0\}$, using the same Lamé parameters in both models. Are σ and S equal? Why?

   We consider $u^T = \{X, 0, 0\}$. First, we apply this displacement into σ stress:

   We know, by (a), that $W_\epsilon = \frac{1}{2}\lambda I_1^2 + \mu I_2$, and

$$\epsilon = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

   where $tr(\epsilon) = 1$ and $tr(\epsilon^2) = 1/2$.
   This implies that:

$$W_\epsilon = \frac{1}{2}(\lambda + \mu).$$

Now, we apply the displacement in S:
We have

$$F_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$C = FF^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$E = \frac{1}{2}(C - I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix},$$

where $tr(E) = -1$ and $tr(\epsilon^2) = 1/2$. This implies that:

$$W(E) = \frac{1}{2}\lambda(-1)^2 + \mu\frac{1}{2} = \frac{1}{2}(\lambda + \mu).$$

It's the same as we got before. Therefore, the stress σ and S for a displacement $u^T = \{X, 0, 0\}$ is the same in the both cases.

This is because, for small deformations, after neglecting the higher order terms, two W are equal.

(c) Do you think that St. Venant's model for large strains is it invariant under a rigid body rotation R? Answer this question by applying a rigid body rotation to a displacement field u, and checking whether W_E and S vary under R.

We apply rigid rotation bodies and calculate W. For example $F = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$C = FF^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$E = \frac{1}{2}(C - I) = 0 \Longrightarrow W_E = 0.$$

For each rigid body rotation, $W_E = 0$, because if we have a F with rigid rotation, $FF^T = I$. Therefore, Venant's model for large strains is invariant under a rigid body rotation.

2. Consider a motion defined by the displacements $u_X(X, Y)$ and $u_Y(X, Y)$, in the directions X and Y respectively, of the square domain $\Omega = [0, 1] \times [0, 1] \in \mathbb{R}^2$, with:

$$u_X = X^2 \qquad u_Y = -\beta XY.$$

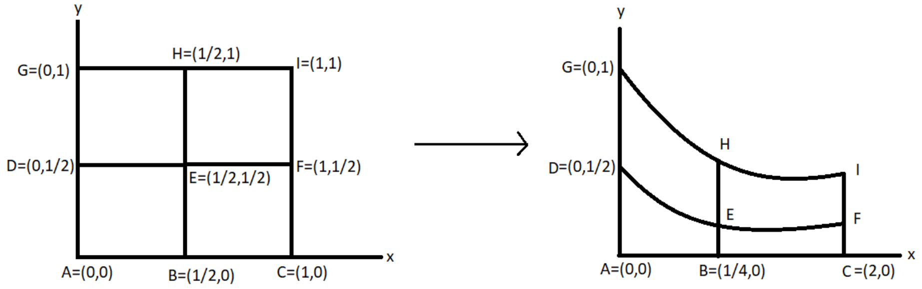
We are assuming plane strain and small deformations. The material is isotropic elastic with Young modulus $E > 0$ and $\nu = 0$. Then,

(a) Draw the deformed shape of a square domain formed by a mesh of 2×2 quadrilaterals, indicating the position of the interior nodes.

The new nodes are:

$$(x, y) = (X + u_X, Y + u_Y) = (X + X^2, Y - \beta XY).$$

We have:



And new nodes E, F, H, and I are:

$$E = \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{4}\beta\right)$$
$$H = \left(\frac{1}{2}, 1 - \frac{1}{2}\beta\right)$$
$$F = \left(1, \frac{1}{2} - \frac{1}{2}\beta\right)$$
$$I = (1, 1 - \beta)$$

(b) Write the expression of the Cauchy stress tensor as a function of X and Y . We want a relation between ϵ and X,Y.

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ 0 \\ \sigma_{xy} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\nabla \cdot f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_13} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}.$$

$$\epsilon = \begin{bmatrix} \epsilon_x & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_y \end{bmatrix} = \nabla^S \cdot u = \frac{1}{2}(\nabla u^T + \nabla u) = \frac{1}{2}\begin{bmatrix} 4x & -\beta y \\ -\beta y & -2\beta x \end{bmatrix} = \begin{bmatrix} 2x & \frac{-\beta y}{2} \\ \frac{-\beta y}{2} & -\beta x \end{bmatrix}$$

Then,
$\sigma_x = (\lambda + 2\mu)2x + \lambda(-\beta x) = x(4\mu + (2 - \beta)\lambda)$,
$\sigma_y = \lambda 2x + (\lambda + 2\mu)(-\beta x) = x(-2\beta\mu + (2 - \beta)\lambda)$,
$\sigma_{xy} = \sigma_{yx} = \mu\epsilon_{xy} = -\frac{1}{2}\beta\mu y$.
Considering $\nu = 0 \implies \lambda = 0$ and $\mu = E/2$. Therefore

$$\sigma = \begin{bmatrix} 2Ex & -\frac{1}{4}\beta Ey & 0 \\ \frac{1}{4}\beta Ey & -\beta Ex & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Evaluate the stress tensor on the four edges of the boundary of Ω.

$$\sigma(Y = 0) = \begin{bmatrix} 2EX & 0 & 0 \\ 0 & -\beta EX & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma(X = 0) = \begin{bmatrix} 0 & -\frac{1}{4}\beta EY & 0 \\ -\frac{1}{4}\beta EY & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma(Y=1) = \begin{bmatrix} 2EX & -\frac{1}{4}\beta E & 0 \\ -\frac{1}{4}\beta E & -\beta EX & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma(X=1) = \begin{bmatrix} 2E & -\frac{1}{4}\beta EY & 0 \\ -\frac{1}{4}\beta EY & -\beta E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

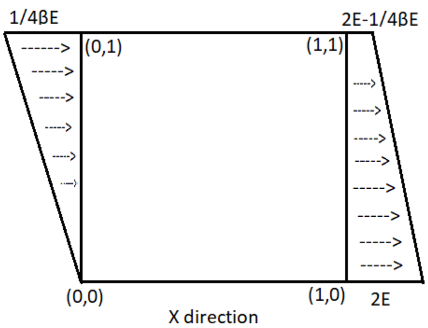
(d) Represent graphically the traction vectors on the four edges forming the boundary of Ω.

$t_1 = \sigma n_1 = \sigma(1,0,0)^T = (2EX - \frac{1}{4}\beta EY)e_1$,
$t_2 = \sigma n_2 = \sigma(0,1,0)^T = (-\beta EX - \frac{1}{4}\beta EY)e_2$.

Therefore, t_1 in:
$X = 0, Y = 0 \Longrightarrow t_1 = 0$,
$X = 1, Y = 0 \Longrightarrow t_1 = 2E$
$X = 0, Y = 1 \Longrightarrow t_1 = \frac{1}{4}\beta E$
$X = 1, Y = 1 \Longrightarrow t_1 = 2E - \frac{1}{4}\beta E$

And t_2 in
$X = 0, Y = 0 \Longrightarrow t_2 = 0$,
$X = 1, Y = 0 \Longrightarrow t_2 = -\beta E$
$X = 0, Y = 1 \Longrightarrow t_2 = -\frac{1}{4}\beta E$
$X = 1, Y = 1 \Longrightarrow t_2 = -\beta E - \frac{1}{4}\beta E$

The schema in X direction is:



And Y direction is similar.

(e) Verify that the sum of the tractions on the boundary of Ω is different from zero.

Traction in:

$$x = 0: \quad \int_0^1 \frac{1}{4}\beta EY dY = \frac{1}{8}\beta E$$

$$x = 1: \quad \int_0^1 2E - \frac{1}{4}\beta EY dY = 2E - \frac{1}{8}\beta E$$

$$y = 0: \quad \int_0^1 \beta EX dX = \frac{1}{2}\beta E$$

$$y = 1: \quad \int_0^1 -\beta EX - \frac{1}{4}\beta E dX = -\frac{3}{4}\beta E$$

The sum is $2E - \frac{1}{4}\beta E \neq 0$

(f) If we know that the body is in static equilibrium, why is the sum of the tensions on the boundary of Ω not zero?

The sum of the tensions is not zero because for the elasticity of the material. If the tensions were zero, the material would be in its initial position or, it would not remain in the final position because the material is elastic and needs a tension to stay static.

3. A specimen with an undeformed configuration given by the square domain $[1,1]^2$ and made of an isotropic elastic material with young modulus E and Poisson ratio ν is being compressed according to the following boundary conditions:

$y = 1 : u = 0$
$x = 1 : t = 0$
$y = 1 : u = \{0, \text{-}\overline{u}\}$
$x = 1 : t = 0.$

(a) If $E > 0$ and $0 < \nu < 0.5$, indicate which components of the small strain tensor ϵ and the stress tensor σ at points $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 1)$ are zero and why.

We have $E > 0$, $0 < \nu < 0.5$ and we are in a plane strain. Therefore,

$$\epsilon_{zz} = 0 = \epsilon_{xz} = \epsilon_{yz}.$$

The equations of σ are

$$\begin{aligned}
\sigma_{xx} =& \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\epsilon_{yy}, \\
\sigma_{yy} =& \frac{E\nu}{(1+\nu)(1-2\nu)}\epsilon_{xx} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{yy}, \\
\sigma_{zz} =& \nu(\sigma_{xx} + \sigma_{yy}), \\
\sigma_{xy} =& \frac{E}{1+\nu}\epsilon_{xy}, \\
\sigma_{xz} =& \epsilon_{xy} 2G = 0, \\
\sigma_{yz} =& \epsilon_{yz} 2G = 0.
\end{aligned}$$

We know that on top of the square there is a force of -u. Therefore, all points have this force down but with a different intensity. If the intensity decreases linearly, the equation of u_y, respecting each point will be the following:

$$u_y = -\frac{u}{2}y - \frac{u}{2}.$$

In this way, when you are at the top (y = 1), the force is -u and when you are down the force is 0.

Horizontally there is deformation and except in the middle of the square, we also obtain forces.
Now, we will study every point:

i. Point A = (0,0): We know that $u = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Therefore, $u_y = -u/2$ and $\epsilon_{yy} \neq 0$. Since there is no force horizontally $\epsilon_x = 0$. And $\epsilon_{xy} = 0$ because there is not diagonal force.
Let's calculate σ, $\sigma_{xx} \neq 0$ because $\epsilon_{yy} \neq 0$. The same happens with σ_{yy}. And, therefore, $\sigma_{zz} \neq 0$ because is the sum of σ_{xx} and σ_{yy}. All others are zero.
Finally, we obtain: $\epsilon_{xx} = 0, \epsilon_{yy} \neq 0, \epsilon_{xy} = 0, \sigma_{xx} \neq 0, \sigma_{yy} \neq 0, \sigma_{zz} \neq 0, \sigma_{xy} = 0, \sigma_{xz} = 0$ and $\sigma_{yz} = 0$.

ii. Point B=(1,0): In this case, we have a shift to the right and that implies that $\epsilon_{xy} \neq 0$. In other cases, everything is the same. $\epsilon_{yy} \neq 0$ and $\epsilon_{xx} = 0$. Therefore $\sigma_{xx} \neq 0, \sigma_{yy} \neq 0, \sigma_{zz} \neq 0, \sigma_{xy} \neq 0, \sigma_{xz} = 0$ and $\sigma_{yz} = 0$.

iii. Point C= (1,1): This case is equal to point A because we only have vertical strength and vertical displacement. The difference is the amount of force that is exercised on each point. At point C it will be higher.

(b) If $E > 0$ and $\nu = 0$, which components of the small strain tensor ϵ and stress tensor σ at points A, B and C are zero? Whenever it is possible, predict also the sign of the non-zero components.

In this case, since $\nu = 0$, we can know the sign of the components. We have:

$$\begin{aligned}
\sigma_{xx} =& E\epsilon_{xx} \\
\sigma_{yy} =& E\epsilon_{yy} \\
\sigma_{zz} =& 0 \\
\sigma_{xy} =& \epsilon_{xy} \\
\sigma_{xz} =& 0 \\
\sigma_{yz} =& 0.
\end{aligned}$$

Let's see what happens at each point:

i. Point A=(0,0): We know that $\epsilon_{xx} = \epsilon_{xy} = \epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$, and $\epsilon_{yy} = -u/2 < 0$. Also, $\sigma_{xx} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$ and $\sigma_{yy} = E(-u/2) < 0$.

ii. Point B=(1,0):
$\epsilon_{xx} = \epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$, and $\epsilon_{yy} = -u/2 < 0$ and $\epsilon_{xy} < 0$. Also, $\sigma_{xx} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, $\sigma_{yy} = E(-u/2) < 0$ and $\sigma_{xy} < 0$.

iii. Point C=(1,1). The same as A.

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Assignment 2: DYNAMICS

[redacted]

March 2019

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1. We aim to analyse the dynamic response of the undamped two bar system depicted in Figure 1a. Since points A and C are fixed ($u_A = u_C = 0$), the motion is fully described by the displacement of point B, and therefore there are only two degrees of freedom. The two normalised eigen modes a_1 and a_2, with eigen frequencies ω_1 and ω_2, are respectively indicated in Figures 1b and 1c.

   We aim to compute the solution $u_B(t)$ using modal analysis. The system is unloaded, initially at rest ($u(0) = 0$), and subjected to the initial velocity v_0 shown in Figure 1a.

   (a) Write the system of uncoupled ODEs that need to be solved in order to find $u_B(t)$.

   The system only has a movement u_y, therefore, u will only have this component different from zero:

$$u = \begin{bmatrix} u_{B_x} \\ u_{B_y} \end{bmatrix}.$$

   Also, we have undamped and non-loaded system. This implies that

$$M\ddot{u} + Ku = 0.$$

   And we have

$$\sum_{i=1}^{2} \ddot{\phi}_i + \omega_i^2 \phi_i = 0$$

   Therefore, the system to solve u is:

$$\left.\begin{aligned} \ddot{\phi}_1 + \omega_1^2 \phi_1 = 0 \\ \ddot{\phi}_2 + \omega_2^2 \phi_2 = 0 \end{aligned}\right\}$$

   OK

   (b) Indicate the initial conditions of the previous ODEs as a function of v_0, $a_{1,2}$, and the diagonal mass matrix M.

   Since $u_0 = \sum_i \phi_i(0) a_i$ and $\dot{u}_0 = \sum_i \dot{\phi}_i(0) a_i$. The initial conditions of each ODE in the system are given by,

$$\phi_i(0) = a_i \cdot M u_0$$
$$\dot{\phi}_i(0) = a_i \cdot M \dot{u}_0.$$

   In our case, we have $u_0 = 0$ and $\dot{u}_0 = v_0$. Therefore,

$$\phi_i(0) = 0$$
$$\dot{\phi}_i(0) = a_i \cdot M v_0.$$

   =0

   (c) Deduce the solution $u_B(t)$ as a function of v_0, the mass matrix M, $a_{1,2}$ and $\omega_{1,2}$. Which will be the frequency of the oscillations at point B?

   We have to solve the system:

$$\left.\begin{aligned} \ddot{\phi}_1 + \omega_1^2 \phi_1 = 0 \\ \ddot{\phi}_2 + \omega_2^2 \phi_2 = 0. \end{aligned}\right\}$$

The two equations are equivalent. We simplify the notation to solve the system:

$$\ddot{x} + ax = 0 \quad \Longleftrightarrow$$
$$\ddot{x} = -ax.$$

Therefore,

$$x(t) = C_1 sin(\sqrt{a}t) + C_2 cos(\sqrt{a}t).$$

And, since $a = \omega^2$,

$$\phi_i(t) = C_1 sin(\omega_i t) + C_2 cos(\omega_i t).$$

Now, the initial conditions are

$$\phi_i(0) = 0$$
$$\dot{\phi}_i(0) = a_i \cdot Mv_0.$$

OK

$$\phi_i(0) = C_1 sin(0) + C_2 cos(0) = C_2 = 0.$$

Therefore $C_2 = 0$. And $\dot{\phi}_i(t) = C_1 cos(\omega_i t)\omega_i$. This implies that

$$\dot{\phi}_i(0) = C_1 cos(0)\omega_i = C_1 \omega_i.$$

Therefore

$$C_1 = \frac{M a_i v_0}{\omega}.$$

And we get:

$$\dot{\phi}_i(t) = \frac{M a_i v_0}{\omega_i} sin(t\omega_i).$$

Finally,

$$u(t) = \sum_{i=1}^{2} \phi_i(t) a_i = \frac{a_1^T M v_0}{\omega_1} sin(t\omega_1) a_1 + \frac{a_2^T M v_0}{\omega_2} sin(t\omega_2) a_2.$$

The question: *Which will be the frequency of the oscillations at point B?*:

The velocity at point B is horizontal. This implies that the term $Ma_i v_0 = 0$. Therefore, for i=1

$$u(t) = \frac{a_2^T M v_0}{\omega_2} sin(t\omega_2) a_2.$$

From this expression we can isolate ω and replace it to the expression:

omega_1 or omega_2?

$$f = \frac{\omega}{2\pi}.$$

2/2

2. We aim to find the eigen-frequencies of a an elastic two-dimensional square $[0,1] \times [0,1]$.

   The Dirichlet boundary conditions of the problem are,

   $$u_x = 0, \quad \text{on} \quad x = 0$$

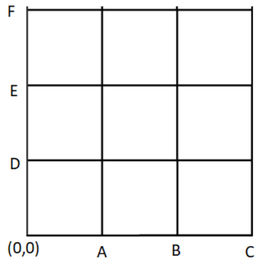
   $$u_y = 0, \quad \text{on} \quad y = 0.$$

   The domain is discretised with bilinear quadrilateral finite elements, using 3 divisions along each direction (total 9 elements).

(a) How many eigen-frequencies should be expect?

The eigenvalues of a system are eigen-frequencies of the same system. Therefore, how many eigenvalues are in the system?. There are 16 nodes and each one can have at most two degrees of freedom. The system will have a maximum of $16 \cdot 2 = 32$ eigen-values.

We have the following scheme:



where nodes A, B and C only have horizontal movement, nodes D, E and F only have vertical movement and node (0,0) has no movement. Therefore, nodes A, B, C, D, E, F have one degree of freedom and node (0,0) don't have any degree of freedom. The other points have vertical and horizontal movement (2 degrees of freedom). In the system there are the following eigen-values:

$$32 - 6 - (2 \cdot 1) = 24.$$

OK

Therefore, there are **24 eigen-frequencies**.

(b) If we double the Young modulus E, how are the eigen-frequencies going to be affected? And if we double the density ρ? Justify your answers.

If we double the Young modulus E, then G is also doubled and K too. Therefore

$$M^{-1}Ku = \lambda u \quad \text{becomes to} \quad 2M^{-1}Ku = \lambda_1 u$$

where $\lambda_1 = 2\lambda = 2\omega^2$ and $\lambda_1 = \omega_1^2$. Therefore $w_1^2 = 2w^2$ and this implies that $w_1 = \sqrt{2}w$. We have seen that the frequency increases because $\sqrt{2} > 1$. OK

If we double the density ρ, then, M is also doubled and M^{-1} becomes to $\frac{1}{2}M^{-1}$. Therefore

$$M^{-1}Ku = \lambda u \quad \text{becomes to} \quad \frac{1}{2}M^{-1}Ku = \lambda_2 u$$

where $\lambda_2 = \frac{1}{2}\lambda = \frac{1}{2}\omega^2$ and $\lambda_2 = \omega_2^2$. Therefore $w_2^2 = \frac{1}{2}w^2$ and this implies that $w_2 = \frac{1}{\sqrt{2}}w$. We have seen that the frequency decreases because $0 < \frac{1}{\sqrt{2}} < 1$. ok

3. In order to know whether a thin rod with length $L = 1m$ is made of copper or steel, we perform a modal analysis. The rod is clamped at the bottom and subjected to free oscillations, as shown in Figure 2(a). We experimentally measure the lowest vibration frequency, which is approximately $f = 810Hz$ (Remark: if is the pulse in rad/s, then $f = \omega/(2\pi)$).

3.5/42

Determine the material of the rod using the simplified model in Figure 2(b). Use two finite elements, and consider only the horizontal displacements (u_1, u_2). The global stiffness and mass matrices of the model, K and M respectively, are given by,

$$K = \frac{2GA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \qquad M = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

In order to know if it is copper or steel we have to do the following steps:

(a) Compute $M^{-1}K$ for copper and for steel.

$$M = \frac{\rho AL}{12}\begin{bmatrix}2 & 1\\ 1 & 4\end{bmatrix}$$
$$M^{-1} = \frac{12}{7\rho AL}\begin{bmatrix}4 & -1\\ -1 & 2\end{bmatrix}.$$
$$M^{-1}K = \frac{12}{7\rho AL}\frac{2GA}{L}\begin{bmatrix}4 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \frac{24G}{7\rho L^2}\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix}.$$

Now, for:

i. Steel:

M^{-1}K, not M

$$M = \frac{24G}{7\rho L^2}\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix} = \frac{24\cdot 82\times 10^9}{7\cdot 7850\cdot 1^2}\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix} = 3,54\times 10^7\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix}$$

ii. Copper:

$$M = \frac{24G}{7\rho L^2}\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix} = \frac{24\cdot 48,5\times 10^9}{7\cdot 8900\cdot 1^2}\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix} = 1,87\times 10^7\begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix}$$

(b) Find eigenvalues for each matrix.

Firts we calculate the eigen-values from $M^{-1}K = \begin{bmatrix}5 & -6\\ -3 & 5\end{bmatrix}$:

$$det\left(\begin{bmatrix}x-5 & -6\\ -3 & x-5\end{bmatrix}\right) = (x-5)^2 - 18 = 0 \Rightarrow$$
$$x_1 = 5 + 3\sqrt{2}$$
$$x_2 = 5 - 3\sqrt{2}.$$

Now, for

i. Steel:

$$x_{s_1} = 3,54\times 10^7\cdot(5+3\sqrt{2})$$
$$x_{s_2} = 3,54\times 10^7\cdot(5-3\sqrt{2}).$$

ii. Copper:

$$x_{c_1} = 1,87\times 10^7\cdot(5+3\sqrt{2})$$
$$x_{c_2} = 1,87\times 10^7\cdot(5-3\sqrt{2}).$$

(c) Check if these eigenvalues match with $\omega^2 = (2\pi f)^2$.

i. Steel

$$f_1 = \frac{\sqrt{x_{s_1}}}{2\pi} = \frac{\sqrt{3,54\times 10^7\cdot(5+3\sqrt{2})}}{2\pi} = 2878,8 = \sout{287,88}Hz$$
$$f_2 = \frac{\sqrt{x_{s_2}}}{2\pi} = \frac{\sqrt{3,54\times 10^7\cdot(5-3\sqrt{2})}}{2\pi} = \sout{8179} = 817,9Hz$$

Lower and upper are swapped

ii. Copper

$$f_1 = \frac{\sqrt{x_{c_1}}}{2\pi} = \frac{\sqrt{1,87\times 10^7\cdot(5+3\sqrt{2})}}{2\pi} = 598 = 59,8Hz$$
$$f_2 = \frac{\sqrt{x_{c_2}}}{2\pi} = \frac{\sqrt{1,87\times 10^7\cdot(5-3\sqrt{2})}}{2\pi} = 2092 = 209,2Hz$$

The closest frequency is the f_2 of the steel. Therefore, the material is **steel**.

Steel is ok, but numbers are mixed.

Total: 8

Assignment 3: PLASTICITY

[redacted]

April 2019

1. Imagine that a small steel sphere is dropped under water at depth H. If it plastifies according to a Von Mises criteria, do you think it may plastify for a given value of H? Justify your answer.

The Von Mises criterion is typically used for metals and we have the following equations:
- Yield function defined as

$$f(\sigma) = \sqrt{\frac{3}{2}}||\sigma'|| - \sigma_Y = \sqrt{3J_2} - \sigma_Y$$

- Equivalent (or von Mises) stress:

$$\sigma_{eq} = \sqrt{3J_2}$$

J are used to denote the invariants of the deviatoric tensor: There are two J: J_1 and J_2:

$$J_1 = tr(\sigma') = tr(\sigma) - 3\sigma_m$$

$$J_2 = \frac{1}{2}\sigma' : \sigma' = \frac{1}{2}I_1^2 - I_1 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2].$$

In our exercise, we have a metal sphere and the components σ_1, σ_2 and σ_3 are equal ($\sigma_1 = \sigma_2 = \sigma_3$). Because the sphere is symmetrical.

Therefore, $J_2 = 0$ and the equation $f(\sigma) = \sqrt{3J_2} - \sigma_Y$ is not fulfilled and **the ball does not plastify** in the Von Mises criteria.

ok. 3

2. A cubic domain $[0,1]^3$ made of a perfect plastic material following a Von-Misees criteria is subjected to two different sets of boundary conditions, (A) and (B).

   (a) Compute the displacement u_x at $x = 1$ in the two cases as a function of the Young modulus E and Poisson ration ν in the elastic range. For a given value of t, in which case will u_x at x $= 1$ is larger?

   Case (A):

   We have

$$\sigma = \epsilon E$$

   and

$$\sigma n = \{t, 0, 0\}^T = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}.$$

   This implies that

$$\sigma_{xx} = t,$$

$$\sigma_{xy} = 0,$$

$$\sigma_{xz} = 0.$$

Since we have $\sigma n = 0$ at $y = 1$ and $z = 1$ we have $\sigma_{yx} = \sigma_{yy} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$.

Therefore, $\sigma = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$\epsilon = \frac{-\nu}{E} Tr(\sigma) I + \frac{1+\nu}{E}\sigma = \left(\frac{-\nu}{E}\right) T + \left(\frac{1+\nu}{E}\right)\sigma = \begin{pmatrix} t/E & 0 & 0 \\ 0 & -\nu t/E & 0 \\ 0 & 0 & -\nu t/E \end{pmatrix}.$$

Now, we know that u_x is linear. This implies that:

$$u_x = (\lambda x, 0, 0).$$

Now,

$$\epsilon = \Delta u^s x$$

and

$$\epsilon_{xx} = \lambda = t/E.$$

Therefore

$$u_x = \epsilon_{xx} = t.$$

Case (B):

Now, for the conditions, we have:

$$\sigma = \begin{pmatrix} t & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

and

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can calculate σ_{yy} and σ_{zz}:

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) = 0$$
$$\epsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) = 0.$$

Therefore,

$$\sigma_{yy} = \nu(t + \sigma_{zz})$$

and

$$\sigma_{zz} = \nu(t + (\nu(t + \sigma_{zz}))) \Longrightarrow \sigma_{zz} = \frac{\nu t}{1-\nu},$$
$$\sigma_{yy} = \nu\left(t + \frac{\nu t}{1-\nu}\right) = \frac{\nu t}{1-\nu}.$$

Therefore:

$$\sigma = \begin{pmatrix} t & 0 & 0 \\ 0 & \frac{\nu t}{1-\nu} & 0 \\ 0 & 0 & \frac{\nu t}{1-\nu} \end{pmatrix}.$$

Now, we have to calcule ϵ_{xx} because $u_{xx} = \{\lambda x, 0, 0\}$.

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] = \frac{1}{E}\left(t - \nu\left(\frac{2\nu t}{1-\nu}\right)\right).$$

$$u_x = \frac{t}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right).$$

ok. 1

To know which is bigger we have to calculate when:

$$\left(1 - \frac{2\nu^2}{1-\nu}\right) = 0.$$

This happens when $\nu = -1$ or $\nu = 1/2$. Therefore, when $-1 < \nu < 1/2$ the case (B) t is larger.

(b) For a given material parameter σ_Y , in which case the material is going to plastify with a smaller value of t?

Wh know that $f(\omega) = \sqrt{3J_2} - \sigma_Y$ and

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2].$$

Case (A):

$$J_{2A} = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2] = \frac{1}{6}(t^2 + t^2) = \frac{t^2}{3}.$$

This implies that:

$$f(\sigma) = \sqrt{3(t^2/3)} - \sigma_Y = t - \sigma_Y.$$

Case (B):

$$J_{2B} = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2] = \frac{1}{6}\left(\left(t - \frac{\nu t}{1-\nu}\right)^2 + \left(t - \frac{\nu t}{1-\nu}\right)^2\right) = \frac{1}{6}\left(2\left(t - \frac{\nu t}{1-\nu}\right)^2\right).$$

This implies that

$$f(\sigma) = \sqrt{(t - \nu/(1-\nu)} - \sigma_Y = t - \frac{\nu t}{1-\nu} - \sigma_Y.$$

Therefore, the displacement is:

Case (A): $J_2 = \frac{t^2}{3}$
Case (B): $J_2 = \frac{1}{6}\left(2\left(t - \frac{\nu t}{1-\nu}\right)^2\right) = \frac{t^2}{3}\left(\frac{1-2\nu}{1-\nu}\right)^2.$

mu<0.5, but Poisson Ratio may be negative.

Therefore, if

$$-1 < \left(\frac{1-2\nu}{1-\nu}\right)^2 < 1$$

then the material (B) is going to plastify with smaller value of t. This implies that, the solutions depends on ν.

$$-1 < \left(\frac{1-2\nu}{1-\nu}\right)^2 < 1$$

if and only if

$$0 < \nu < \frac{2}{3}.$$

In the notes, we can see that $0 < \nu < 0.5$. Therefore, in case (B), J_2 is less than J_2 in case (A).

(c) In which case the loading parameter t will reach a larger magnitude?

We have in case (A)

$$\sigma_Y = t$$

simga_Y=t/3, and absolute value. Plastification is homogeneous, so that this is why we can use this condition and t cannot increase. 0.5

and in case (B)

$$\sigma_Y = t\left(1 - \frac{\nu}{1-\nu}\right).$$

The σ_Y are equal if $\nu = 0$. Therefore, in case (B), the parameter t will reach a larger magnitude.

(d) If we now use $t < 0$, is the response in question (c) going to change?

It is the same case as the previous section because t is always in the square. Therefore, the sign does not matter.

ok. 1

3. Two specimens made of different materials have been subjected to the strain loading cycle $0 \to \epsilon^B \to -\epsilon^B \to 0$. Complete in Figure 1 the approximated plot of the stressstrain curves of the complete loading path for (a) a material with hardening K and (b) a material with hardening $2K$.

(a): We have $f(\sigma) = |\sigma| - (\sigma_Y + K\alpha) = 0 \longrightarrow \alpha = \frac{|\sigma|-\sigma_Y}{K}$.

In the drawing there are four points and each one has a corresponding alpha.

In the point A, we are in the σ_Y, therefore $\sigma_A = \sigma_Y$ and $\alpha_A = \frac{|\sigma_Y|-\sigma_Y}{K} = 0$.

In the point B, we have σ_B and $\alpha_B = \frac{|\sigma_B|-\sigma_Y}{K} > \alpha_A$.

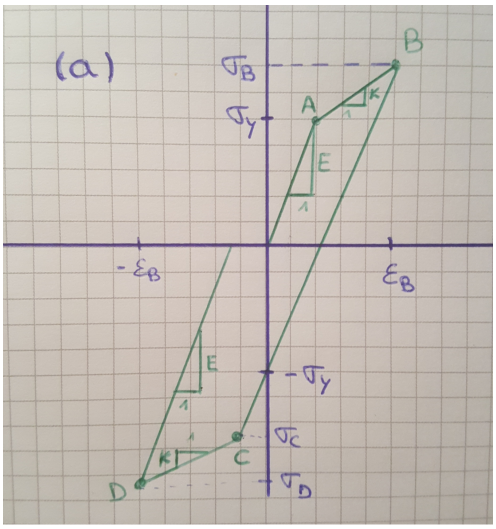
In the point C, $\alpha_B = \alpha_C$ and $\sigma_B = -\sigma_C$.

In the point D, $\alpha_D = \frac{|\sigma_D|-\sigma_Y}{K} > \alpha_C$.

In the drawing we can see the first cycle:

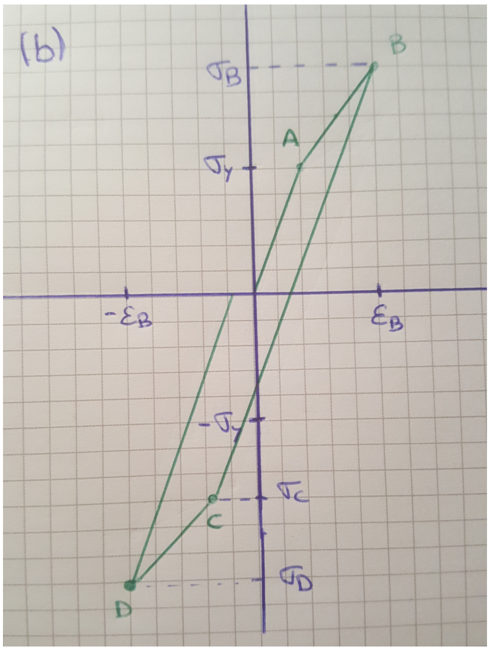
The σ_Y (σ limit) will be getting bigger and bigger. This is because the material will be increasingly elastic and less plastic. The deformation, in each cycle, we will do it up to ϵ_B and ϵ_{-B}, therefore, in the end, the material will no longer be plastic and it will be completely elastic. That is for the hardening plasticity of the material.

The schematic drawing of what will end up happening is the following:



Should start plastifying in the return at previous maximum stress: sigma_C=sigma_B.
2

(b): The scheme is the following:



In this case, it is the same as the previous case but the elasticity will become faster. We will have to do less repetitions so that the material becomes completely elastic.

Assignment 4: FLUIDS

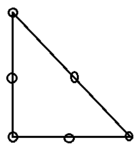
[redacted]

May 2019

1. We are modelling an incompressible Stokes flow, and we use a P_2^+/P_1 interpolation for velocities and pressures. This triangle element consists on a continuous quadratic interpolation of velocities (6 nodes per element) and a discontinuous bubble pressure with 3 nodes per element.
   Consider a two-dimensional square domain $[0,1]\times[0,1]$, discretised with n divisions per side and with $2n^2$ triangles (n^2 squares with two triangles each).

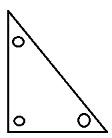
   (a) Compute the limit,
   $$\lim_{n\to\infty}\frac{n_v}{n_p} = \lim_{n\to\infty}\frac{Dim(Q^h)}{Dim(P^h)}$$
   with n_v and n_p the number of degrees of freedom for the velocity and pressure fields, respectively.

   The triangle element consists on a continuous quadratic interpolation of velocities (6 nodes per element) are triangles with form:

   and therefore,
   $$n_v = (2n+1)^2 2 = 2(4n^2+4n+1)^2.$$
   and we multiply by 2 because the velocity has two degrees of freedom.

   The discontinuous bubble pressure with 3 nodes per element are:

   and therefore,
   $$n_p = 3\cdot 2n^2\cdot 1 = 6n^2.$$
   We multiply by 1 because pressure had 1 degrees of freedom.

   Now, we can compute the limit:
   $$\lim_{n\to\infty}\frac{n_v}{n_p} = \lim_{n\to\infty}\frac{2(4n^2+4n+1)^2}{6n^2} = \lim_{n\to\infty}\frac{4n^2}{3n^2} = \frac{4}{3}.$$

(b) Does this element satisfy the necessary stability requirements regarding the dimensions of Q^h and P^h? Justify your answer.

The necessary requeriments are

$$Dim(Q^h) \leq Dim(P^h).$$

We know that

$$Q^h = 6n^2$$

and

$$P^h = 2(4n^2 + 4n + 1)^2.$$

Therefore, the condition is true because, in the limit,

$$2(4n^2 + 4n + 1)^2 \leq 12n^2.$$

2. We aim to study the flow around cylinders by analyzing the behavior of different characteristic lengths of the problem. We propose studying the two problems in the figure, considering in both cases the same value for the viscosity ν. Is it necessary to solve both problems? Why?

To study the flow through a cylinder, you must calculate the number of Reynolds. The Reynolds number is the ratio of inertial forces to viscous forces within a fluid which is subjected to relative internal movement due to different fluid velocities.

The Reynolds number is:

$$Re = \frac{VL}{\nu}$$

where V is the velocity of the fluid, L is the lenght of the body (in our case the cilynder) and ν is the viscosity. In this exercice, the viscosity ν is the same in both cases.

In the first case we have $L = 0,6m$ and $V = 10m/s$. Therefore

$$Re = \frac{VL}{\nu} = \frac{10 \cdot 0,6}{\nu} = \frac{6}{\nu}.$$

In the second case we have $L = 0,3m$ and $V = 20m/s$. Therefore

$$Re = \frac{VL}{\nu} = \frac{20 \cdot 0,3}{\nu} = \frac{6}{\nu}.$$

The Raynolds number is the same in both cases. Therefore, the flow is the same too. It is for this reason that you do not have to solve the two problems. Both will give the same result and we can say that the flow is the same without doing any other operation.

3. Explain the main differences between the Stokes and Navier-Stokes equations. That is, comment the differences in the physical assumptions and in the numerical solution. Must the LBB condition be taken into account in both cases?

STOKES EQUATION:

In the case of the Stokes equations, we have that the viscosity is much higher compared to the speed of the flow and the dimensions of the object. This implies that, the Reynolds number is low, (i.e. $Re << 1$). This is a typical situation in flows where the fluid velocities are very slow.

The equation in the solw flow is:

$$v_t + (v \cdot \nabla)v - \nu\nabla^2 v + \nabla p = f$$

$$\nabla \cdot v = 0.$$

$$\nu\nabla^2 v + \nabla p = f \quad \text{in} \quad \Omega$$
$$\nabla v = 0 \quad \text{in} \quad \Omega$$
$$v = v_D \quad \text{in} \quad \partial\Omega.$$

Velocity and pressure belong to different spaces.

LBB condition: In numerical partial differential equations, LBB condition is a necessary condition for a saddle point problem to have a unique solution.

NAVIER-STOKES EQUATIONS:

In physics, the Navier–Stokes equations describe the motion of viscous fluid substances.

Navier–Stokes equations are useful because they describe the physics of many phenomena of scientific and engineering interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing.

Weak form and discretization: Problem statement:

$$-\nu\nabla^2 v + (v \cdot \nabla)v + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla v = 0 \quad \text{in} \quad \Omega$$

$$v = v_D \quad \text{in} \quad \partial\Omega.$$

Weak form: find p, v such that $v = v_D$ on the boundary.

An iterative algorithm has to be used to solve the system of non-linear equations arising from the discretization of Navier-Stokes equations.

Iterations needed to achieve convergence ($tol = 0.5e - 08$) depending on Reynolds number.

Non-linear solver convergence: A fine enough mesh (or adapted to features) is necessary for convergence of the non-linear solver.

For high Re, the solver may not converge if the initial guess is not good enough. Usual strategy:

- Solve for low Re and use the solution as initial guess for the higher Re.

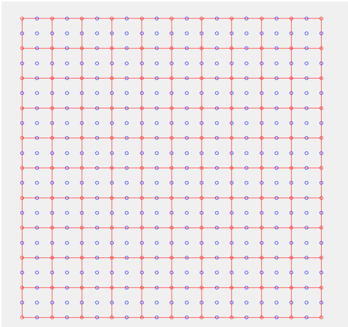
- Compute several intermediate Re if necessary.

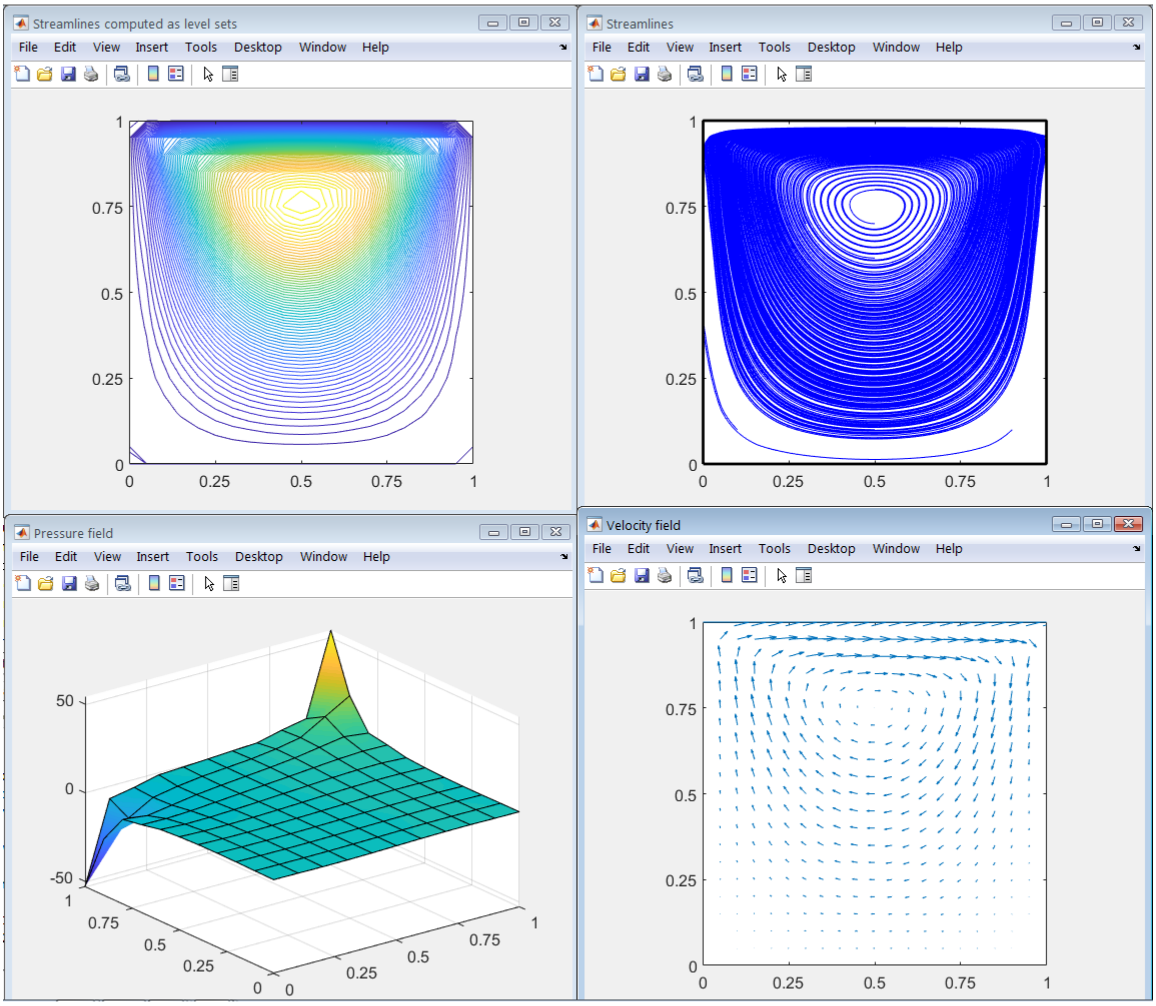
The LBB condition is only for Stokes Equations because is for the discretization the problem. The Navier–Stokes equations are not linear.

4. The code at *http://www.lacan.upc.edu/huerta/exercises/Incompressible/Incompressible Ex2.htm* solves the steady incompressible Navier-Stokes equations for the cavity flow problem. Show the streamlines and the pressure for $Re = 1$ and for $Re = 2000$, and comment on the result in each case. Explain how did you compute the solution in each case and how many iterations of the non-linear solver were necessary.

Here are the results obtained with Matlab:

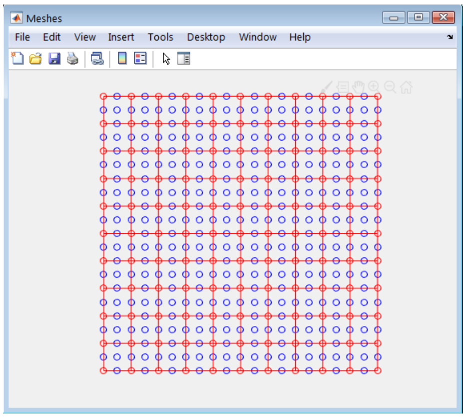
(a) Reynolds number = 1, the element for the solution : Q2,Q1, Number of nodes in X direction (default: 21)= 21

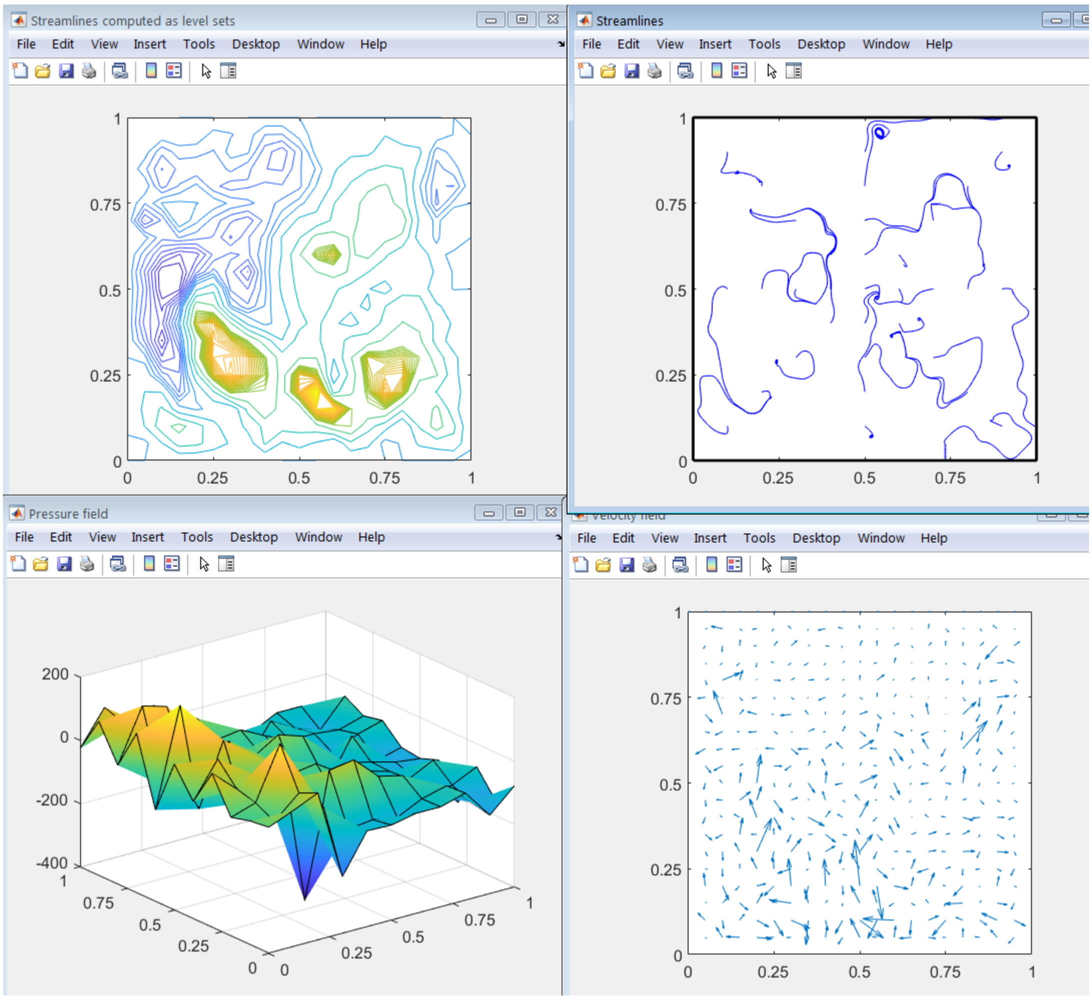




Number of iterations: 2.

(b) Reynolds number = 2000, the element for the solution : [3] Q2,Q1, Number of nodes in X direction (default: 21)= 21.

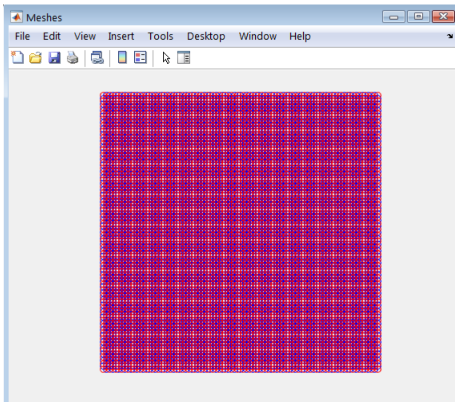


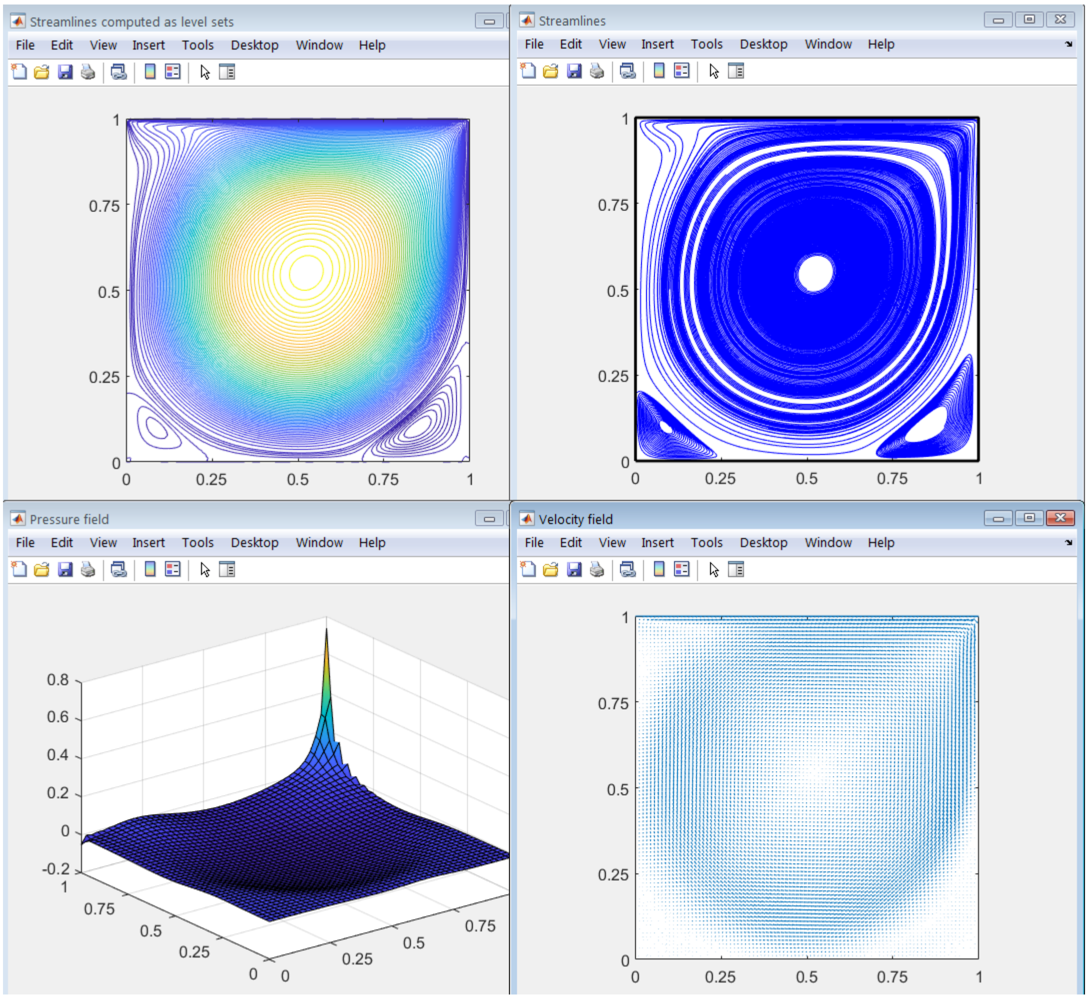


In this case, there is not converge. The merge is really small.

(c) The previous case has not converged. The mesh was too small. We tested with a mesh of 91:

Reynolds number = 2000, the element for the solution : Q2,Q1, Number of nodes in X direction (default: 21)= 91.





Number of iterations: 39

(d) Bigger Re: Testing different numbers of Reynolds, we have verified that the largest is $Re = 4000$ with a mesh of 111.

We note that in (a), the pressure is smaller than the pressure in (c). This is because the viscosity in (a) is less than in (b).