

1. In small strains, the constitutive law can be deduced from the strain energy function  $W_\varepsilon = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$  as  $\boldsymbol{\sigma} = \frac{\partial W_\varepsilon}{\partial \boldsymbol{\varepsilon}}$ .
- Deduce the expression of the strain energy function  $W_\varepsilon$  for a linear isotropic material. Write the expression in terms of the invariants  $I_1(\boldsymbol{\varepsilon}) = \text{trace}(\boldsymbol{\varepsilon})$  and  $I_2(\boldsymbol{\varepsilon}) = \text{tr}(\boldsymbol{\varepsilon}^2)$ , and the Lamé parameters  $\lambda$  and  $\mu$ .
  - St Venant's constitutive model is obtained by replacing in the previous expression of  $W_\varepsilon$  the invariants  $I_i(\boldsymbol{\varepsilon})$  by  $I_i(\mathbf{E})$ , with  $\mathbf{E}$  the Green-Lagrange strain tensor. The resulting strain energy function  $W_E$  allows to compute the (2nd Piola) stress tensor  $\mathbf{S}$  as  $\mathbf{S} = \frac{\partial W_E}{\partial \mathbf{E}}$ . Compare the stresses  $\boldsymbol{\sigma}$  and  $\mathbf{S}$  for a displacement  $\mathbf{u}^T = \{X, 0, 0\}$ , using the same Lamé parameters in both models. Are  $\boldsymbol{\sigma}$  and  $\mathbf{S}$  equal? Why?
  - Do you think that St. Venant's model for large strains is it invariant under a rigid body rotation  $\mathbf{R}$ ? Answer this question by applying a rigid body rotation to a displacement field  $\mathbf{u}$ , and checking whether  $W_E$  and  $\mathbf{S}$  vary under  $\mathbf{R}$ .

**Solution:**

- From the expression of  $\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}$  we get  $W = \frac{1}{2} \lambda I_1(\boldsymbol{\varepsilon}) + \mu I_2(\boldsymbol{\varepsilon})$ .
  - $\mathbf{S} = \lambda \text{tr}(\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}$ . For  $\mathbf{u} = \{X, 0, 0\}$ ,  $\boldsymbol{\varepsilon} = \text{diag}(1, 0)$ ,  $\mathbf{F} = \text{diag}(2, 1)$ , and thus  $\mathbf{E} = 0.5(\mathbf{F}^T \mathbf{F} - \mathbf{I}) = \text{diag}(1.5, 0)$ .  
Therefore  $\boldsymbol{\sigma}$  and  $\mathbf{S}$  are different.  $\mathbf{E}$  is non-linear, and consequently its linearisation  $\boldsymbol{\varepsilon}$  is different.
  - $\mathbf{x}' = \mathbf{R}(\mathbf{X} + \mathbf{u})$ .  $\mathbf{F}' = \mathbf{R}\mathbf{F}$ .  $\mathbf{E}' = \mathbf{E}$ , and thus  $\mathbf{S}' = \mathbf{S}$  and  $W_E$  does not change.
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2. Consider a motion defined by the displacements  $u_X(X, Y)$  and  $u_Y(X, Y)$ , in the directions  $X$  and  $Y$  respectively, of the square domain  $\Omega = [0, 1] \times [0, 1] \in \mathbb{R}^2$ , with:

$$u_X = X^2 \qquad u_Y = -\beta XY$$

We are assuming plane strain and small deformations. The material is isotropic elastic with Young modulus  $E > 0$  and  $\nu = 0$ . Then,

- Draw the deformed shape of a square domain formed by a mesh of  $2 \times 2$  quadrilaterals, indicating the position of the interior nodes.
- Write the expression of the Cauchy stress tensor  $\boldsymbol{\sigma}$  as a function of  $X$  and  $Y$ .
- Evaluate the stress tensor on the four edges of the boundary of  $\Omega$ .

- d) Represent graphically the traction vectors on the four edges forming the boundary of  $\Omega$ .
- e) Verify that the sum of the tractions on the boundary of  $\Omega$  is *different from zero*.
- f) If we know that the body is in static equilibrium, why is the sum of the tensions on the boundary of  $\Omega$  not zero?

Solution:

a)

$$\nabla_{\mathbf{x}}\mathbf{U} = \begin{bmatrix} 2X & 0 \\ -\beta Y & -\beta X \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} 2X & -\beta Y/2 \\ -\beta Y/2 & -\beta X \end{bmatrix} \Rightarrow$$

$$\boldsymbol{\sigma} = \begin{bmatrix} 4\mu X & -\beta\mu Y \\ -\beta\mu Y & -2\mu\beta X \end{bmatrix} \text{ with } \mu = E/2.$$

b) ■  $Y = 0$ :

$$\boldsymbol{\sigma} = \begin{bmatrix} 4\mu X & 0 \\ 0 & -2\mu\beta X \end{bmatrix}.$$

■  $X = 1$ :

$$\boldsymbol{\sigma} = \begin{bmatrix} 4\mu & -\beta\mu Y \\ -\beta\mu Y & -2\mu\beta \end{bmatrix}.$$

■  $Y = 1$ :

$$\boldsymbol{\sigma} = \begin{bmatrix} 4\mu X & -\beta\mu \\ -\beta\mu & -2\mu\beta X \end{bmatrix}.$$

■  $X = 0$ :

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & -\beta\mu Y \\ -\beta\mu Y & 0 \end{bmatrix}.$$

c) ■  $Y = 0$ :

$$\mathbf{n} = (0, -1) \Rightarrow \mathbf{t} = (0, 2\mu\beta X)$$

■  $X = 1$ :

$$\mathbf{n} = (1, 0) \Rightarrow \mathbf{t} = (4\mu, -\beta\mu Y)$$

■  $Y = 1$ :

$$\mathbf{n} = (0, 1) \Rightarrow \mathbf{t} = (-\beta\mu, -2\mu\beta X)$$

■  $X = 0$ :

$$\mathbf{n} = (-1, 0) \Rightarrow \mathbf{t} = (0, \beta\mu Y)$$

d)

$$\begin{aligned} \text{sum of tractions} &= \left(0, \frac{1}{2} \cdot 2\mu\beta\right) + \left(4\mu, -\frac{1}{2}\beta\mu\right) + \left(-\beta\mu, \frac{1}{2}(-2\mu\beta)\right) + \left(0, \frac{1}{2}\beta\mu\right) \\ &= (\mu(4 - \beta), 0) \neq 0 \end{aligned}$$

e) Body force.

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3. A specimen with an undeformed configuration given by the square domain  $[-1, 1]^2$  and made of an isotropic elastic material with young modulus  $E$  and Poisson ratio  $\nu$  is being compressed according to the following boundary conditions:

- $y = -1: \mathbf{u} = \mathbf{0}$
- $x = 1: \mathbf{t} = \mathbf{0}$
- $y = 1: \mathbf{u} = \{0, -\bar{u}\}$
- $x = -1: \mathbf{t} = \mathbf{0}$

with  $\bar{u}$  a small positive constant. The initial and deformed configurations are represented in Figure 1. Assuming plane strain and small deformations, answer the following questions:

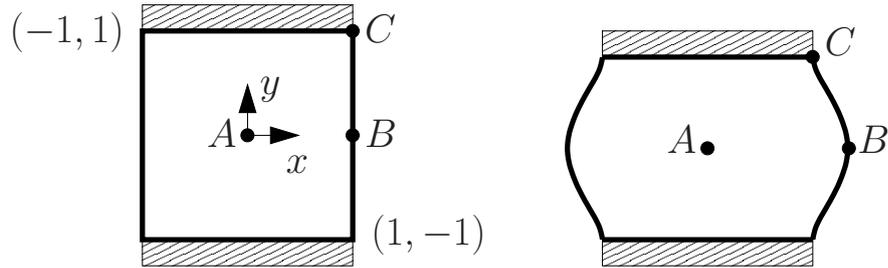


Figure 1: Initial and deformed configuration of the compressed square.

- a) If  $E > 0$  and  $0 < \nu < 0.5$ , indicate which components of the small strain tensor  $\boldsymbol{\varepsilon}$  and the stress tensor  $\boldsymbol{\sigma}$  at points  $A = (0, 0)$ ,  $B = (1, 0)$  and  $C = (1, 1)$  are zero and why.
- b) If  $E > 0$  and  $\nu = 0$ , which components of the small strain tensor  $\boldsymbol{\varepsilon}$  and stress tensor  $\boldsymbol{\sigma}$  at points  $A$ ,  $B$  and  $C$  are zero? Whenever it is possible, predict also the sign of the non-zero components.

Solution: Symmetry means  $u_x^+(x) = -u_x^-(-x)$ ,  $u_y^+ = u_y^-$ ,  $t_x^+(x) = -t_x^-(-x)$ ,  $t_y^+ = t_y^-$ :

- a)
- Point A:  $\sigma_{xy} = 0$ ,  $\varepsilon_{xy} = 0$  (symmetry). ( $\varepsilon_{xx}$  may be  $> 0$ )
  - Point B:  $\sigma_{xx} = \sigma_{xy} = 0$  (free surface),  $\varepsilon_{xy} = 0$  (Hooke's law).  $\varepsilon_y = (\nu - 1)\varepsilon_x/\nu$ ,  $\sigma_y = E\varepsilon_x/(1 + \nu)\nu$ .
  - Point C:  $\sigma_{xx} = \sigma_{xy} = 0$  (free surface),  $\varepsilon_{xx} = 0$  (boundary condition),  $\varepsilon_{xy} = \varepsilon_{yy} = 0$  (Hooke's law),  $\sigma_{yy} = 0$  (Hooke's law).
- b) From the exact solution  $u_x = 0$ ,  $u_y = -\bar{u}(y + 1)/2$ ,  $\varepsilon_{xx} = \varepsilon_{xy} = \sigma_{xx} = \sigma_{xy} = 0$  at points A, B, and C.  
 $\varepsilon_{yy} = -\bar{u}/2 < 0$ ,  $\sigma_{yy} = -\mu\bar{u} < 0$ .

1. We aim to analyse the dynamic response of the undamped two bar system depicted in Figure 1a. Since points  $A$  and  $C$  are fixed ( $\mathbf{u}_A = \mathbf{u}_C = \mathbf{0}$ ), the motion is fully described by the displacement of point  $B$ , and therefore there are only two degrees of freedom. The two normalised eigen modes  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , with eigen frequencies  $\omega_1$  and  $\omega_2$ , are respectively indicated in Figures 1b and 1c.

We aim to compute the solution  $\mathbf{u}_B(t)$  using modal analysis. The system is unloaded, initially at rest ( $\mathbf{u}(0) = \mathbf{0}$ ), and subjected to the initial velocity  $\mathbf{v}_0$  shown in Figure 1a.

- Write the system of uncoupled ODEs that need to be solved in order to find  $\mathbf{u}_B(t)$ .
- Indicate the initial conditions of the previous ODEs as a function of  $\mathbf{v}_0$ ,  $\mathbf{a}_{1,2}$ , and the *diagonal mass matrix*  $\mathbf{M}$ .
- Deduce the solution  $\mathbf{u}_B(t)$  as a function of  $\mathbf{v}_0$ , the mass matrix  $\mathbf{M}$ ,  $\mathbf{a}_{1,2}$  and  $\omega_{1,2}$ . Which will be the frequency of the oscillations at point  $B$ ?

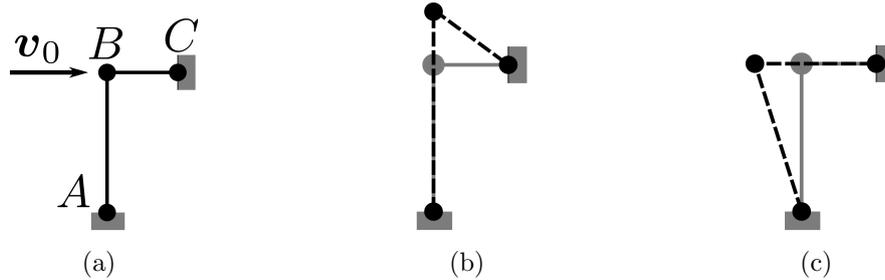


Figure 1: (a) 2 bar system in Problem 1. (b) Eigen mode  $\mathbf{a}_1$ . (c) Eigen mode  $\mathbf{a}_2$ .

**Solution:**

- (a) Since we have no loads nor damping,

$$\begin{aligned} \ddot{\phi}_1 + \omega_1^2 \phi_1 &= 0 \\ \ddot{\phi}_2 + \omega_2^2 \phi_2 &= 0 \end{aligned}$$

- (b) Since  $\mathbf{u}(0) = \mathbf{0}$  and  $\mathbf{v}(0)^T \mathbf{M} \mathbf{a}_1 = 0$ , the structure will satisfy the following two ICs:

$$\begin{aligned} ODE1 : \phi_1(0) = \dot{\phi}_1(0) &= 0 \\ ODE2 : \phi_2(0) = 0, \dot{\phi}_2(0) &= \mathbf{v}_0^T \mathbf{M} \mathbf{a}_2 \end{aligned}$$

- (c) Solving the previous ODEs gives:

$$\begin{aligned} ODE1 : \phi_1(t) &= 0 \\ ODE2 : \phi_2(t) &= \mathbf{v}_0^T \mathbf{M} \mathbf{a}_2 \sin(\omega_2 t) / \omega_2 \end{aligned}$$

and  $\mathbf{u}_B(t) = \phi_2(t) \mathbf{a}_2$ . Thus, point  $B$  will oscillate with frequency  $\omega_2$ .

2. We aim to find the eigen-frequencies of a an elastic two-dimensional square  $[0, 1] \times [0, 1]$ . The Dirichlet boundary conditions of the problem are,

$$\begin{aligned} \bullet u_x &= 0, \text{ on } x = 0 \\ \bullet u_y &= 0, \text{ on } y = 0 \end{aligned}$$

The domain is discretised with bilinear quadrilateral finite elements, using 3 divisions along each direction (total 9 elements).

- (a) How many eigen-frequencies should be expect?  
 (b) If we double the Young modulus  $E$ , how are the eigen-frequencies going to be affected? And if we double the density  $\rho$ ? Justify your answers.

**Solution:**

- (a)  $16 \cdot 2 - 4 - 4 = 24$  degrees of freedom = number of eigen-freq.  
 (b) Eigen frequencies are the solution of the generalised eigen problem  $\omega^2 \mathbf{M} \mathbf{a} = \mathbf{K} \mathbf{a}$ . If  $E$  is doubled (and also  $\mathbf{K}$ ), then we have the same eigen-value problem but with  $\omega_{new}/\sqrt{2} = \omega_{orig}$ , that is the new frequency is multiplied by  $\sqrt{2}$ . If  $\rho$  is double instead, then  $\omega_{new} = \omega_{orig}/\sqrt{2}$ .
3. In order to know whether a thin rod with length  $L = 1$  m is made of copper or steel, we perform a modal analysis. The rod is clamped at the bottom and subjected to free oscillations, as shown in Figure 2(a). We experimentally measure the lowest vibration frequency, which is approximately  $f = 810$  Hz (Remark: if  $\omega$  is the pulse in rad/s, then  $f = \omega/(2\pi)$ ).



Figure 2: Clamped thin rod (a) and simplified model with 2 elements (b).

Determine the material of the rod using the simplified model in Figure 2(b). Use two finite elements, and consider only the horizontal displacements  $(u_1, u_2)$ . The global stiffness and mass matrices of the model,  $\mathbf{K}$  and  $\mathbf{M}$  respectively, are given by,

$$\mathbf{K} = \frac{2GA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{M} = \frac{\rho AL}{12} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix},$$

where  $A$ ,  $\rho$  and  $G$  are the cross-section area, the density and the shear modulus, respectively. The values of  $\rho$  and  $G$  for the steel and copper are given in Table 1.

	$\rho[\text{Kg} \cdot \text{m}^{-3}]$	$G[\text{N} \cdot \text{m}^{-2}]$
Steel	7850	$82.0 \times 10^9$
Copper	8900	$48.5 \times 10^9$

Table 1: Density  $\rho$  and shear modulus for copper and steel.

**Solution:**

By solving the generalized eigenvalue problem defined by  $\mathbf{K}$  and  $\mathbf{M}$ , we get the lowest frequencies as

$$\begin{aligned}\omega_{\text{steel}} &= 5209 \text{ rad/s}, & f_{\text{steel}} &= \omega_{\text{steel}}/(2\pi) = 829\text{Hz} \\ \omega_{\text{copper}} &= 3734 \text{ rad/s}, & f_{\text{copper}} &= 599\text{Hz}.\end{aligned}$$

So the material is steel.

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4. Show that in the HHT- $\alpha$  algorithm with  $\gamma = 1/2$  and  $\beta = 1/4$ , and no external forces, the total energy  $T = \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} + \frac{1}{2}\mathbf{v}^T \mathbf{M} \mathbf{v}$  decreases if  $\alpha < 0$ , that is,  $T_{n+1} < T_n$ . Assume that,

$$(\mathbf{u}_{n+1} - \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_n - \mathbf{u}_{n-1}) \approx (\mathbf{u}_{n+1} - \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_n).$$

**Solution:**

$$\mathbf{M} \mathbf{a}_{n+1} + \mathbf{K} \mathbf{u}_{n+1-\alpha} = \mathbf{M} \mathbf{a}_{n+1} + \mathbf{K} \mathbf{u}_{n+1} - \alpha \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_n) = \mathbf{0}$$

$$\mathbf{M} \mathbf{a}_n + \mathbf{K} \mathbf{u}_{n-\alpha} = \mathbf{M} \mathbf{a}_n + \mathbf{K} \mathbf{u}_n - \alpha \mathbf{K} (\mathbf{u}_n - \mathbf{u}_{n-1}) = \mathbf{0}$$

$$\mathbf{M} \mathbf{a}_{n+1/2} + \mathbf{K} \mathbf{u}_{n+1/2} = \frac{\alpha}{2} \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_{n-1})$$

$$\Delta T = \Delta t \mathbf{v}_{n+1/2}^T (\mathbf{M} \mathbf{a}_{n+1/2} + \mathbf{K} \mathbf{u}_{n+1/2}) = \frac{\Delta t \alpha}{2} \frac{\mathbf{u}_{n+1}^T - \mathbf{u}_n^T}{\Delta t} \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_{n-1}) \quad (1)$$

$$= \frac{\alpha}{2} (\mathbf{u}_{n+1} - \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_n) + \frac{\alpha}{2} (\mathbf{u}_{n+1} - \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_n - \mathbf{u}_{n-1}) \quad (2)$$

$$\approx \alpha (\mathbf{u}_{n+1} - \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_n) \quad (3)$$

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5. We consider the dynamics of a solid in three-dimensions (3D). We discretize it into a mesh with 1000 eight-node hexahedral finite elements and 1500 nodes. We impose Dirichlet boundary conditions such that at 123 nodes the displacement vector is zero. How many eigenmodes does this discretized system have?

**Solution:**

$$3 \times (1500 - 123) = 4131$$


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6. The elastic energy  $U_n$  and kinetic energy  $K_n$  at time  $t_n$  of a discretised linear elastic solid are respectively given by:

$$U_n = \frac{1}{2} \mathbf{u}_n^T \mathbf{K} \mathbf{u}_n$$

$$K_n = \frac{1}{2} \mathbf{v}_n^T \mathbf{M} \mathbf{v}_n$$

with  $\mathbf{K}$  and  $\mathbf{M}$  the stiffness and mass matrices of the solid. Show that if the solid has no external forces, and the equations of motion are solved using the Newmark scheme with  $\gamma = 1/2$  and  $\beta = 1/4$ , the total energy  $T_n = U_n + K_n$  is preserved, that is  $T_n = T_{n+1}$ .

HINT: you can demonstrate the energy preservation by following the next steps:

- (i) Rewrite the Newmark scheme by expressing  $\Delta \mathbf{v}$  as a function of  $\mathbf{a}_{n+1/2}$  and  $\Delta \mathbf{u}$  as a function of  $\mathbf{v}_{n+1/2}$ . The following notation has been used:  $\Delta \mathbf{x} = \mathbf{x}_{n+1} - \mathbf{x}_n$ , and  $\mathbf{x}_{n+1/2} = \frac{1}{2}(\mathbf{x}_n + \mathbf{x}_{n+1})$ .
- (ii) From the discretised equilibrium at  $t_n$  and  $t_{n+1}$ , deduce the equilibrium equations at time  $t_{n+1/2}$ .
- (iii) By exploiting the symmetry of the matrices  $\mathbf{K}$  and  $\mathbf{M}$ , express the increment of energy  $\Delta T = T_{n+1} - T_n$  as a function of  $\Delta \mathbf{u}$ ,  $\mathbf{u}_{n+1/2}$ ,  $\Delta \mathbf{v}$  and  $\mathbf{v}_{n+1/2}$ . From the results in (i) and (ii), show that  $\Delta T = 0$ .

**Solution:**

(i)

$$\Delta \mathbf{v} = \Delta t \mathbf{a}_{n+1/2}$$

$$\Delta \mathbf{u} = \Delta t \mathbf{v}_n + \frac{\Delta t^2}{2} \mathbf{a}_{n+1/2} = \Delta t \mathbf{v}_{n+1/2}$$

(ii)

$$\mathbf{M} \mathbf{a}_{n+1/2} + \mathbf{K} \mathbf{u}_{n+1/2} = \mathbf{0}$$

(iii)

$$\begin{aligned} \Delta T &= \frac{1}{2} (\mathbf{u}_{n+1}^T \mathbf{K} \mathbf{u}_{n+1} - \mathbf{u}_n^T \mathbf{K} \mathbf{u}_n + \mathbf{v}_{n+1}^T \mathbf{M} \mathbf{v}_{n+1} - \mathbf{v}_n^T \mathbf{M} \mathbf{v}_n) \\ &= \Delta \mathbf{u}^T \mathbf{K} \mathbf{u}_{n+1/2} + \Delta \mathbf{v}^T \mathbf{M} \mathbf{v}_{n+1/2} \\ &= \Delta t \mathbf{v}_{n+1/2}^T \mathbf{K} \mathbf{u}_{n+1/2} + \Delta t \mathbf{a}_{n+1/2}^T \mathbf{M} \mathbf{v}_{n+1/2} \\ &= \Delta t \mathbf{v}_{n+1/2}^T (\mathbf{K} \mathbf{u}_{n+1/2} + \mathbf{M} \mathbf{a}_{n+1/2}) = \Delta t \mathbf{v}_{n+1/2}^T \mathbf{0} = 0 \end{aligned}$$


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7. A clamped beam is subjected to an axial dynamic load (see Figure 3). We are interested in studying the influence of the density on the dynamic response of the beam.

- (a) Determine, by using the program `cantilever.m`, the lowest eigenfrequency and the associated eigenmode for three different beams with densities 1, 10 and 1/10.

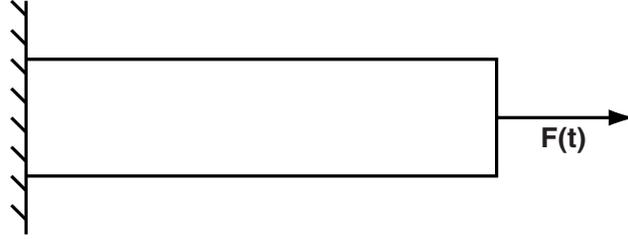


Figure 3: Clamped beam

(b) What is the relation between the frequencies? And between the modes? Why?

**Solution:**

(a)  $(\rho, f_{min}[Hz]) = (1, 75.061), (10, 23.73), (0.1, 237.36)$

(b)  $\mathbf{K}\mathbf{a} = \omega^2\mathbf{M}\mathbf{a}$ , where  $\mathbf{M}$  is proportional to  $\rho$ , i.e.  $\mathbf{M} = \rho\tilde{\mathbf{M}}$ . Therefore,  $\omega^2\rho = \text{constant}$ , that is,

$$\omega = 2\pi f \propto 1/\sqrt{\rho}$$

8. Given a structure with  $n$  degrees of freedom, we know the first eigenmode  $\mathbf{a}_1$  and the associated eigenfrequency  $\omega_1$ . The structure is subjected to the following time varying load:

$$\mathbf{f} = \mathbf{M}\mathbf{a}_1 P \sin(\bar{\omega}t)$$

where  $\mathbf{M}$  is the mass matrix,  $P$  is a constant and  $\bar{\omega}$  is the load frequency such that  $0 < \bar{\omega} < \omega_1$ . We assume that the structure has no damping, and we consider the following initial conditions:

$$\mathbf{u}_0 = \mathbf{0} \quad ; \quad \dot{\mathbf{u}}_0 = v\mathbf{a}_1$$

with  $v$  a known constant. According to this data,

- (a) Deduce the time evolution of the displacements  $\mathbf{u}$ . Hint: decompose  $\mathbf{u}$  into its eigenmodes and decouple the equation of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

into harmonic oscillators.

- (b) How does the structure behave when  $\bar{\omega} \simeq \omega_1$ ?

**Solution:**

- (a) Setting

$$\mathbf{u}(t) = \sum_{i=1}^n \phi_i(t)\mathbf{a}_i$$

and substituting into

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f},$$

we have

$$\sum_{i=1}^n \left( \ddot{\phi}_i \mathbf{M}\mathbf{a}_i + \phi_i \mathbf{K}\mathbf{a}_i \right) = \mathbf{M}\mathbf{a}_1 P \sin(\bar{\omega}t).$$

Multiplying  $\mathbf{a}_j$ ,  $1 \leq j \leq n$ , from the left we have

$$\ddot{\phi}_j + \omega_j^2 \phi_j = \delta_{j1} P \sin(\bar{\omega}t).$$

The initial conditions can be obtained in a similar fashion:

$$\phi_j(0) = 0, \quad \dot{\phi}_j(0) = v\delta_{j1}.$$

Hence for  $j \neq 1$  the solution is trivial. For  $j = 1$ , we have

$$\ddot{\phi}_1 + \omega_1^2 \phi_1 = P \sin(\bar{\omega}t), \quad (4)$$

$$\phi_1(0) = 0, \quad \dot{\phi}_1(0) = v. \quad (5)$$

We then let

$$\phi_1 = A_1 \sin(\omega_1 t - \delta_1) + \bar{A}_1 \sin(\bar{\omega}t - \bar{\delta}_1).$$

Substituting into (4) yields

$$\bar{A}_1 (\omega_1^2 - \bar{\omega}^2) \sin(\bar{\omega}t - \bar{\delta}_1) = P \sin(\bar{\omega}t), \quad \forall t.$$

Hence,

$$\bar{A}_1 = \frac{P}{\omega_1^2 - \bar{\omega}^2}, \quad \bar{\delta}_1 = 0.$$

Therefore,

$$\phi_1 = A_1 \sin(\omega_1 t - \delta_1) + \frac{P}{\omega_1^2 - \bar{\omega}^2} \sin(\bar{\omega}t).$$

From (5),

$$-A_1 \sin \delta_1 = 0, \quad \omega_1 A_1 \cos \delta_1 + \frac{\bar{\omega}P}{\omega_1^2 - \bar{\omega}^2} = v.$$

Without loss of generality, we set  $\delta_1 = 0$ , then

$$A_1 = \frac{1}{\omega_1} \left( v - \frac{\bar{\omega}P}{\omega_1^2 - \bar{\omega}^2} \right).$$

As a result,

$$\phi_1 = \frac{1}{\omega_1} \left( v - \frac{\bar{\omega}P}{\omega_1^2 - \bar{\omega}^2} \right) \sin(\omega_1 t) + \frac{P}{\omega_1^2 - \bar{\omega}^2} \sin(\bar{\omega}t),$$

and

$$\mathbf{u} = \left[ \frac{1}{\omega_1} \left( v - \frac{\bar{\omega}P}{\omega_1^2 - \bar{\omega}^2} \right) \sin(\omega_1 t) + \frac{P}{\omega_1^2 - \bar{\omega}^2} \sin(\bar{\omega}t) \right] \mathbf{a}_1$$

(b) When  $\bar{\omega} \rightarrow \omega_1$ , we have, from L'Hôspital's rule, for any fixed  $t$ ,

$$\mathbf{u} = \left[ \frac{v}{\omega_1} \sin(\omega_1 t) + P \frac{\omega_1 t \cos(\omega_1 t) - \sin(\omega_1 t)}{2\omega_1^2} \right] \mathbf{a}_1.$$

Hence, for a large  $t$ , the amplitude of the oscillation is  $Pt/(2\omega_1)\mathbf{a}_1$ , which is ever increasing.

9. We want to solve the following dynamic problem:

$$\rho \frac{d^2 \mathbf{u}}{dt^2} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x})$$

+ boundary conditions

- (a) Briefly describe the steps to follow in order to solve this problem using an eigen-analysis.
- (b) Explain in a few words the essence of direct time integration.
- (c) Mention the advantages and drawbacks of each one of the previous approaches.

10. The discretization of an undamped mechanical system leads to the following (consistent) mass and stiffness matrices:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

We wish to integrate the system in time with the central difference method.

- (a) Calculate the eigenfrequencies of the system with the matrices given. (Hint:  $\sqrt{2} \approx 1.414$ .)
- (b) Replace the mass matrix by the following diagonal mass matrix:

$$\tilde{\mathbf{M}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Calculate the eigenfrequencies of this new system.

- (c) Recall that the critical time step in either case is given by  $\Delta t_{\text{crit}} = 2/\omega_{\text{max}}$ . What is the advantage of the diagonal mass matrix compared to the consistent mass matrix for this particular example?
- (d) Name one additional advantage of the diagonal mass matrix in the case with many degrees of freedom.

**Solution:**

(a)

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 0,$$

or

$$\lambda^2 - 6\lambda + 1 = 0.$$

Hence

$$\lambda_{1,2} = 3 \pm 2\sqrt{2}, \quad \omega_{1,2} = \sqrt{3 \pm 2\sqrt{2}}.$$

$$\omega_1 = 2.4142, \quad \omega_2 = 0.4142.$$

(b) The eigenequation becomes

$$6\lambda^2 - 7\lambda + 1 = 0.$$

Hence,

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{6}.$$
$$\omega_1 = 1, \quad \omega_2 = \sqrt{\frac{1}{6}} = 0.4082$$

(c) The diagonal mass matrix allows a bigger time step.

(d) Easy to invert.

---

11. The two noded bar element with only axial displacements (one degree of freedom per node,  $u_x$ ) has the following elemental mass and stiffness matrices,

$$\mathbf{M} = \frac{LA\rho}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad \mathbf{K} = \frac{E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (6)$$

where  $L$ ,  $\rho$ ,  $A$  and  $E$  are respectively the length, density, cross-section area and Young modulus. In order to study the stability of the explicit Euler time-integration, answer the following questions:

a) Write the equilibrium ODE equations of a free single bar element,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

as a first order differential equation. Note that,

$$\mathbf{M}^{-1}\mathbf{K} = \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

with  $\alpha = \frac{6E}{\rho AL^2}$ . Which are the dimensions of the vector of unknowns?

b) The bar is now subjected to homogeneous Dirichlet condition on the left node ( $u_x = 0$ ), and homogeneous Newman condition on the right node. Rewrite the first order differential equation imposing these boundary conditions. Which are the dimensions of the vector of unknowns?

c) Apply the explicit Euler time-integration algorithm as a function of time-step size  $h$ . Write the time-stepping algorithm in the following form:

$$\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n$$

where vector  $\mathbf{y}$  contains the unknowns of the problem. Give the explicit expression of matrix  $\mathbf{A}$  as a function of  $\alpha$  and  $h$ .

d) Deduce whether, for an arbitrary non-zero initial vector  $\mathbf{y}_0$ , the explicit Euler algorithm may yield positions and velocities with increasing or decreasing amplitudes.

**Solution:**

a)

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \mathbf{y} \quad (7)$$

with  $\mathbf{y} = \{u_1, u_2, v_1, v_2\} \in \mathbb{R}^4$  (2 dof per node, position  $u$  and velocity  $v$ ).

b) Removing first and third rows and columns we get,

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \mathbf{y} \quad (8)$$

with  $\mathbf{y} = \{u_2, v_2\}$ .

c) Applying explicit Euler we get:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \mathbf{y}_n = \begin{bmatrix} 1 & h \\ -\alpha h & 1 \end{bmatrix} \mathbf{y}_n = \mathbf{A} \mathbf{y}_n$$

d) The eigenvalues  $\lambda_{1,2} \in \mathbb{C}$  of  $\mathbf{A}$  are given by

$$(1 - \lambda)^2 + \alpha h^2 = 0$$

i.e.

$$\lambda = 1 \pm i\alpha h^2$$

It follows that  $|\lambda| > 1, \forall h, \alpha > 0$ , and thus the amplitude of the response will never decrease.

---

1. (Imagine that a small steel sphere is dropped under water at depth  $H$ . If it plastifies according to a Von Mises criteria, do you think it may plastify for a given value of  $H$ ? Justify your answer.

**Solution:**

In this case the stress state is hydrostaic, which means the  $J_2 = 0$  for all values of  $H$ . Therefore it will never plastify.

2. A cubic domain  $[0, 1]^3$  made of a perfect plastic material following a Von-Misees criteria is subjected to two different sets of boundary conditions, (A) and (B), given in the next table:

	$x = 0$	$x = 1$	$y = 0$	$y = 1$	$z = 0$	$z = 1$
(A)	$u_x = 0$	$\boldsymbol{\sigma n} = \{t, 0, 0\}^T$	$u_y = 0$	$\boldsymbol{\sigma n} = \mathbf{0}$	$u_z = 0$	$\boldsymbol{\sigma n} = \mathbf{0}$
(B)	$u_x = 0$	$\boldsymbol{\sigma n} = \{t, 0, 0\}^T$	$u_y = 0$	$u_y = 0$	$u_z = 0$	$u_z = 0$

with  $t > 0$  a loading parameter.

- (a) Compute the displacement  $u_x$  at  $x = 1$  in the two cases as a function of the Young modulus  $E$  and Poisson ration  $\nu$  in the elastic range. For a given value of  $t$ , in which case will  $u_x$  at  $x = 1$  is larger? NOTE: In the two cases, the solution of the elastic problem has a constant deformation and stress field.
- (b) For a given material parameter  $\sigma_Y$ , in which case the material is going to plastify with a smaller value of  $t$ ?
- (c) In which case the loading parameter  $t$  will reach a larger magnitude?
- (d) If we now use  $t < 0$ , is the response in question (c) going to change?

**Solution:**

- (a)

$$(A) : \varepsilon_{xx}^A = t/E.$$

$$(B) : \varepsilon_{yy} = 0 \Rightarrow \sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz})$$

$$\varepsilon_{zz} = 0 \Rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\Rightarrow \sigma_{yy} = \sigma_{zz} = \frac{\nu}{1 - \nu} \sigma_{xx}$$

$$\Rightarrow \varepsilon_{xx}^B = 1/E(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) = t/E(1 - 2\frac{\nu^2}{1 - \nu}) < \varepsilon_{xx}^A$$

So  $u_x$  will be larger in (A).

(b)

$$(A) \boldsymbol{\sigma}' = t/3 \text{diag}(2, -1, -1) \Rightarrow \boldsymbol{\sigma}' : \boldsymbol{\sigma}' = 2t^2/3$$

(B) From above:

$$\Rightarrow \boldsymbol{\sigma} = t/(1-\nu) \text{diag}(1-\nu, \nu, \nu)$$

$$\Rightarrow \text{tr}(\boldsymbol{\sigma}) = \left( \frac{1+\nu}{1-\nu} \right) t$$

$$\Rightarrow \boldsymbol{\sigma}' = t/3/(1-\nu) \text{diag}(2-4\nu, 2\nu-1, 2\nu-1)$$

$$\Rightarrow \boldsymbol{\sigma}' : \boldsymbol{\sigma}' = \frac{2}{3} t^2 \left( \frac{2\nu-1}{1-\nu} \right)^2 = \frac{2}{3} t^2 \left( \frac{\nu}{1-\nu} - 1 \right)^2$$

Then case (B) will plastify first for  $\nu < 0$ , since  $\boldsymbol{\sigma}'_A : \boldsymbol{\sigma}'_A < \boldsymbol{\sigma}'_B : \boldsymbol{\sigma}'_B \forall t$  in this case, and later if  $\nu > 0$ .

(c)  $\nu < 0$ : Since case (B) plastifies first, and the stress value is homogeneous (stresses are constant for each  $t$ , so that the whole domain plastifies in the same manner), case (A) will reach larger value of  $t$ . For  $\nu > 0$ , larger values of  $t$  for case (B).

(d) No, Von-Mises is symmetric if we reverse all the stress values (only depends on  $\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$  which depends on  $t^2$ ).

3. Two specimens made of different materials have been subjected to the strain loading cycle  $0 \rightarrow \varepsilon^B \rightarrow -\varepsilon^B \rightarrow 0$ . Complete in Figure 1 the approximated plot of the stress-strain curves of the complete loading path for (a) a material with hardening  $K$  and (b) a material with hardening  $2K$ .

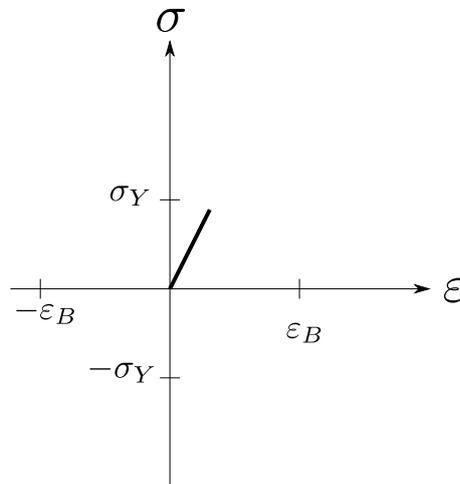


Figure 1: Stress-strain curve for a material with isotropic hardening (Problem 3)

**Solution:**

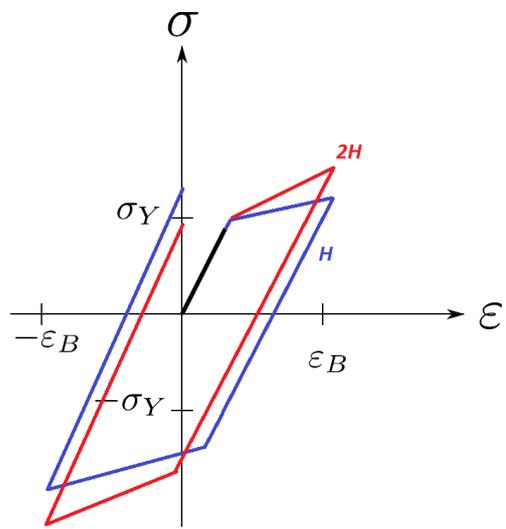


Figure 2: Stress-strain curve for a material with isotropic hardening (Solution problem 3)

1. We are modelling an incompressible Stokes flow, and we use a  $P_2^+/P_{-1}$  interpolation for velocities and pressures. This triangle element consists on a continuous quadratic interpolation of velocities (6 nodes per element) and a discontinuous bubble pressure with 3 nodes per element.

Consider a two-dimensional square domain  $[0, 1] \times [0, 1]$ , discretised with  $n$  divisions per side and with  $2n^2$  triangles ( $n^2$  squares with two triangles each).

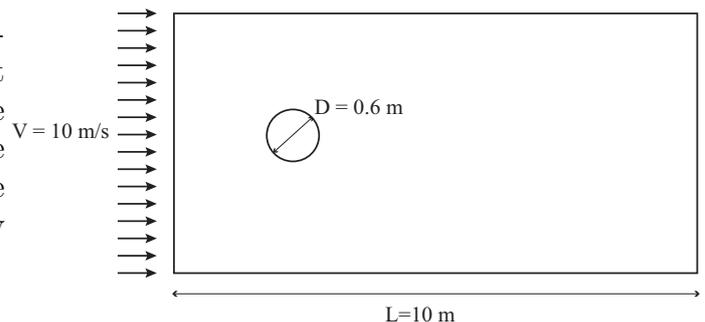
- (a) Compute the limit,

$$\lim_{n \rightarrow \infty} \frac{n_v}{n_p} = \lim_{n \rightarrow \infty} \frac{\dim(Q^h)}{\dim(P^h)}$$

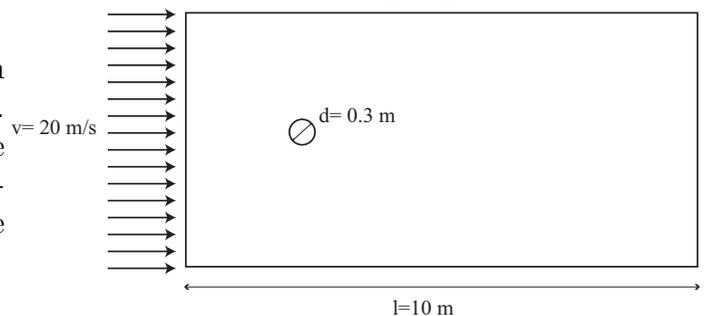
with  $n_v$  and  $n_p$  the number of degrees of freedom for the velocity and pressure fields, respectively.

- (b) Does this element satisfy the necessary stability requirements regarding the dimensions of  $Q^h$  and  $P^h$ ? Justify your answer.

2. We aim to study the flow around cylinders by analyzing the behavior of different characteristic lengths of the problem. We propose studying the two problems in the figure, considering in both cases the same value for the viscosity  $\nu$ . Is it necessary to solve both problems? Why?



3. Explain the main differences between the Stokes and Navier-Stokes equations. That is, comment the differences in the physical assumptions and in the numerical solution. Must the LBB condition be taken into account in both cases?



4. The code at

[http://ww2.lacan.upc.edu/huerta/exercices/Incompressible/Incompressible\\_Ex2.htm](http://ww2.lacan.upc.edu/huerta/exercices/Incompressible/Incompressible_Ex2.htm) solves the steady incompressible Navier-Stokes equations for the cavity flow problem. Show the streamlines and the pressure for  $Re = 1$  and for  $Re = 2000$ , and comment on the result in each case. Explain how did you compute the solution in each case and how many iterations of the non-linear solver were necessary.

5. For the design of a sailboat we are interested in analyzing the hydrodynamics. The study is for a boat with a length of 20 m that sails at a velocity of 50 km/h. In the design stage we consider a simulation with the finite element method.

- (a) With what differential equations should the problem be modeled? Note: the kinematic viscosity of water is  $1.2 \times 10^{-6} \text{ m}^2/\text{s}$ .
  - (b) What may be the reason if the pressure solution from the finite element method has significant oscillations? How to avoid them?
- 

6. The velocity components of a flow are given by,

$$v_x = -\omega y - \frac{Ay}{x^2 + y^2} \quad ; \quad v_y = \omega x + \frac{Ax}{x^2 + y^2}$$

with  $A$  and  $\omega$  constants.

- a) Compute the tensors of the velocity gradient, deformation rate and spin, and also write the vorticity vector.
  - b) Interpret geometrically the constants  $A$  and  $\omega$ . Which type of flow do we obtain if  $A = 0$ ? And if  $\omega = 0$ ?
- 

7. The velocity field of a fluid with respect to a Cartesian reference system  $xyz$  is given by,

$$v_x = \frac{x - ut}{\tau} \quad ; \quad v_y = -\frac{y}{\tau} \quad ; \quad v_z = 0$$

with  $u$  and  $\tau$  positive constants.

- (a) Obtain the Lagrangian description of the motion. Draw the trajectories of points  $(X, Y, Z) = (0, 0, 0)$  and  $(X, Y, Z) = (u\tau, u\tau, 0)$ .
  - (b) At point  $(u\tau, u\tau, 0)$  and from time  $t = 0$ , a dye has been injected into the fluid. Describe the successive shapes of the dye for  $t > 0$ .
- 

8. The Navier-Stokes equations may be written as,

$$-\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt}$$

Decide which terms can be neglected or simplified in each one of the following cases:

- (a) Steady problem.
  - (b) Incompressible fluid,  $Re = 20\,000$ .
  - (c) Incompressible fluid,  $Re = 15$ .
-

9. The motion of a fluid is characterised by the following velocity field:

$$v_x = \alpha_x + \beta y^2$$

$$v_y = \beta_x + \alpha y^2$$

where  $\alpha$  and  $\beta$  are non-zero constants. Determine if the flow is compressible or not. Justify your answer.

10. El programa `NavierStokes.m` resol l'equació de Navier-Stokes per un fluid incompressible en un domini quadrat amb condicions de contorn Dirichlet en tot el contorn, tal i com es pot veure a la figura.

Executeu el programa amb una malla de  $15 \times 15$  elements Q2Q1 i amb nombre de Reynolds  $Re = 10, 250, 500$  i  $1000$ . Comenteu els resultats obtinguts.

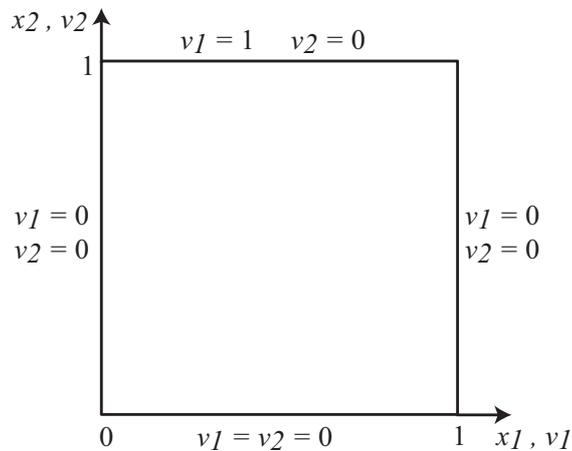


Figure 1: Representació del domini i les condicions de contorn de l'exercici 1

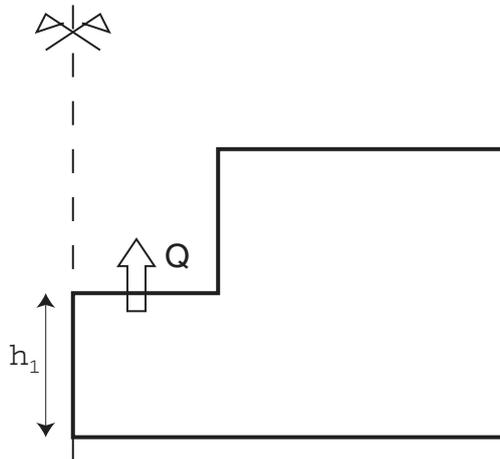


Figure 2: Representació del domini de l'exercici 2

11. Comment on the main differences and analogies between the Euler and Navier-Stokes equations.

---



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12. Interessa determinar el cabal  $Q$  d'aigua que cal bombar del fons d'una rasa (vegeu la figura). Amb aquest objectiu, es resol un problema de flux en medi pors amb l'ajut del programa `rasa.dgibi`.

- a) Estudieu mitjanant experiments numrics, la influncia del gruix de l'estrat de sl,  $h_1$ , en el cabal  $Q$ . A partir de quin gruix no hi ha diferncies significatives en  $Q$ ?
- b) ¿Quina s la influncia de la permeabilitat en el cabal? Compareu la soluci, en termes del camp d'altures piezomtriques  $h$  i del cabal  $Q$ , per un sl de permeabilitat  $k$  i un altre de permeabilitat  $2k$ .

---

13. Proponer razonadamente un ejemplo de aplicacin de las ecuaciones de Stokes, de las ecuaciones de flujo potencial y de las ecuaciones de Euler para flujo compresible.

---

14. Comentar las similitudes y diferencias entre las ecuaciones constitutivas para slidos elsticos y para fluidos newtonianos.

---

15. Se considera el problema de flujo en medio poroso representado en la figura ???. La permeabilidad del suelo es ortótropa y dada por la matriz  $\mathbf{K}$ .

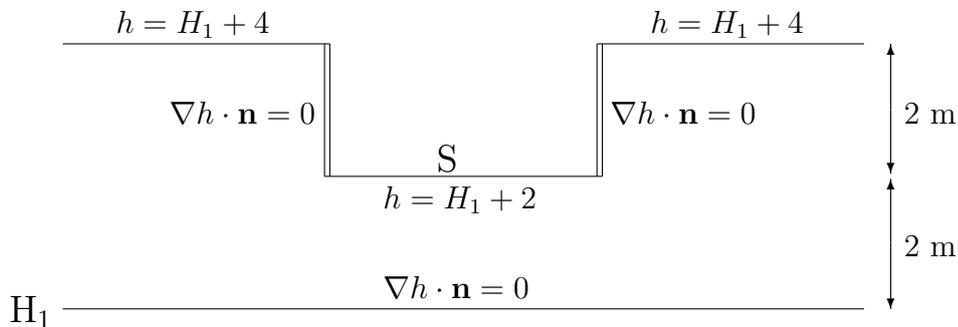


Figure 3: Dominio y condiciones de contorno

Determinar razonadamente la veracidad de las siguientes afirmaciones:

- (i) Un cambio en la altura geométrica  $H_1$  influye en la altura piezométrica  $h(x, y)$  pero no en el flujo que sale por la solera, S,

$$\phi_S = \int_S \mathbf{K} \cdot \nabla h \cdot \mathbf{n} \, d\Gamma$$

donde  $\mathbf{n}$  es el vector normal unitario exterior en S.

- (ii) La multiplicación de la matriz de permeabilidades  $\mathbf{K}$  por una constante influye tanto en la altura piezométrica como en el flujo por la solera.
- 

16. The flow in a pipe is assumed to be laminar, parallel to the walls, steady, irrotational and incompressible, with a constant velocity for all the points on the same cross-section. When the cross section is reduced by 50% for a fluid that enters at  $v = 2\text{m/s}$ , the pressure varies 6%.
- (a) If the pressure of the fluid at the entry of the pipe is equal to the atmospheric pressure ( $1e5\text{Pa}$ ), which is the density of the fluid?
- (b) Explain the pressure variation in terms of the energy of the fluid.
-

1. Consider a one-dimensional acoustic problem with Robin condition applied at the right boundary. In this case, the condition reads:

$$v = AP$$

Deduce that  $A = \frac{1}{\rho_0 c}$  gives the fully absorbing condition by following the next steps:

- Use the balance of linear momentum in order to express the Robin condition as a function of solely the pressure variable  $P$ .
- Manipulate the previous expression and impose that the wave equation is satisfied at the Robin boundary.

**Solution:**

- Balance of linear momentum in one dimension reads:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}$$

which inserted in the time-derivative of Robin condition,  $\partial_t v = A \partial_t P$  gives,

$$-\frac{\partial P}{\partial x} = \rho_0 A \frac{\partial P}{\partial t} \quad (1)$$

- Wave equation in one dimension reads:

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \quad (2)$$

By deriving wrt time and space the Robin condition in (1) we get

$$\begin{aligned} -\frac{\partial^2 P}{\partial x \partial t} &= \rho_0 A \frac{\partial^2 P}{\partial t^2} \\ -\frac{\partial^2 P}{\partial x^2} &= \rho_0 A \frac{\partial^2 P}{\partial t \partial x} \end{aligned}$$

which after equalising the cross derivatives gives,

$$\frac{\partial^2 P}{\partial x^2} = (\rho_0 A)^2 \frac{\partial^2 P}{\partial t^2}$$

Comparing the last equation with the wave equation we have that,

$$\frac{1}{c^2} = (A \rho_0)^2$$

i.e.

$$A = \frac{1}{c \rho_0}$$

2. A new heat pump has managed to reduce the transmitted noise in 3 dB. How much is the acoustic pressure reduced?

**Solution:**

$$L = 20 \log_{10}(P/P_0), L - L' = 3 = 20(\log_{10}(P/P_0) - \log_{10}(P'/P_0)) = 20 \log_{10}(P/P')$$

$$\Rightarrow P/P' = 10^{(3/20)} = 1.4125 \Rightarrow P' = 0.7079P$$


---

3. The fundamental equation in acoustics is the wave equation

$$\Delta P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \quad (3)$$

where  $P(\mathbf{x}, t)$  is the acoustic pressure and  $c$  is the speed of sound in the air. The usual strategy when solving such equation is to write the solution in the form  $P(\mathbf{x}, t) = p(\mathbf{x}) \exp(-i\omega t)$ , where  $i$  is the imaginary unit and  $\omega$  is the angular frequency. Let's consider the general case with complex angular frequencies,  $\omega = \omega_{\mathbb{R}} + i\omega_{\mathbb{I}}$ .

- a) Obtain *Helmholtz equation*, in terms of spatial variations of the acoustic pressure,  $p(\mathbf{x})$ .
- b) Interpret physically the real part  $\omega_{\mathbb{R}}$  and the imaginary part  $\omega_{\mathbb{I}}$  of the angular frequency. What does it mean  $\omega_{\mathbb{I}} > 0$ ? And  $\omega_{\mathbb{I}} < 0$ ?

**Solution:**

- (a) If  $P(\mathbf{x}, t) = p(\mathbf{x})e^{i\omega t}$ , then,

$$\begin{aligned} \Delta P &= e^{i\omega t} \Delta p \\ \partial_t P &= -i\omega e^{i\omega t} p \\ \partial_{tt} P &= -\omega^2 e^{i\omega t} p \end{aligned}$$

Replacing in the wave equation we get:

$$e^{i\omega t} \Delta p = -e^{-i\omega t} \omega^2 / c^2 p$$

That is, by setting  $k = \omega/c$ , Helmholtz equation:

$$\Delta p + k^2 p = 0$$

- (b) By setting

$$\begin{aligned} P(\mathbf{x}, t) &= e^{i\omega t} \Delta p(\mathbf{x}) \\ e^{-i\omega t} &= \cos(\omega t) - i \sin(\omega t) \\ \omega &= \omega_R + i\omega_I \Rightarrow e^{i\omega t} = e^{-i\omega_R t} e^{\omega_I t} \end{aligned}$$

we arrive to the following real part of  $P(\mathbf{x}, t)$ :

$$Re(P(\mathbf{x}, t) = e^{\omega_I t} (Re(p(\mathbf{x})) \cos(\omega_R t) + Im(p(\mathbf{x})) \sin(\omega_R t))$$

so that by setting  $\phi = \text{Re}(p)/\text{Im}(p)$ , it can be expressed as:

$$\text{Re}(P(\mathbf{x}, t)) = |p|e^{\omega_I t} \sin(\omega_R t + \phi)$$

Therefore, we can interpret  $\omega_R$  as the angular frequency, and  $|p|e^{\omega_I t}$  as the amplitude, which increases if  $\omega_I > 0$  and viceversa.

4. (15 marks) Show that in acoustics, the Robin condition with  $A = \frac{1}{\rho c}$  is equivalent to having no reflections at the boundaries, that is, that the velocity  $\mathbf{v}$  at the boundary satisfies  $\mathbf{v} \cdot \mathbf{n} = c$  with  $c$  the speed of the sound.

**Solution:** Robin:  $\mathbf{v} \cdot \mathbf{n} = AP$

Since  $P = c^2 \rho$ , if  $A = \frac{1}{\rho c}$ , then the Robin condition reads,  $\mathbf{v} \cdot \mathbf{n} = AP = \frac{1}{\rho c} P = c$ , as we wanted.

5. A secret meeting is taking place in an area controlled by a radar. A group of spies is planning to take aerial pictures of a set of classified documents using a convenient airplane. From previous experiences, it is advised to fly at least below 500 meters. The spies can choose one of the airplanes listed in Table ??, and the properties of the radar are given in Table ?. If we know that the cost of the flight increases as the RCS decreases, which airplane should the spies select in order to minimise the operational costs?

Airplane	RCS (dBsm)
F-22 Raptor	-21.8709
F-117 Nighthawk	-25.2288
B-2 Spirit	-28.2391
AGM-129 ACM	-30
Boeing Bird of Prey	-70

Table 1: Available airplanes and RCS

Radar parameter	Value
Gain	100
Transmitted power	100 W
Wave length	1 km
Minimum power in order to be detected	$10^{-6}$ W

Table 2: Radar properties

**Solution:**

The received power is given by  $P_r = P_t G^2 \lambda^2 \sigma / ((4\pi)^2 R^4)$ . By introducing the given values, we get:

$$P_t = 100 * 100^2 * 1000^2 / 2^4 / 5^4 / 100^4 * \sigma / \pi^2 = \sigma / \pi^2$$

Airplane	$P_r$
F-22 Raptor	6.58E-4
F-117 Nighthawk	3.03E-4
B-2 Spirit	1.52E-4
AGM-129 ACM	1.01E-4
Boeing Bird of Prey	1.01E-8

Table 3: Available airplanes and RCS

For each plane,  $P_r$  is the one given in Table ??.

Therefore, since the minimum power to be detect is  $10^{-6}W$ , we can only use Boeing Bird of Prey.

---

6. Maxwell equations in a medium with permittivity and magnetic permeability  $\epsilon$  and  $\mu$ , respectively, can be reduced to one dimension. In this case,
- Indicate the relative orientation of the magnetic and electric oscillations with respect to the direction of the wave propagation. Are the electric and the magnetic fields coupled?
  - If we write the components  $E_y$  and  $H_z$  in one vector  $\mathbf{U}^T = \{E_y, H_z\}$ , Maxwell equations in the void can be written in a conservative form given by equation (??). Deduce the expression of  $\mathbf{F}(\mathbf{U})$ .

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0} \quad (4)$$

- Deduce the wave equations associated to the components  $E_y$  and  $H_z$ . Which is the speed of the wave for each one of the components?

**Solution:**

- We assume direction of propagation =  $x$ . Then  $\partial_z = \partial_y = 0$ . From,

$$\epsilon \partial_t E_y = -\partial_x H_z \quad (5)$$

$$\mu \partial_t H_z = \partial_x E_y \quad (6)$$

and

$$\epsilon \partial_t E_z = -\partial_x H_y \quad (7)$$

$$\mu \partial_t H_y = \partial_x E_z \quad (8)$$

Then the components  $(E_y, H_z)$  are coupled, and  $(H_y, E_z)$  as well, but each pair is uncoupled.

- Maxwell equation reads:

$$\partial_t \begin{Bmatrix} E_y \\ E_z \end{Bmatrix} + \partial_x \begin{Bmatrix} H_z/\epsilon \\ E_y/\mu \end{Bmatrix} = \mathbf{0} \Rightarrow \mathbf{F}(\mathbf{U}) = \begin{bmatrix} 0 & 1/\epsilon \\ 1/\mu & 0 \end{bmatrix} \mathbf{U}$$

- By applying  $\partial_t$  on (??) and  $\partial_x$  on (??), we have:

$$\partial_{tt} E_y = \frac{1}{\epsilon \mu} \partial_{xx} E_y$$

This is the wave equation for  $E_y$ , with speed  $c = 1/\sqrt{\epsilon \mu}$ . By applying  $\partial_t$  on (??) and  $\partial_x$  on (??), we have:

$$\partial_{tt} H_z = \frac{1}{\epsilon \mu} \partial_{xx} H_z$$

Wave equation for  $H_z$ , same speed as  $E_y$ .

7. Las ecuaciones de Maxwell en un medio sin fuentes de carga eléctrica se pueden escribir como

$$\begin{cases} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \\ \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \end{cases} \quad (9)$$

donde  $\mathbf{E}$  es el campo eléctrico y  $\mathbf{H}$  el campo magnético. Este sistema de ecuaciones en derivadas parciales debe ir acompañado por tres leyes constitutivas, donde intervienen los parámetros materiales  $\varepsilon$ ,  $\mu$  y  $\sigma$ .

En problemas de *scattering* se suele descomponer el campo eléctrico y el magnético total en incidente y *scattered* ( $\mathbf{E} = \mathbf{E}^I + \mathbf{E}^S$  y  $\mathbf{H} = \mathbf{H}^I + \mathbf{H}^S$ ), donde el campo incidente es conocido y satisface las ecuaciones de Maxwell en el vacío (es decir con  $\varepsilon = \mu = 1$  y  $\sigma = 0$ ).

- (a) Comprueba que el campo *scattered* satisface el mismo sistema que el campo total (??), pero con término fuente distinto. ¿Cuál es el nuevo término fuente?
- (b) Nombra alguna ventaja de plantear la descomposición de los campos en incidente y *scattered* a la hora de resolver el problema numéricamente.

**Solution:**

- (a) The first component of Maxwell equations reads:

$$\partial_t(\mu H_x^I) + \partial_t(\mu H_x^S) = \partial_y E_z^I - \partial_z E_y^I + \partial_y E_z^S - \partial_z E_y^S$$

Since  $\partial_t H_x^I = \partial_t(\mu - 1)H_x^I$ , and  $H^I$  and  $E^I$  satisfy Maxwell eq., the above eqn. reduces to,

$$\partial_t(\mu H_x^S) = \partial_y E_z^S - \partial_z E_y^S + \partial_t(1 - \mu)H_x^S$$

Doing similar manipulations for the other components we get:

$$\partial_t(\mu \mathbf{H}^S) = -\nabla \times \mathbf{E}^S + \partial_t(1 - \mu)\mathbf{H}^S$$

The last term is the additional source term  $\mathbf{s}_H$ . The source field may have a fine mesh next to the object and coarse far from it.

- (b) The first component of Maxwell equations reads:

$$\partial_t(\epsilon E_x^I) + \partial_t(\epsilon E_x^S) = \partial_z H_y^I - \partial_y H_z^I + \partial_z H_y^S - \partial_y H_z^S - \sigma E_x$$

Since  $\partial_t \epsilon E_x^I = \partial_t E_x^I + \partial_t(\epsilon - 1)E_x^I$ , and  $H^I$  and  $E^I$  satisfy Maxwell equations in the void, we have:

$$\partial_t(\epsilon E_x^S) = \partial_z H_y^S - \partial_y H_z^S - \sigma E_x - \partial_t((\epsilon - 1)E_x^S)$$

Doing similar manipulations on the other components, we obtain:

$$\partial_t(\epsilon \mathbf{E}^S) = \nabla \times \mathbf{H}^S - \sigma \mathbf{E} + \partial_t((1 - \epsilon) \mathbf{E}^S)$$

The last term is the source term  $\mathbf{S}_E$ . In summary, the source term depends on  $\mathbf{E}^I$  and  $\mathbf{E}^S$ , that is

$$\begin{Bmatrix} \mathbf{s}_H \\ \mathbf{s}_E \end{Bmatrix} = \begin{Bmatrix} \partial_t(1 - \mu) \mathbf{H}^I \\ -\sigma(\mathbf{E}^I + \mathbf{E}^S) + \partial_t((1 - \epsilon) \mathbf{E}^S) \end{Bmatrix}$$

With the scattered formulation, the boundary condition at the object can be imposed exactly. With the total formulation is not possible, and thus boundary condition at object is less accurate.

8. What is scattering of electromagnetic waves? Is this phenomenon exclusive to electromagnetic waves?

**Solution:** The scattering is the analysis of the reflected waves after reaching an object. This is not exclusive of electromagnetic waves, since it No, also in acoustics with pressure waves (bats) or mechanical waves (earthquakes).

9. Describir brevemente las condiciones de contorno en los problemas de vibroacústica y su significado físico.

**Solution:**

$$\Delta p + k^2 p = 0$$

- CC Dirichlet:  $p = p^*$  in  $\gamma_D$ , not much used.
  - CC Neumann:  $\partial_n p = i\rho\omega \mathbf{v}^s \cdot \mathbf{n}$ , vibrating panel.
  - CC Robin:  $\partial_n p = i\rho\omega A p$ , absorbing panel, with  $A$  absorption coefficient.
10. ¿Qué tipo de condición de contorno se utiliza en la interfase entre el fluido acústico y el sólido elástico en un problema vibroacústico? ¿Cuál es el significado físico de dicha condición de contorno?