Hopf, Caccioppoli and Schauder, reloaded

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April 28, 2023

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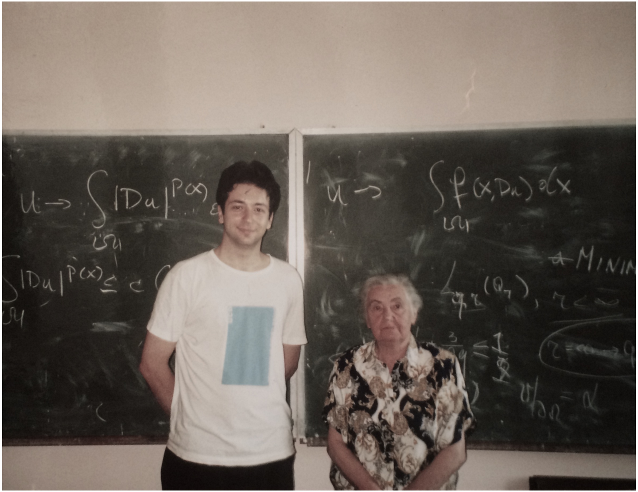
With Olga Ladyzhenskaya

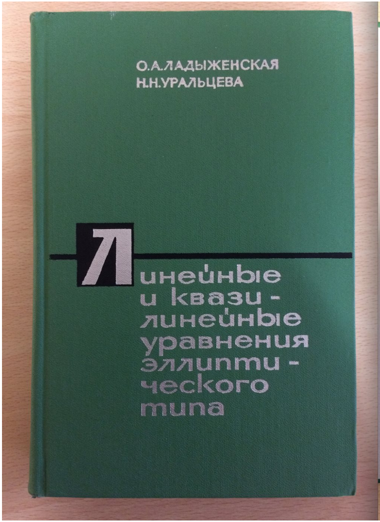


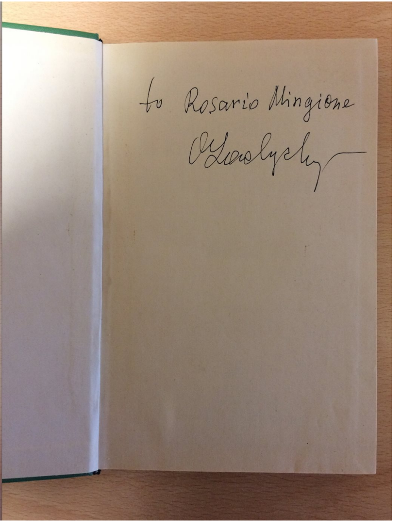
St. Petersburg 2001

With Gregory Seregin and Nina Uraltseva

St. Petersburg 2001

The 1973 legendary Russian edition





Part 1: Notions of Nonuniform Ellipticity

Functionals and equations

- We consider both integral functionals

$$v \mapsto \int_\Omega F(x, Dv)\, dx$$

- and equations of the type

$$-\operatorname{div} A(x, Du) = 0$$

- The catch is of course given by the Euler-Lagrange equation

$$-\operatorname{div} \partial_z F(x, Du) = 0\,.$$

Ellipticity

- Here by ellipticity mean that a conditions of the type

$$g_1(x,|z|)\mathbb{I}_d \leq \partial_z A(x,z) \leq g_2(x,|z|)\mathbb{I}_d$$

  is verified for non-negative functions $g_1, g_2 \colon [0,\infty) \to \mathbb{R}$.
- That is, in the case of functionals

$$g_1(x,|z|)\mathbb{I}_d \leq \partial_{zz} F(x,z) \leq g_2(x,|z|)\mathbb{I}_d\,.$$

- Autonomous case

$$g_1(|z|)\mathbb{I}_d \leq \partial_z A(z) \leq g_2(|z|)\mathbb{I}_d\,.$$

Uniform Ellipticity

- Uniform ellipticity means
$$\limsup_{|z|\to\infty} \frac{g_2(x,|z|)}{g_1(x,|z|)} < \infty.$$
uniformly with respect to x.
- When $\partial_z A(\cdot)$ is symmetric (for instance in the variational case)
$$\sup_{|z|\geq 1} \frac{\text{highest eigenvalue of } \partial_z A(x,z)}{\text{lowest eigenvalue of } \partial_z A(x,z)} \leq c\,.$$

Important remark

- The condition
$$\limsup_{|z|\to\infty} \frac{g_2(|z|)}{g_1(|z|)} < \infty.$$
has nothing to do with the fact of being non-degenerate, that means
$$\inf_z g_1(|z|) > 0\,.$$
- For instance, the p-Laplacean operator
$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div} A(Du) = 0$$
is such that
$$\partial_z A(z) \approx |z|^{p-2}\mathbb{I}_d$$
and it is therefore a degenerate, uniformly elliptic operator.

Additional uniformly elliptic operators

- Uniformly elliptic equations are not only of polynomial type. For instance
$$-\mathrm{div}\,(\tilde{a}(|Du|)Du) = f$$
is uniformly elliptic provided
$$\begin{cases} -1 < i_a \leq \dfrac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_a < \infty \quad \text{for every } t > 0 \\ \tilde{a}\colon (0,\infty) \to [0,\infty) \text{ is of class } C^1_{\mathrm{loc}}(0,\infty)\,. \end{cases}$$
- Here it is
$$\begin{cases} |\partial_z a(z)| \lesssim \max\{1, s_a+1\}\tilde{a}(z) \\ \min\{1, i_a+1\}\tilde{a}(|z|)|\xi|^2 \leq \partial_z a(z)\xi\cdot\xi \end{cases}$$
therefore
$$\mathcal{R}(z) \lesssim \frac{\max\{1, s_a+1\}}{\min\{1, i_a+1\}}\,.$$

Additional uniformly elliptic operators

In this case Schauder estimates for solutions to

$$-\mathrm{div}\,(c(x)\tilde{a}(|Du|)Du) = 0$$

are an achievement of Lieberman (CPDE 1991).

Nonuniform ellipticity

- It is a classical topic.
- When dealing with equations of the type

$$-\operatorname{div} A(x, Du) = [\text{right-hand side}]$$

nonuniform ellipticity means that

$$\limsup_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial_z A(x_0, z)}{\text{lowest eigenvalue of } \partial_z A(x_0, z)} = \infty$$

holds for at least one point x_0 [needless to say, $\partial_z A(\cdot)$ is symmetric here].

Nonuniform ellipticity

- Similar definitions occur when considering non-divergence form equations of the type

$$A^{ij}(x, u, Du)D_{ij}u = [\text{right-hand side}]$$

Under the assumption

$$g_1(|z|)\mathbb{I}_d \leq A(u, v, z) \leq g_2(|z|)\mathbb{I}_d\,.$$

- Nonuniform ellipticity then occurs when

$$\limsup_{|z|\to\infty} \frac{g_2(|z|)}{g_1(|z|)} = \infty\,.$$

Nonuniformly elliptic classics $\leq 70s$

- Ladyzhenskaya & Uraltseva (Book + CPAM 1970)
- Gilbarg (1963)
- Stampacchia (CPAM 1963)
- Hartman & Stampacchia (Acta Math. 1965)
- Ivočkina & Oskolkov (Zap. LOMI 1967)
- Oskolkov (Trudy Mat. Inst. Steklov 1967)
- Serrin (Philos. Trans. Roy. Soc. London Ser. A 1969)
- Ivanov (Proc. Steklov Inst. Math. 1970)
- Trudinger (Thesis, Bull. AMS 1967, ARMA 1971)
- Leon Simon (Indiana Univ. Math. J. 1976)

Nonuniformly elliptic classics $\geq 80s$

- Trudinger (Invent. Math. 1981)
- Zhikov (papers from the 80s)
- N.N. Ural'tseva & A. B. Urdaletova (Vestnik Leningrad Univ. Math. 1984)
- Lieberman (Indiana Univ. Math. J. 1983)
- Ivanov (Proc. Steklov Inst. Math. Book 1984)
- Marcellini (ARMA 1987, JDE 1991, Ann. Pisa 1996)
- Zhikov (Math. of USSR-Izvestia. 1995, Russian J. Math. Phys. 1997)

A Ladyzhenskaya & Uraltseva classic

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XXIII, 677–703 (1970)

Local Estimates for Gradients of Solutions of Non-Uniformly Elliptic and Parabolic Equations

O. A. LADYZHENSKAYA AND N. N. URAL'TSEVA
Leningrad University

Various classes of non-uniformly elliptic (and parabolic) equations of second order of the form

$$\sum_{i,j=1}^{n} a_{ij}(x, u, u_x) u_{ij} = a(x, u, u_x), \qquad a_{ij}(x, u, u_x)\xi_i\xi_j > 0 \quad \text{for} \quad |\xi| = 1, \tag{1.1}$$

for all solutions $u(x)$ of which $\max_\Omega |u_x|$ can be estimated by $\max_\Omega |u|$ and $\max_{\partial\Omega} |u_x|$, were discussed in [1] (see also [2]).[1] The method used was introduced in [3]. In the same paper a method was suggested for obtaining local estimates of $|u_x|$, i.e., estimates of $\max_{\Omega'} |u_x|$ in terms of $\max_\Omega |u|$ and the distance $d(\Omega', \partial\Omega)$ of $\Omega' \subset \Omega$ from the boundary $\partial\Omega$. In a series of papers (concerning these see [4] and [5]) we have shown that this method is applicable to the whole class of uniformly elliptic and parabolic equations. In the present paper we investigate the possibility of applying it to non-uniformly elliptic and parabolic equations. It turns out that it is applicable, roughly speaking, to those classes of [1] for which the order of nonuniformity of the quadratic form $a_{ij}(x, u, u_x)\xi_i\xi_j$ is less than two. The first part of this paper is devoted to the proof of this assertion.

In the second part we analyze a different method of obtaining local estimates for $|u_x|$ which is applicable to elliptic equations of the form

$$-\sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, u, u_x) + a(x, u, u_x) = 0, \tag{1.2}$$

and embraces such interesting cases as equations for the mean curvature of a

[1] We shall use the notation

$$u_{x_i} = u_i, \quad u_x = (u_1, \cdots, u_n), \quad u_{x_i x_j} = u_{ij},$$
$$u_x^2 = \sum_{i=1}^{n} u_i^2, \quad |u_x| = \sqrt{u_x^2}, \quad u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2, \quad |u_{xx}| = \sqrt{u_{xx}^2}.$$

Trudinger's classics

THE DIRICHLET PROBLEM FOR NONUNIFORMLY ELLIPTIC EQUATIONS[1]

BY NEIL S. TRUDINGER

Communicated by F. John, January 23, 1967

Introduction. Let Ω be a bounded domain in E^n. The operator

$$Qu = a^{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x)$$

acting on functions $u(x) \in C^2(\Omega)$ is *elliptic* in Ω if the minimum eigenvalue $\lambda(x, u, p)$ of the matrix $[a^{ij}(x, u, p)]$ is positive in $\Omega \times E^{n+1}$. Here

$$u_x = (u_{x_1}, \cdots u_{x_n}), \qquad p = (p_1, \cdots p_n)$$

and repeated indices indicate summation from 1 to n. The functions $a^{ij}(x, u, p)$, $a(x, u, p)$ are defined in $\Omega \times E^{n+1}$. If furthermore for any $M > 0$, the ratio of the maximum to minimum eigenvalues of $[a^{ij}(x, u, p)]$ is bounded in $\Omega \times (-M, M) \times E^n$, Qu is called *uniformly elliptic*. A solution of the *Dirichlet problem* $Qu = 0$, $u = \phi(x)$ on $\partial\Omega$ is a $C^0(\bar{\Omega}) \cap C^2(\Omega)$ function $u(x)$ satisfying $Qu = 0$ in Ω and agreeing with $\phi(x)$ on $\partial\Omega$.

When Qu is elliptic, but not necessarily uniformly elliptic, it is referred to as *nonuniformly elliptic*. In this case it is well known from two dimensional considerations, that in addition to smoothness of the boundary data $\partial\Omega$, $\phi(x)$ and growth restrictions on the coefficients of Qu, geometric conditions on $\partial\Omega$ may play a role in the solvability of the Dirichlet problem. A striking example of this in higher dimensions is the recent work of Jenkins and Serrin [**4**] on the minimal surface equation, mentioned below.

On the Regularity of Generalized Solutions of Linear, Non-Uniformly Elliptic Equations

NEIL S. TRUDINGER

Communicated by J. C. C. NITSCHE

1. Introduction

We consider in this paper the simplest form of a second order, linear, divergence structure equation in n variables, namely

$$\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \tag{1.1}$$

where the coefficients a^{ij}, $1 \leqq i, j \leqq n$, are measurable functions on a domain Ω in Euclidean n space E^n. Following the usual summation convention, repeated indices will indicate summation from 1 to n. We assume always that $n \geqq 2$.

Equation (1.1) is *elliptic* in Ω if the coefficient matrix $\mathscr{A}(x) = [a^{ij}(x)]$ is positive almost everywhere in Ω. Let $\lambda(x)$ denote the minimum eigenvalue of $\mathscr{A}(x)$ and set

$$\mu(x) = \sup_{1 \leqq i, j \leqq n} |a^{ij}(x)| \tag{1.2}$$

so that

$$\lambda(x)|\xi|^2 \leqq a^{ij}(x)\xi_i\xi_j \leqq n^2\mu(x)|\xi|^2 \tag{1.3}$$

for all $\xi \in E^n$, $x \in \Omega$. We will say that equation (1.1) is *uniformly elliptic* in Ω if the function $\gamma(x) = \mu(x)/\lambda(x)$ is essentially bounded in Ω. If γ is not necessarily bounded, then equation (1.1) is referred to as *non-uniformly elliptic*. We note here that uniformly elliptic equations for which λ^{-1} is unbounded have sometimes been referred to as *degenerate elliptic* [9].

Uniformly elliptic equations of the form (1.1), with bounded λ^{-1} and μ, have been extensively studied in the literature, two of the major results being a *Hölder estimate* for generalized solutions, due to DE GIORGI [1] and NASH [11], and a *Harnack inequality*, due to MOSER [7]. The purpose of this paper is to extend these results to a class of non-uniformly elliptic equations. In order to accomplish this, our methods differ substantially from those previously proposed and hence may be considered as new proofs of the original results. Various features of our proofs do coincide, however, with techniques in MOSER's two papers [6], [7]. An essential difference is that in order to obtain the stronger results we need to extract more information from the equation.

Interior Gradient Bounds for Non-uniformly Elliptic Equations

LEON SIMON

In [1] Bombieri, De Giorgi and Miranda were able to derive a local interior gradient bound for solutions of the minimal surface equation with n independent variables, $n \geqq 2$, thus extending the result previously established by Finn [2] for the case $n = 2$. Their method was to use test function arguments together with a Sobolev inequality on the graph of the solution (Lemma 1 of [1]). A much simplified proof of their result was later given by Trudinger in [12].

Since the essential features of the test function arguments given in [1] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided appropriate analogues for the Sobolev inequality of [1] could be established. Ladyzhenskaya and Ural'tseva obtained such inequalities ([4], Lemma 1) for a rather large class of equations, including the minimal surface equation as a special case. They were thus able to obtain gradient bounds for this class of equations.

In §2 of [5] a general Sobolev inequality was established on certain generalized submanifolds of Euclidean space. In the special case of nonparametric hypersurfaces in $\mathbf{R}^{n+1}$ of the form $x_{n+1} = u(x)$, where u is a C^2 function defined on an open subset $\Omega \subset \mathbf{R}^n$, the inequality of [5] implies

$$(1) \qquad \left\{\int_\Omega h^{n/(n-1)} v \, dx\right\}^{(n-1)/n} \leqq c \int_\Omega \left[\left\{\sum_{i,j=1}^n g^{ij} h_{x_i} h_{x_j}\right\}^{1/2} + h\,|H|\right] v \, dx$$

for each non-negative C^1 function h with compact support in Ω, where

$$v = (1 + |Du|^2)^{1/2}$$

$$g^{ij} = \delta_{ij} - \nu_i \nu_j\,, \qquad \nu_i = u_{x_i}/v, \qquad i, j = 1, \cdots, n$$

$$H = \frac{1}{n} v^{-1} \sum_{i,j=1}^n g^{ij} u_{x_i x_j}\,,$$

and where c is a constant depending only on n. (See the discussion in §2 below.) The quantity H appearing in this inequality is in fact the mean curvature of the hypersurface $x_{n+1} = u(x)$ and in the special case when $H \equiv 0$ (*i.e.* when u

The variational setting

- For local estimates the variational setting

$$v \mapsto \int_\Omega F(x, Dv)\, dx$$

turns out to be the most appropriate one.

- The Euler-Lagrange reads as

$$-\mathrm{div}\, \partial_z F(x, Du) = f\,.$$

- Nonuniform ellipticity reads as

$$\lim_{|z|\to\infty} \mathcal{R}_{\partial_z F(x,\cdot)}(z) = \lim_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial_{zz}F(x,z)}{\text{lowest eigenvalue of } \partial_{zz}F(x,z)} = \infty\,.$$

Polynomial Nonuniform Ellipticity

- This happens, when, for $|z|$ is large,

$$\mathcal{R}_{\partial_z F(x,\cdot)}(z) \approx |z|^{\delta} \qquad \text{for some } \delta \geq 0$$

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- These are usually formulated prescribing

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(x,z) \lesssim |z|^{q-2}\mathbb{I}_d$$

so that

$$\mathcal{R}_{\partial_z F(x,\cdot)}(z) \lesssim |z|^{q-p}, \ \text{ for } |z| \text{ large}\,, \quad 1 < p \leq q\,.$$

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- These are called (p,q)-growth conditions in Marcellini's terminology. They describe a general situation in which polynomial nonuniform ellipticity occurs. They usually couple with growth conditions on $F(\cdot)$

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1 \qquad \text{and} \qquad 1 < p \leq q\,.$$

A basic condition

- Consider

$$W^{1,1} \ni v \mapsto \int_\Omega F(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1 \qquad \text{and} \qquad 1 < p \leq q$$

then

$$\frac{q}{p} < 1 + o(n)$$

is a sufficient (Marcellini) and necessary (Giaquinta and Marcellini) condition for regularity.

- Several pioneering papers of Marcellini from the 80-90s.
- Holds in the parabolic case, see recent papers by Bögelein & Duzaar & Marcellini.

Gap bounds miscellanea

- Bella & Schäffner, CPAM 2021 – Analysis & PDE 2021

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{\infty}_{\mathrm{loc}} \quad [n > 2, \text{scalar case}]\,.$$

- Schäffner, Calc. Var. & PDE 2021

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{q}_{\mathrm{loc}} \quad [n > 2, \text{vectorial case}]\,.$$

- Hirsch & Schäffner, Comm. Cont. Math. 2020

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n-1} \Longrightarrow u \in L^{\infty}_{\mathrm{loc}}\,.$$

- De Filippis & Kristensen & Koch, in preparation

$$\frac{q}{p} < 1 + \frac{2}{n-2} \Longrightarrow Du \in L^{\infty}_{\mathrm{loc}}$$

by duality methods, under special assumptions.

Two different ellipticity ratios

For nonautonomous functionals of the type

$$v \mapsto \int_\Omega F(x, Dv)\, dx\,.$$

We have the pointwise ellipticity ratio

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0, z)}\,.$$

The nonlocal ellipticity ratio

- The nonlocal ellipticity ratio is defined by

$$\mathcal{R}_{\partial_z F}(z,B) = \frac{\sup_{x\in B} \ \text{highest eigenvalue of } \partial_{zz}F(x,z)}{\inf_{x\in B} \ \text{lowest eigenvalue of } \partial_{zz}F(x,z)}$$

  where B is a ball.

- In general it is

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) \lesssim \mathcal{R}_{\partial_z F}(z,B) \qquad \forall\, x_0 \in B\,.$$

- The second ratio usual detects milder, non traditional forms of nonuniform ellipticity

Two different notions

- The integrand $F(\cdot)$ is nonunifomly elliptic if

$$\sup_{x_0,|z|\geq 1} \mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \infty\,.$$

- We call it softly nonuniformly elliptic if

$$\sup_{x_0,|z|\geq 1} \mathcal{R}_{\partial_z F(x_0,\cdot)}(z) < \infty \quad \text{but} \quad \lim_{|z|\to\infty} \mathcal{R}_{\partial_z F}(z,B) = \infty$$

  for at least one ball B.

- We call it uniformly elliptic if

$$\sup_{B,|z|\geq 1} \mathcal{R}_{\partial_z F}(z,B) < c\,.$$

- Discussion in De Filippis & Min. ARMA 2021.
- See also Beck & Min. CPAM 2020.

A functional of Zhikov

Zhikov considered, between the 80s and the 90s, the following functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\,dx \qquad a(x) \geq 0$$

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc.

Soft nonuniform ellipticity

The double phase functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \equiv \int_\Omega F(x, Dv)\, dx$$

allows for treating Hölder coefficients but is pointwise uniformly elliptic, in the sense that, whenever we freeze x_0

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0, z)} < \infty\,.$$

Soft nonuniform ellipticity

In the double phase case

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \equiv \int_\Omega F(x, Dv)\, dx$$

we have (with $B \cap \{a(x) = 0\} \neq \emptyset$)

$$\mathcal{R}_{\partial_z F}(z, B) = \frac{\sup_{x\in B} \ \text{highest eigenvalue of } \partial_{zz} F(x,z)}{\inf_{x \in B} \ \text{lowest eigenvalue of } \partial_{zz} F(x,z)}$$
$$\approx 1 + \|a\|_{L^\infty(B)} |z|^{q-p} \to \infty$$

vs

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0,z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0,z)} < \infty\,.$$

A counterexample

Theorem (Fonseca, Malý & Min. ARMA 2004)

Take $n \geq 2$, $B \subset \mathbb{R}^n$ *and* $\varepsilon, \sigma > 0$, $0 < \alpha \leq 1$. *There exists* $a(\cdot) \in C^{0,\alpha}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min_{v} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ v \in u_0 + W_0^{1,p}(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \sigma$.

See also Esposito, Leonetti & MIn. JDE 2004 and Balci, Diening & Surnachev Calc. Var. 2020.

Many years later

Theorem

Let $u \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \qquad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega)$$

with

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}\,.$$

Then

Du is locally Hölder continuous.

- Colombo & Min. ARMA (2 papers) 2015
- Baroni, Colombo & Min. Calc. Var. 2018.

Soft Nonuniform ellipticity and Special structures

$$w \mapsto \int_\Omega F(x, Dw)\, dx$$

$$\mathcal{R}(x,z) \lesssim 1, \quad \mathcal{R}(z,B) \to \infty.$$

- Acerbi & Min. *ARMA* 2001.
- Baasandorj & Byun & Oh *JFA* 2020; *Calc. Var. & PDE* 2021.
- Baroni *JDE* 202?.
- De Filippis & Oh *JDE* 2019.
- Hästo & Ok *JEMS* 2022; *ARMA* 2022.
- Karppinen & Lee *IMRN* (2021).

The **frozen integrand** $z \mapsto F(x_0, z)$ is **uniformly elliptic**.

Pointwise nonuniform ellipticity

- This does not happen for basic model examples as

$$v \mapsto \int_\Omega c(x) F(Dv)\, dx \equiv \int_\Omega \bar{F}(x, Dv)\, dx$$

under genuine non-uniform ellipticity

$$|z|^{p-2}\mathbb{I}_d \leq \partial_{zz} F(z) \leq |z|^{q-2}\mathbb{I}_d$$

- Freezing yields

$$\begin{aligned}
\mathcal{R}_{\partial_z \bar{F}(x_0,\cdot)} &\approx \frac{\text{highest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)} \\
&\approx \frac{\text{highest eigenvalue of } \partial_{zz}F(z)}{\text{lowest eigenvalue of } \partial_{zz}F(z)} \approx |z|^{q-p}\,.
\end{aligned}$$

Part 2: Schauder estimates

Classical Schauder estimates

- Solutions to

$$-\triangle u = -\mathrm{div}\,(Du) = 0$$

are smooth.

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are smooth.
- How much of such regularity is preserved when adding coefficients (ingredients)?
$$-\mathrm{div}\,(A(x)Du) = -(A^{ij}(x)D_j u)_{x_i} = 0\,.$$

Classical Schauder estimates

- Solutions to

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are smooth.

- How much of such regularity is preserved when adding coefficients (ingredients)?

$$-\mathrm{div}\,(A(x)Du) = -(A^{ij}(x)D_j u)_{x_i} = 0\,.$$

- The matrix $A(\cdot)$ is bounded and elliptic

$$\nu\mathbb{I}_d \leq A(x) \leq L\mathbb{I}_d\,.$$

Classical Schauder estimates

- As Du and $A(x)$ stick together we have

$$A(\cdot) \in C^{0,\alpha} \Longrightarrow Du \in C^{0,\alpha} \qquad 0 < \alpha < 1\,.$$

- Similar results hold for equations not in divegence form
- This kind of results where first obtained by Hopf (1929), Caccioppoli (1934) and Schauder (1934), in various forms, and are today known as Schauder estimates (see also some contributions of Giraud). They have also parabolic analogs.
- This is a basic tool in elliptic and parabolic PDES and in the Calculus of Variations.

Classical Schauder estimates

- The original proofs involve heavy potential theory, as many others early elliptic results.
- Similar results hold for equations not in divergence form.
- Modern proofs have been given by Campanato, Trudinger and Leon Simon. All these proof rely, in a way or in another, on perturbation/comparison methods.
- In turn, all these proofs require that the estimates involved are homogeneous. This is the crucial point.

Classical Schauder estimates (here following Campanato)

- One considers solutions to the "frozen" problems on a ball $B_r(x_0)$

$$\begin{cases} -\operatorname{div}(A(x_0)Dv) = 0 \\ v - u \in W_0^{1,2}(B_r(x_0)) \,. \end{cases}$$

- and combines reference estimates

$$\fint_{B_\varrho(x_0)} |Dv - (Dv)_{B_\varrho(x_0)}|^2 \, dx \lesssim \left(\frac{\varrho}{r}\right)^2 \fint_{B_r(x_0)} |Dv - (Dv)_{B_r(x_0)}|^2 \, dx$$

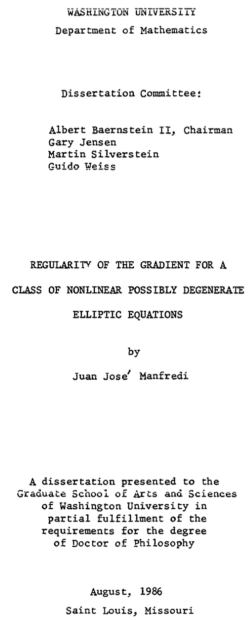
- and comparison estimates

$$\fint_{B_r(x_0)} |Du - Dv|^2 \, dx \lesssim r^{2\alpha} \fint_{B_r(x_0)} |Du|^2 \, dx \,.$$

Nonlinear Schauder estimates

- Nonlinear theory is a more recent story, dating back to the 80s, by Manfredi (see also papers by Giaquinta & Giusti, DiBenedetto).
- A model example is given by the p-Laplacean equation with coefficients

$$-\mathrm{div}\,(\mathrm{c}(x)|Du|^{p-2}Du) = 0\,, \qquad \mathrm{c}(\cdot) \in C^{0,\alpha}\,.$$


WASHINGTON UNIVERSITY

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REGULARITY OF THE GRADIENT FOR A

CLASS OF NONLINEAR POSSIBLY DEGENERATE

ELLIPTIC EQUATIONS

by

Juan José Manfredi

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

August, 1986

Saint Louis, Missouri

Nonlinear Schauder estimates

- The possibility of nonlinear Schauder estimates in the nonlinear case relies on the fact that solutions v to frozen equations
$$-\mathrm{div}\left(c(x_0)|Dv|^{p-2}Dv\right) = 0$$
still enjoy good regularity estimates (this is Uraltseva-Uhlenbeck theory).
- In this case we can say that Du is Hölder continuous for some exponent. The results extend to general uniformly elliptic equations.

Non-differentiable functionals

- What happens when dealing for instance with classical model functionals as

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{h}(x, v)]\, dx$$

when $y \mapsto \mathtt{h}(\cdot, y)$ is not differentiable, but only Hölder?

- As $\mathtt{h}(\cdot)$ is not differentiable, the Euler-Lagrange equation

$$-\mathrm{div}\, \partial_z F(Du) + \partial_u \mathtt{h}(x, u) = 0$$

does not exists.

- This is done in classical papers by Giaquinta & Giusti from the beginning of the 80s.

Invent. math. 72, 285–298 (1983)

Inventiones mathematicae

Differentiability of Minima of Non-Differentiable Functionals

Mariano Giaquinta and Enrico Giusti
Istituto Matematico, Università di Firenze, Viale Morgagni 67/A, I-50134 Firenze, Italia

In this paper we shall consider the problem of the regularity of the derivatives of functions minimizing a variational integral

$$F(u;\Omega)=\int_\Omega f(x,u,Du)\,dx \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$, $u:\Omega\to\mathbb{R}^N$, $Du=\{D_\alpha u^i\}$ $\alpha=1,\dots,n$; $i=1,\dots,N$, and $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}$ is a Carathéodory function (i.e. measurable in x and continuous in u,p) satisfying

$$\lambda|p|^2-a\leqq f(x,u,p)\leqq\Lambda|p|^2+a,\qquad \lambda>0. \tag{1.2}$$

A local minimum for the functional F is a function $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that for every $\phi\in W^{1,2}(\Omega,\mathbb{R}^N)$ with $\operatorname{supp}\phi\subset\subset\Omega$ we have

$$F(u;\operatorname{supp}\phi)\leqq F(u+\phi;\operatorname{supp}\phi).$$

In a recent article [6], we have proved basic regularity results for the local minima of the functional (1.1). In the scalar case ($N=1$) we have shown that every local minimum of F with condition (1.2) is Hölder-continuous in Ω.

In the general case $N\geqq 1$ such a result cannot hold; we proved however that $Du\in L^{2+\sigma}_{\mathrm{loc}}$ for some $\sigma>0$. More generally these results hold for Q-minima (see [7]).

In this paper we investigate the regularity of the first derivatives of the minima of F, under additional hypotheses on the function $f(x,u,p)$. Roughly speaking, we assume that f is twice differentiable and strictly convex in p, and Hölder-continuous in (x,u). We remark that we do not assume the existence of the derivative f_u, and therefore our functionals are in general non differentiable.

As usual, our results will take different form in the scalar and in the vector case. When $N=1$, we prove that every local minimum has Hölder-continuous derivatives in Ω. When $N\geqq 1$, we obtain that for every local minimum u there exists an open set $\Omega_0\subset\Omega$, with mean $s(\Omega-\Omega_0)=0$ such that $u\in C^{1,\alpha}(\Omega_0,\mathbb{R}^N)$

Non-differentiable functionals

- These arguments can be carried through up to general functionals of the type

$$v \mapsto \int_\Omega \mathrm{c}(x, v) F(Dv)\, dx$$

therefore falling outside the realm of traditional Schauder estimates.

- In this case crucial use is made of the fact that u s a priori known to be Hölder continuous by other means, so that

$$x \mapsto \mathrm{c}(x, u(x))\,.$$

This extra information directly comes from growth conditions

$$|z|^p \lesssim F(x, v) \lesssim |z|^p + 1\,.$$

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$$|z|^p \lesssim F(x, v) \lesssim |z|^p + 1\,.$$

- The main point is that the comparison argument relies directly on minimality.

- Classical fact 1. For solutions to

$$-\text{div}\left(\mathrm{c}(x)|Du|^{p-2}Du\right)=0$$

and, more in general, uniformly elliptic equations with p-growth we have

$$\mathrm{c}(\cdot) \text{ is Hölder} \Longrightarrow Du \text{ is Hölder.}$$

- Classical fact 2. For minima of non-differentiable functionals

$$v \mapsto \int_\Omega [\mathrm{c}(x)|Dv|^p + \mathtt{h}(x,v)]\,dx$$

and, more in general, uniformly elliptic integrals with p-growth, we have

$$\mathrm{c}(\cdot), \mathtt{h}(x,\cdot) \text{ are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

Two classical issues

- Open problem 1. Schauder for nonuniformlly elliptic. For solutions to

$$-\mathrm{div}\,(\mathrm{c}(x)A(Du)) = 0\,, \qquad \mathcal{R}_A(z) \approx |z|^{q-p}$$

and more general, equations with polynomial nonuniform ellipticity

$$\text{coefficients (like } \mathrm{c}(\cdot)\text{) are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

- Open problem 2. Non-differentiable functionals. For minima of non-differentiable functionals

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{h}(x, v)]\,dx$$

and, more in general, of integrals with polynomial nonuniform ellipticity, it holds that

$$\text{coefficients (like } \mathtt{h}(\cdot,\cdot)\text{) are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

Publications results for "Author=(giaquinta) AND Reviewer=(lieberman)"

MR0749677 (85k:35077) Reviewed

Giaquinta, M. (I-FRNZ); Giusti, E. (I-FRNZ)

Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form.

J. Reine Angew. Math. 351 (1984), 55–65.

35J60 (35B65 49A22)

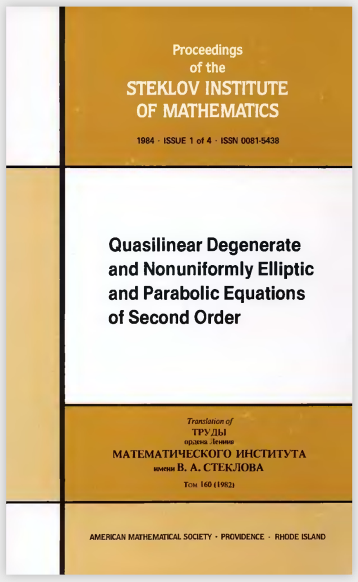
It is by now classical that solutions of the Dirichlet problem for a divergence form elliptic equation: $\operatorname{div} A(x,u,Du) = B(x,u,Du)$ in Ω, $u=\varphi$ on $\partial\Omega$, are $C^{k,\alpha}$ if $\varphi \in C^{k,\alpha}$ for any nonnegative integer $k \neq 1$, under suitable hypotheses on the coefficients A and B. Moreover, the reviewer has proved this result for $k=1$ [Comm. Partial Differential Equations 6 (1981), no. 4, 437–497; MR0612553] assuming, among other things, that A has Holder continuous first derivatives and that B is Holder continuous. The present paper provides an alternative proof of this regularity result for $k=1$ by means of some interesting techniques developed by the authors to study the regularity of minima of functionals [Invent. Math. 72 (1983), no. 2, 285–298; MR0700772]. Basically, they freeze the coefficient vector A at a point and then use a perturbation argument.

As well as being applicable to minimization problems, their method allows weaker smoothness hypotheses, namely, A is C^1 in Du and $C^{0,\alpha}$ in x and u, and B is bounded and measurable. In addition, bounded weak solutions of the Dirichlet problem can be studied directly when certain growth properties are imposed on the coefficients for large Du.

A comment needs to be made concerning their brief application to equations when their growth properties fail. As they point out, such equations fall under their considerations provided a global gradient bound has been established; however, this gradient bound has only been proved when A is differentiable with respect to all its arguments, and in many cases more smoothness of the coefficients is needed. The results of this paper are thus much more striking when applied to uniformly elliptic equations than to nonuniformly elliptic ones.

Reviewed by Gary M. Lieberman

Discussion. Ivanov's book.



Discussion. Ivanov's book.

terms of the majorants $\mathscr{E}_1$ and $\mathscr{E}_2$. Here it is also important to note that the structure of these conditions and the character of the basic a priori estimates established for solutions of (2) do not depend on the "parabolicity constant" of the equation. This determines at the outset the possibility of using the results obtained to study in addition boundary value problems for quasilinear degenerate parabolic equations. In view of the results of Ladyzhenskaya and Ural′tseva (see [80]), the proof of classical solvability of the first boundary value problem for equations of the form (2) can be reduced to establishing an a priori estimate of $\max_Q |\nabla u|$, where ∇u is the spatial gradient, for solutions of a one-parameter family of equations (2) having the same structure as the original equation (see §2.1).

Discussion. Ivanov's book.

THEOREM OF LADYZHENSKAYA AND URAL'TSEVA [83]. *Suppose that a function* $u \in C^2(\bar{\Omega})$ *satisfying the condition*

$$\max_{\bar{\Omega}}|u| \leqslant m, \quad \max_{\bar{\Omega}}|\nabla u| \leqslant M, \tag{2.5}$$

is a solution of (1.1) *in a bounded domain* $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$, *and that equation* (1.1) *is elliptic at this solution in the sense that*

$$a^{ij}(x, u(x), \nabla u(x))\xi_i\xi_j \geqslant \nu\xi^2, \qquad \nu = \text{const} > 0, \xi \in \mathbf{R}^n, x \in \bar{\Omega}. \tag{2.6}$$

Suppose that on the set $\mathscr{F}_{\Omega,m,M} \equiv \bar{\Omega} \times \{|u| \leqslant m\} \times \{|p| \leqslant M\}$ *the functions* $a^{ij}(x, u, p)$, $i, j = 1, \ldots, n$, *and* $a(x, u, p)$ *satisfy the condition*

$$|a^{ij}| + \left|\frac{\partial a^{ij}}{\partial x}\right| + \left|\frac{\partial a^{ij}}{\partial u}\right| + \left|\frac{\partial a^{ij}}{\partial p}\right| + |a| \leqslant M_1 \equiv \text{const} > 0 \quad \text{on } \mathscr{F}_{\Omega,m,M}. \tag{2.7}$$

Then there exists a number $\gamma \in (0,1)$, *depending only on* n, ν, M *and* M_1, *such that for any subdomain* Ω', $\bar{\Omega}' \subset \Omega$,

$$\|\nabla u\|_{C^\gamma(\bar{\Omega}')} \leqslant c_1, \tag{2.8}$$

where c_1 *depends only on* n, ν, M, M_1, *and the distance from* Ω' *to* $\partial\Omega$. *If the domain* Ω *belongs to the class* C^2 *and* $u = \varphi(x)$ *on* $\partial\Omega$, *where* $\varphi \in C^2(\bar{\Omega})$, *then*

$$\|\nabla u\|_{C^\gamma(\bar{\Omega})} \leqslant c_2, \tag{2.9}$$

Our technique is a nonlinear version of the well-known method of freezing the coefficients A^j at a point x_0, and then using a perturbation argument. A special form of De Giorgi's theorem is needed that requires linear growth for the A^i and at most a quadratic growth for B. However, the general case of coefficients A^i and B of arbitrary growth can easily be reduced to this once a gradient bound has been proved. This happens for instance for the minimal surface equation

$$\operatorname{div}\left\{\frac{Du}{\sqrt{1+|Du|^2}}\right\}=0$$

for which a gradient estimate for $C^{1,\alpha}$ boundary values has been proved in [10] (see also [13]).

- Solutions in a paper by Cristiana De Filippis (Parma) & Min. (Arxiv 2022).
- Catches both cases of non-differentiable functionals and equations with Hölder continuous coefficients.
- Introduces a hybrid perturbation approach.
- Crucial point in the proof is to get L^∞-bounds for the gradient.

Theorem (De Filippis & Min. #1)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{h}(x, v)] \, dx$$

where

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $|\mathtt{h}(x, y_1) - \mathtt{h}(x, y_2)| \lesssim |y_1 - y_2|^\alpha \,, \quad \alpha \in (0, 1]$
- *and*
$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n} \,.$$

Then Du is locally Hölder continuous in Ω.

Non-differentiable functionals 2

Theorem (De Filippis & Min. #2)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{g}(x, v, Dv) + \mathtt{h}(x, v)]\, dx$$

where $F(\cdot)$ and $\mathtt{h}(\cdot)$ are in Theorem 1, and

- $z \mapsto \mathtt{g}(\cdot, z)$ *is convex and* $|\partial_{zz}\mathtt{g}(\cdot, z)| \lesssim |z|^{\gamma-2}$
- $|\mathtt{g}(x, y_1, z) - \mathtt{g}(x, y_2, z)| \lesssim (|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha)\, |z|^\gamma$
- $\alpha + \gamma < p$
- *and*
$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha+\gamma}{p}\right)\frac{\alpha}{n}\,.$$

Then Du is locally Hölder continuous in Ω.

Theorem (De Filippis & Min. #3)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega c(x) F(Dv)\, dx$$

where

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $0 < c(\cdot) \in C^{0,\alpha}(\Omega)$
- *and*
$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^2 .$$

Then Du is locally Hölder continuous in Ω.

Theorem (Hopf, Caccioppoli & Schauder, reloaded)

If, in addition, we have

- $(|z|+1)^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim (|z|+1)^{q-2}\mathbb{I}_d$
- *and*
$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^2 .$$

Then
$$\mathrm{c}(\cdot) \in C^{0,\alpha}(\Omega) \Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega).$$

Non-differentiable functionals 4

Theorem (De Filippis & Min. #4)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega c(x,v)F(Dv)\,dx$$

where

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $0 < c(\cdot) \in C^{0,\alpha}(\Omega)\,, \quad \alpha \in (0,1]$
- $p > n$
- *and*
$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{n}{p}\right)\left(\frac{\alpha}{n}\right)^2.$$

Then Du is locally Hölder continuous in Ω.

- The role of the assumption

$$p > n$$

compensates the lack of a priori continuity of u which is known in the case $p = q$.

- When $p = q$ by De Giorgi's theory the local Hölder continuity of u is just implied by

$$|z|^p \lesssim F(x, v, z) \lesssim |z|^p + 1$$

with no convexity used.

- $p > n$ implies that u is Hölder continuous, but this fact is not used in the proof.
- This assumption could be optimal.

Integrability problems

- In the case of p-harmonic maps, distributional solutions u are so-called energy solutions when

$$\int_\Omega |Du|^p \, dx < \infty \, .$$

- This is necessary to use u-based test functions in the weak form

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \, , \qquad \varphi \approx u \, .$$

- In the case of (p, q)-equations being an energy solution means

$$\int_\Omega |Du|^q \, dx < \infty \, .$$

Being an energy solution is necessary for being regular:

Theorem (Colombo & Tione, JEMS 202?)

For every $p \neq 2$, there exists $\varepsilon \equiv \varepsilon(p) \in (0,1)$ and $u \in W^{1,p-1+\varepsilon}(B)$, $B \subset \mathbb{R}^2$ being the unit ball, such that u is a distributional solution to

$$-\mathrm{div}\left(|Du|^{p-2}Du\right) = 0$$

in B, but

$$\int_{\tilde{B}} |Du|^p \, dx = \infty \qquad \textit{for every ball } \tilde{B} \Subset B\,.$$

Construction based on Faraco's staircase laminate (a convex integration method).

(not really) "Safe" conjectures

Colombo & Tione's result disproves the following safe conjecture of Iwaniec & Sbordone (Crelle J., 1993):

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0\,. \tag{1.9}$$

In this case, however, there are enough arguments to $\boxed{\text{safely}}$ conjecture that

Conjecture 1. Every weak p-harmonic mapping $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with

$$r > \max\{1, p-1\}$$

belongs to $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.

(From Iwaniec & Sbordone original paper).

Equations: Two possible approaches

- Start with energy solutions $u \in W^{1,q}$.
- Prove existence of more regular solutions, selected via approximation methods.

Schauder estimates for equations

- We consider Dirichlet problems of the type

$$\begin{cases} -\operatorname{div} A(x, Du) = 0 & \text{in } \Omega \\ u \equiv u_0 & \text{on } \partial\Omega\,, \end{cases} \qquad u_0 \in W^{1,\frac{p(q-1)}{p-1}}(\Omega)\,,$$

- Under the assumptions

$$\begin{cases} |z|^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim |z|^{q-2}\mathbb{I}_d \\ |A(x_1,z) - A(x_2,z)| \lesssim |x_1 - x_2|^{\alpha}|z|^{q-1}\,. \end{cases}$$

- For nonuniformly elliptic problems, the concept of energy solutions is not well defined. This leads to prove existence-and-regularity theorems.

Theorem (De Filippis & Min. #5)

If

$$\frac{q}{p} \leq 1 + \frac{p-1}{10p}\left(\frac{\alpha}{n}\right)^2 \,,$$

then there exists a solution u to the above Dirichlet problem such that Du is locally Hölder continuous in Ω.

Theorem (Hopf, Caccioppoli and Schauder, reloaded)

If in addition, $p \geq 2$ and the problem is non-degenerate, i.e.,

$$(|z|+1)^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim (|z|+1)^{q-2}\mathbb{I}_d$$

we have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)\,.$$

- Proofs go via a novel use of Nonlinear Potential Theory via a version of certain nonlinear potentials originally introduced by Havin & Mazya and later on used by Hedberg & Wolff. These are combined with somme renormalized Caccioppoli type inequality.
- For those interested in the technical aspects, some hints on the proofs will be given tomorrow in the Barcelona Analysis Seminar.

More general functionals

- For general functionals of the type

$$v \mapsto \mathcal{F}(v, \Omega) := \int_\Omega F(x, Dv)\, dx$$

we face the possible occurrence of the Lavrentiev phenomenon

$$\inf_{v \in u_0 + W^{1,p}_0(B)} \mathcal{F}(v, B) < \inf_{v \in u_0 + W^{1,p}_0(B) \cap W^{1,q}(B)} \mathcal{F}(v, B)\,.$$

- This is a clear obstruction to regularity.

The Lebesgue-Serrin-Marcellini extension (relaxation)

- We introduce the relaxed functional

$$\overline{\mathcal{F}}(v,B) := \inf_{\{v_k\}\subset W^{1,q}(B)} \left\{ \liminf_k \mathcal{F}(v_k,B) : \ v_k \to v \text{ in } L^1(B) \right\}$$

  for every $v \in W^{1,p}(B)$.

- Accordingly, we introduce the Lavrentiev gap functional

$$\mathcal{L}_{\mathcal{F}}(v,B) := \overline{\mathcal{F}}(v,B) - \mathcal{F}(v,B)$$

  for every $v \in W^{1,1}(B)$ such that $\mathcal{F}(v,B) < \infty$; we set $\mathcal{L}(v,B) = 0$ otherwise.

- In the case $\mathcal{L}_{\mathcal{F}} \equiv 0$ a minimizer of $\mathcal{F}$ is also a minimizer of $\overline{\mathcal{F}}$.
- We therefore prove that every minimizer of $\overline{\mathcal{F}}$ is regular, and then deduce regularity of minima of $\mathcal{F}$.
- For more, see recent De Filippis' papers on JMPA (2022), JFA (2023, with B. Stroffolini).