Characterise and hinder turbulence in shear flows via nonlinear optimization

Stefania Cherubini[1]

Previous and current Ph.D. students: M. Farano, E. Parente, N. Ciola
Coworkers: J.-C. Robinet[2] & P. De Palma[1]

[1] DMMM, Politecnico di Bari
[2]DynFluid Laboratory, École Nationale Supérieure d'Arts et Métiers

stefania.cherubini@poliba.it

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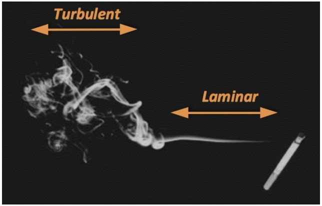
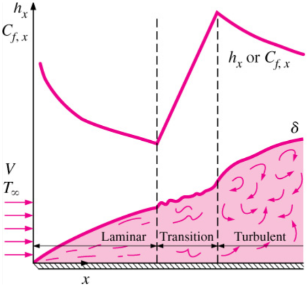
Laminar vs turbulent flows

- **Laminar flow:** fluid particles follow parallel layers
- **Turbulent flow:** flow variables experience chaotic behaviour





The *skin friction drag* is the resistant force exerted on an object moving in a fluid

- **Laminar flow → Low drag**
- **Transitional flow → Sudden drag increase**
- **Turbulent flow → Higher drag**

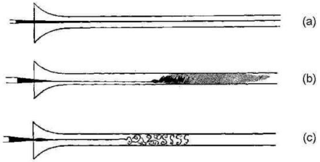
as measured by the skin friction coefficient

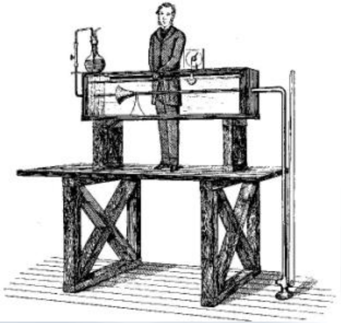
$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty v_\infty^2}$$

The Reynolds number

$$Re = \frac{UL\rho}{\mu}$$

- Low Re: **Laminar flow**
- Increasing Re: **Transition**
- High Re: **Turbulent flow**

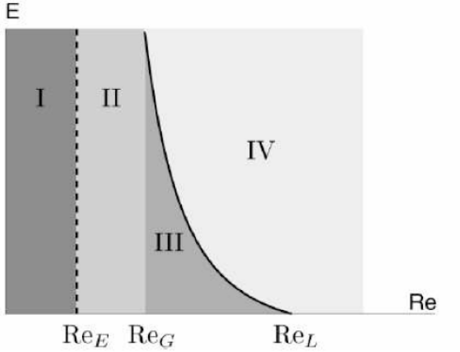




- *"Did steady motion hold up to a critical value and eddies come in?"*
- *"Did the eddies first make their appearance as small, and then increase gradually with the velocity, or did they come in suddenly?"*

O. Reynolds, An experimental investigation of the circumstance which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channel. PROCEEDINGS OF ROYAL SOCIETY OF LONDON, 35(224-226):84-99 (1883)

Stability of the laminar solution



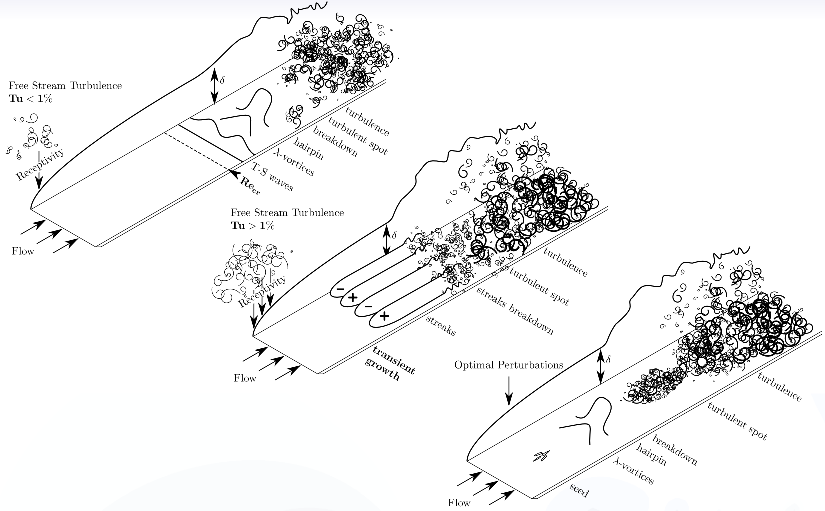
Flow	Re_G	Re_L
Pipe flow	≈ 2700	∞
Plane Couette flow	$\lesssim 415$	∞
Channel flow	$\lesssim 1600$	5772

I. $Re < Re_E$: Monotonic stability region

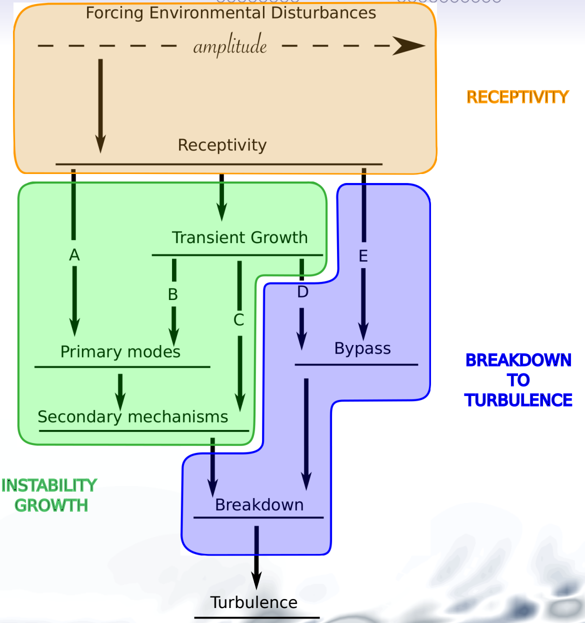
II. $Re_E < Re < Re_G$: Unconditional non monotonic stability region

III. $Re_G < Re < Re_L$: Conditional stability region

IV. $Re > Re_L$: Linear instability region



Transitional flow over a flat plate: (top) low-amplitude modal instablity (middle) nonmodal linear instability (bottom) high-amplitude, nonlinear transition scenario



Adapted from Morkovin et al., 1994

Transient energy growth

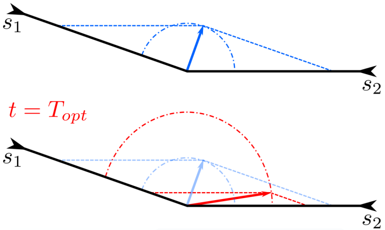
Let us consider the incompressible nondimensional Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{\nabla^2 \mathbf{u}}{Re} = 0 \qquad \nabla \cdot \mathbf{u} = 0$$

and decompose the instantaneous variables in a steady base flow $\mathbf{U}, P$ and a perturbation $\mathbf{u}'$. For the base flow:

$$\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \frac{\nabla^2 \mathbf{U}}{Re} = 0 \qquad \nabla \cdot \mathbf{U} = 0$$

$t = 0$



Injecting $\mathbf{u} = \mathbf{U} + \mathbf{u}'$ in the NS eq.s, using the above and linearising:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}' + \nabla p' - \frac{\nabla^2 \mathbf{u}'}{Re} = 0$$

This linear problem is written as:

$$\frac{\partial \mathbf{u}'}{\partial t} = A\mathbf{u} \rightarrow i\omega \hat{\mathbf{u}} = A\hat{\mathbf{u}}$$

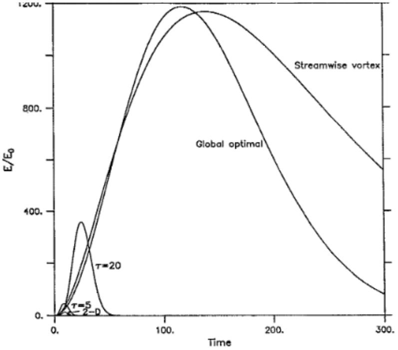
where $\mathbf{u}' = \hat{\mathbf{u}} \exp(i\omega t)$, whose eigendecomosition provides only asymptotically decaying modes

- **However, there can be a transient growth of the perturbation energy due to non-normality of the linearized NS operator (Farrell 1998)**

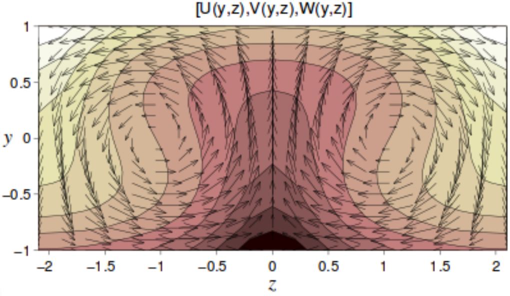
Optimal mechanism of transient growth in shear flows

Maximizing the energy of perturbations of the laminar flow at a given time T provides →

Weak ($O(1/Re)$) streamwise vortices inducing strong ($O(1)$) streamwise **streaks** *by transport of the base flow velocity*: **Lift-up mechanism**



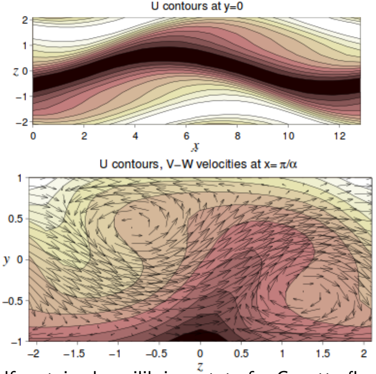
Butler and Farrell *Threedimensional optimal perturbations in viscous shear flow*, Phys. Fluids A (1992).



Landhal, A note on an algebraic instability of inviscid parallel shear flows, J. Fluid Mech., 1980

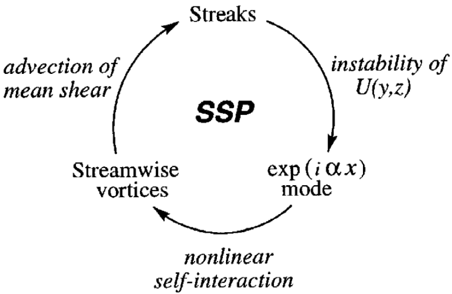
Self-sustained cycle

- **Secondary instability** bends the streaks + **non-linear effects** sustain the vortices

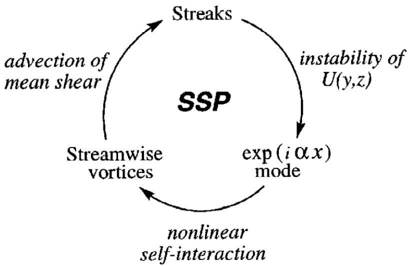


Self-sustained equilibrium state for Couette flow at $Re = 150$ (Waleffe 1998).

- Waleffe (1995) → *Self-sustained process*



Self Sustained Process (SSP)



F. Waleffe. On a self-sustaining process in shear flows. Physics of Fluids 9.4 (1997): 883-900.

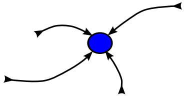
This cycle sustains relative attractors such as:

- equilibria
- travelling waves
- periodic orbits of period T
- relative periodic orbits
- ...

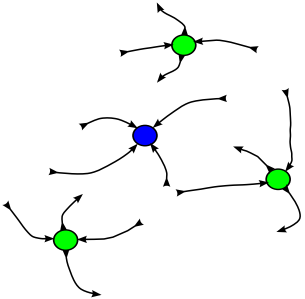
The turbulent region consists of numerous invariant solutions. A turbulent trajectory performs a walk through this forest, resulting in a web of homoclinic and heteroclinic connections [G. Kawahara et al. 2011].

Olga Ladyzhenskaya demonstrated the global existence of stationary solutions to NavierStokes equations in a bounded domain for the Dirichlet boundaries: the trajectories can "eventually" come and remain very close to a particular solution, "the attractor" (lectures series in Rome on "Attractors for Semigroups and Evolution Equations").

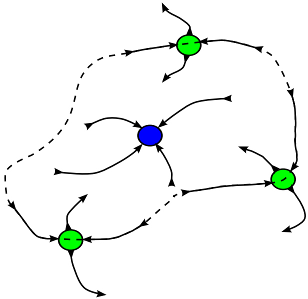
Chaotic saddle



Chaotic saddle



Chaotic saddle



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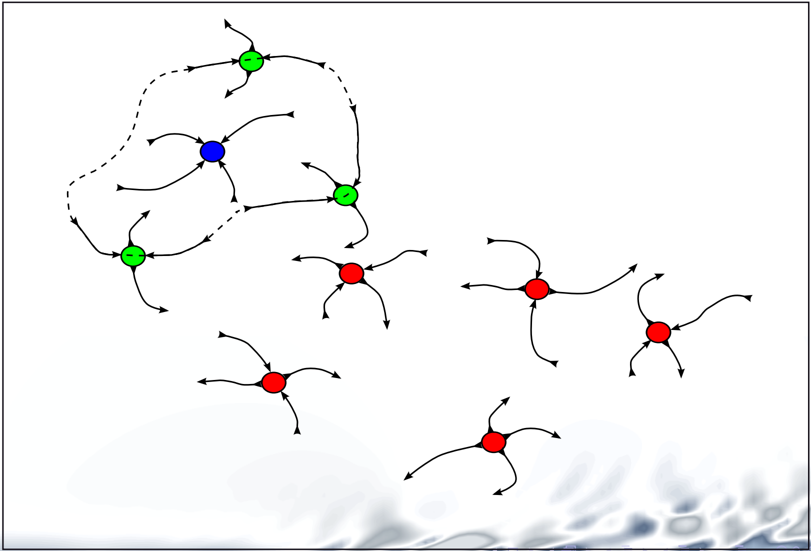
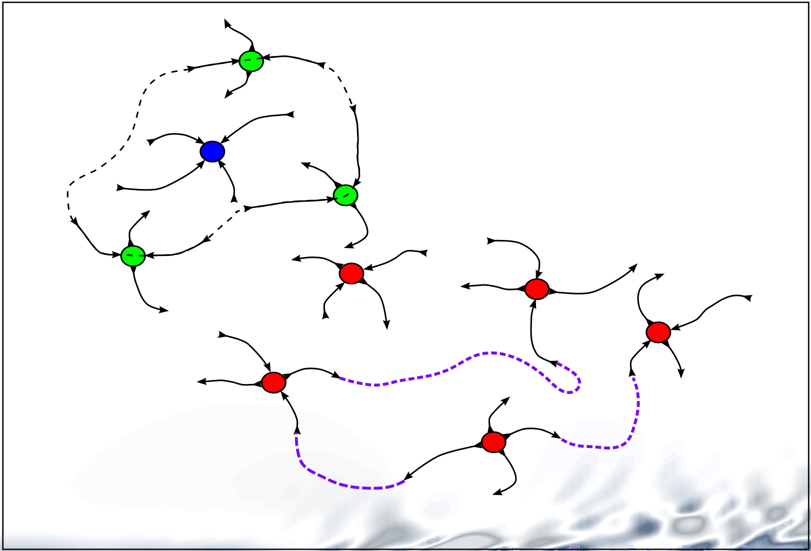
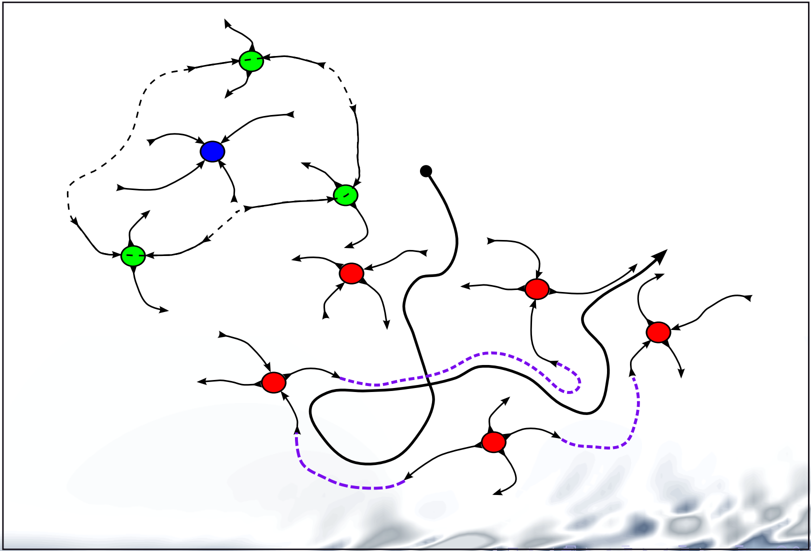


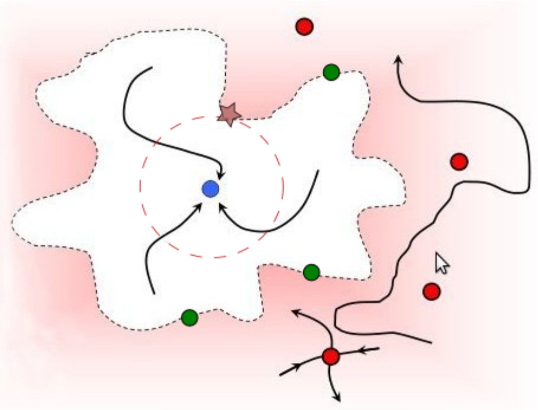
Chaotic saddle



Chaotic saddle



Motivation



Source: B. Eckhardt 2011.

AIM 1: identifying the perturbations of minimal energy on the laminar-turbulent separatrix

1. *Being on the edge of chaos they should be* **attracted by the edge state** *for* $t \to \infty$
2. *For reaching the edge state with minimal initial energy, they experience an* **optimal energy growth**

AIM 2: use nonlinear optimization for explaining highly energetic and dissipative events in turbulent flows

High dissipation events (bursts)

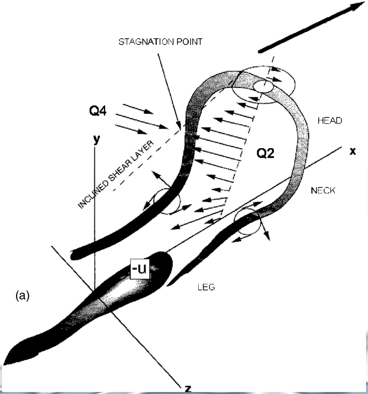
Which is the role of bursting events in this scenario?

Transitional and turbulent shear flows are characterized by:

- The *self sustained process*, mostly characterized by streaky structures
- Highly energetic and dissipative *bursting events*, sometimes in the form of hairpin vortices [Adrian et al., 2000]

Streaky structures are explained by a **linear optimization**, maximising the energy of the fluctuations

Can a nonlinear optimization provide an explanation for the bursting and extreme dissipation events?



Summary

- **Problem formulation**
- Results

1. Mapping the edge of chaos
    - Minimal seeds for fully-developed turbulence
    - Minimal seeds for localised turbulence
2. Investigating the turbulent saddle
    - Modeling recurrent coherent structures in turbulence
    - Explaining extreme dissipation events

Lagrange multipliers method

Goal

Find the initial perturbation $\mathbf{u}_0 = (u_0, v_0, w_0)$ providing the largest energy for a given target time, T_{opt}, and initial energy E_0.

- **Objective function**: the integral kinetic energy at target time T:
$$E(T) = \frac{1}{2V}\int_0^{L_x}\int_0^{L_y}\int_{-Z}^{Z}\left(u^2+v^2+w^2\right)dxdydz = \frac{1}{2V}\{\mathbf{u}(T)\cdot\mathbf{u}(T)\}.$$

- **Constraints**: 1) 3D nonlinear NS equations; 2) Given initial energy E_0
$$\mathfrak{L} = E(T) - \int_0^T\left\{\mathbf{u}^\dagger\cdot\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p-\frac{\nabla^2\mathbf{u}}{Re}\right)\right\}dt$$
$$-\int_0^T\left\{p^\dagger\nabla\cdot\mathbf{u}\right\}dt-\lambda\left(\frac{E_0}{E(0)}-1\right).$$

- **First variation of $\mathfrak{L}$ is set to zero** → adjoint equations plus compatibility condition.
- $\delta\mathfrak{L}/\delta\mathbf{u}_0$ is iteratively nullified by means of gradient rotation algorithm [Foures et al. 2013].

Problem formulation

$$\frac{\delta \mathfrak{L}}{\delta p^\dagger} = \frac{\partial u_i}{\partial x_i} = 0 \text{ Direct equations}$$

$$\frac{\delta \mathfrak{L}}{\delta u_k^\dagger} = \frac{\partial u_k}{\partial t} + \frac{\partial (u_k U_j)}{\partial x_j} + \frac{\partial (U_k u_j)}{\partial x_j} + \frac{\partial p}{\partial x_k} - \frac{1}{Re}\frac{\partial^2 u_k}{\partial x_j^2} + \frac{\partial (u_k u_j)}{\partial x_j} = 0$$

$$\frac{\delta \mathfrak{L}}{\delta p} = \frac{\partial u_i^\dagger}{\partial x_i} = 0 \text{ Adjoint equations}$$

$$\frac{\delta \mathfrak{L}}{\delta u_k} = \frac{\partial u_k^\dagger}{\partial t} + \frac{\partial \left(u_k^\dagger U_j\right)}{\partial x_j} - u_i^\dagger \frac{\partial U_i}{\partial x_k} + \frac{\partial p^\dagger}{\partial x_k} + \frac{1}{Re}\frac{\partial^2 u_k^\dagger}{\partial x_j^2} + \frac{\partial \left(u_k^\dagger u_j\right)}{\partial x_j} + u_i \frac{\partial u_i^\dagger}{\partial x_k} = 0$$

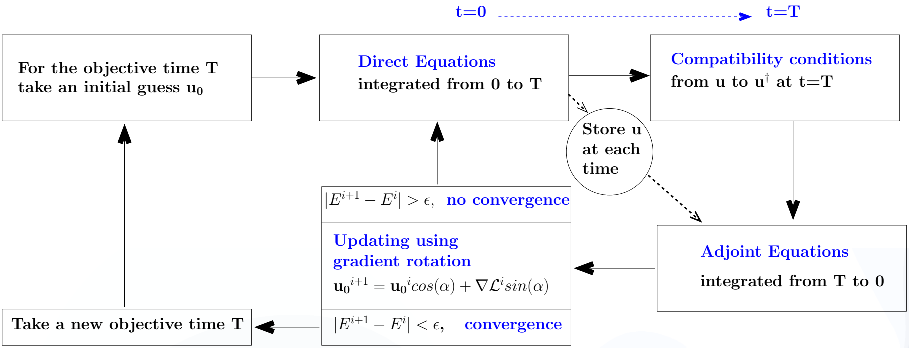
$$\frac{\delta \mathfrak{L}}{\delta u_k(T)} = \frac{u_h(T)}{E(0)} - u_h^\dagger(T) = 0, \text{ Compatibility conditions}$$

$$\frac{\delta \mathfrak{L}}{\delta u_k(0)} = -\frac{E_p(t) - \lambda E_0}{E_0^2} u_k(0) + u_h^\dagger(0) = 0 \text{ Gradient w.r.t. the initial perturbation}$$

$$\frac{\delta \mathfrak{L}}{\delta \lambda} = \frac{E_0}{E(0)} - 1 = 0 \text{ Initial energy constraint}$$

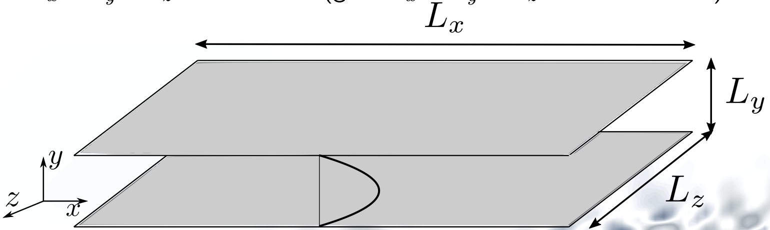
Lagrange multipliers method

Algorithm



Numerical code and flow case

- *Channelflow* open source code developed by J. F. Gibson (Gibson et al. (2008)) solving the perturbative incompressible NS equations
- Spatial discretization $\longrightarrow$ Fourier $\times$ Chebyshev $\times$ Fourier
- Time integration $\longrightarrow$ third-order semi-implicit scheme
- Flow in a channel limited by infinite plates driven by a constant pressure gradient
- Computational domain (periodic b.c. in x, z): $L_x \times L_y \times L_z = 2\pi \times 2 \times \pi$ (grid $N_x \times N_y \times N_z = 64 \times 129 \times 32$)



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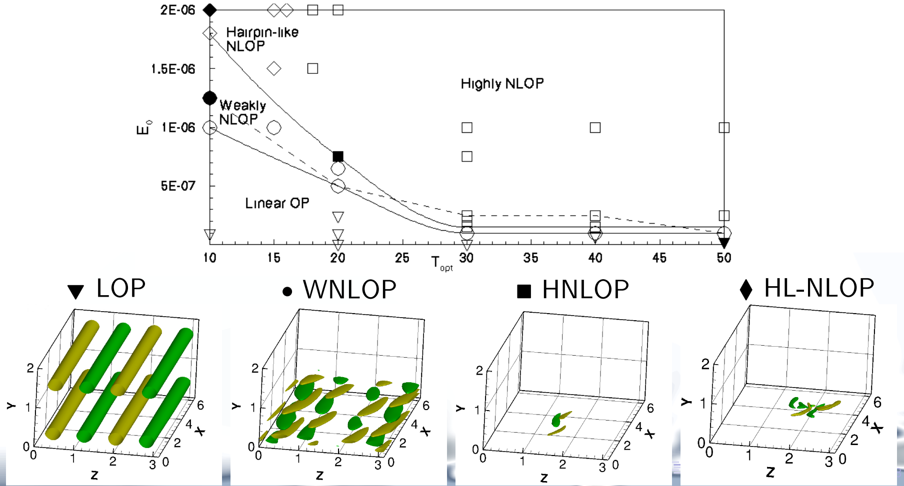
Nonlinear optimal perturbations in channel flow

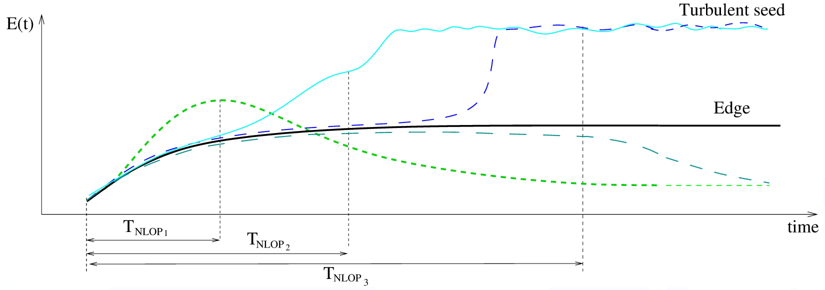
Optimizing at different T_{opt}, E_0 for $Re = 4000$, provides different optimals.



Optimize then bisect procedure

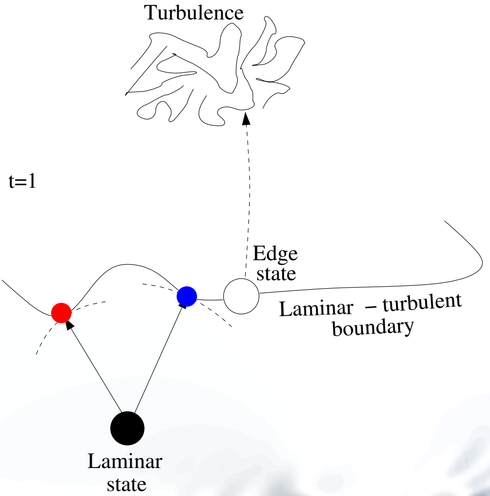
Minimal seed approach:

1. *set an initial energy E_0 and a (large) target time T*
2. *OPTIMIZE a functional linked to turbulence in a* **non-linear framework**
3. *BISECT the value of the initial energy until it approaches the edge of chaos*



The MINIMAL SEED is the disturbance of minimal energy asymptotically approaching the edge state → *its energy is the threshold energy for transition!*

Energy minima on the edge of chaos



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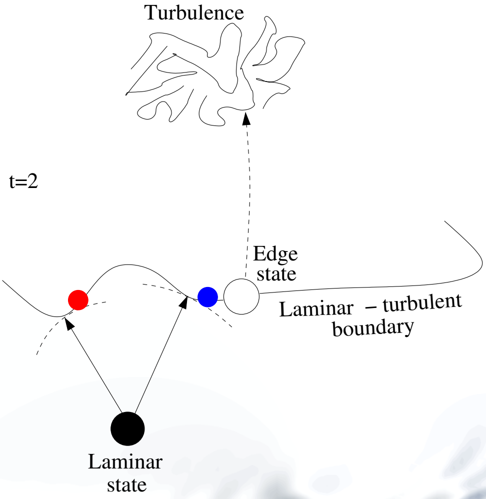

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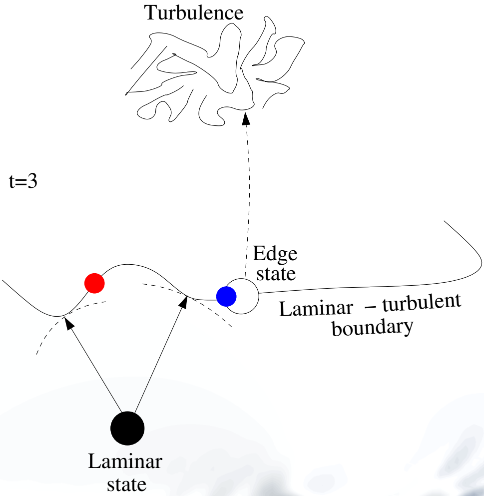

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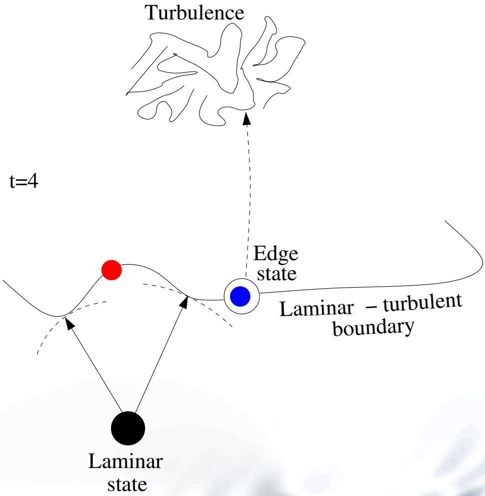

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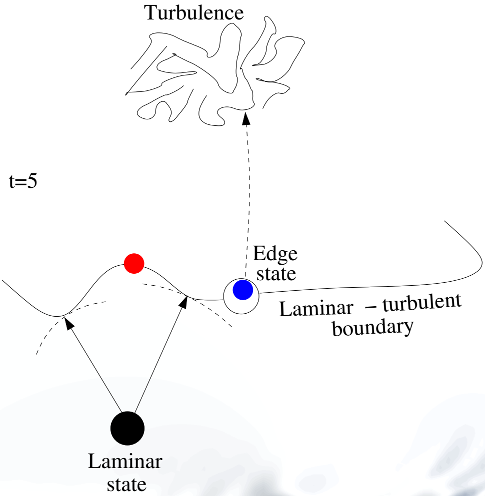

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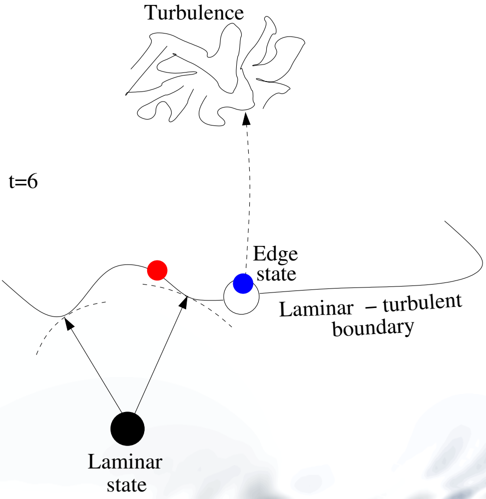

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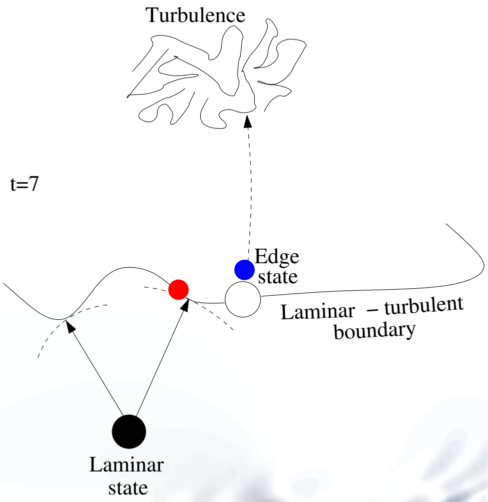

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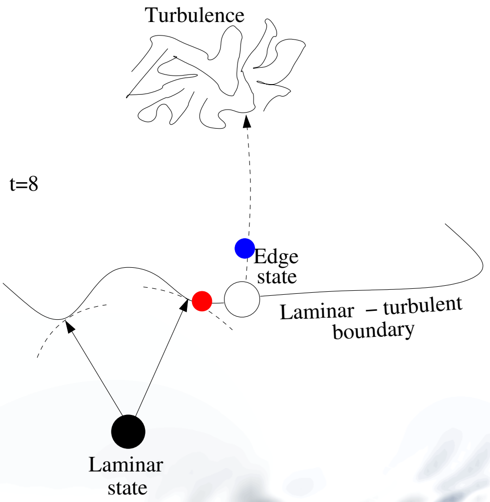

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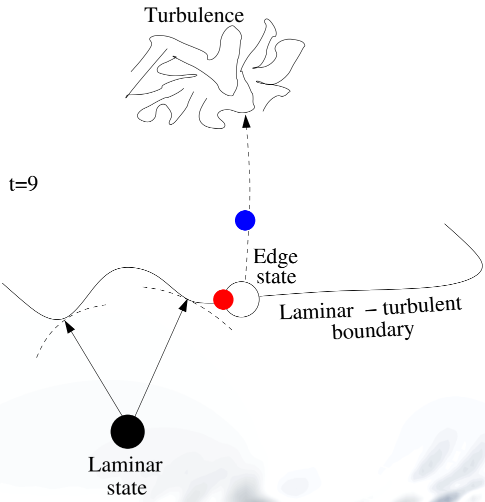

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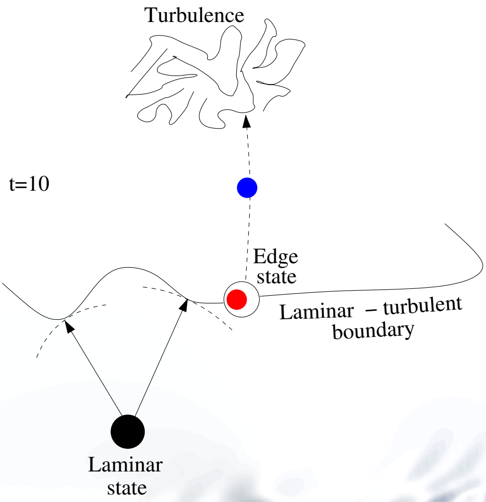

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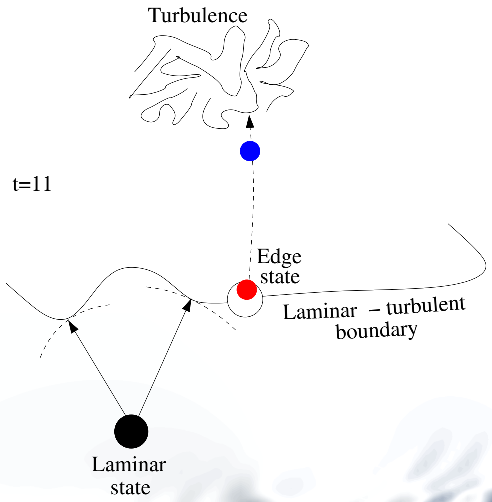

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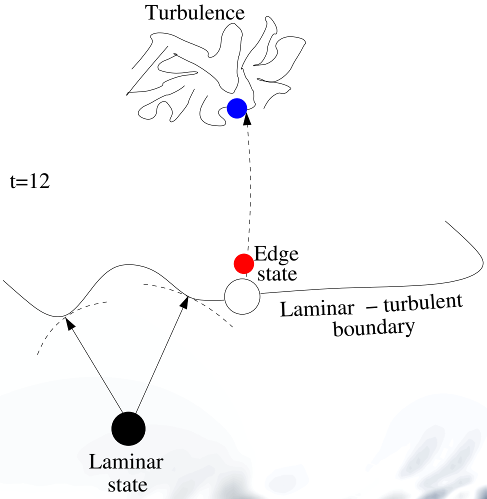

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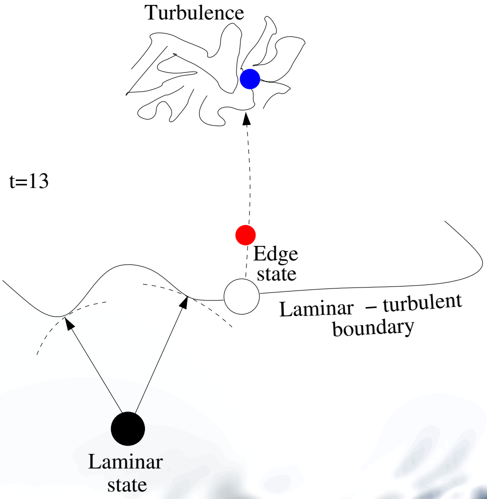

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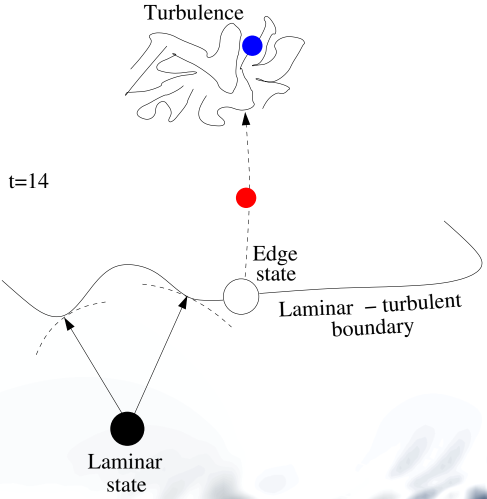

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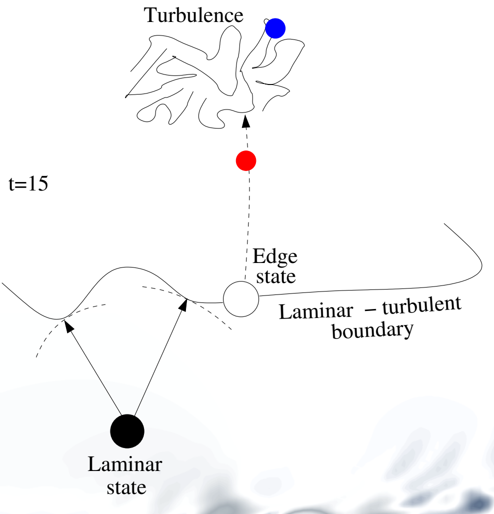
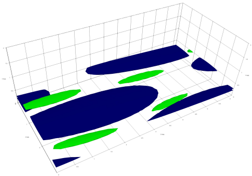
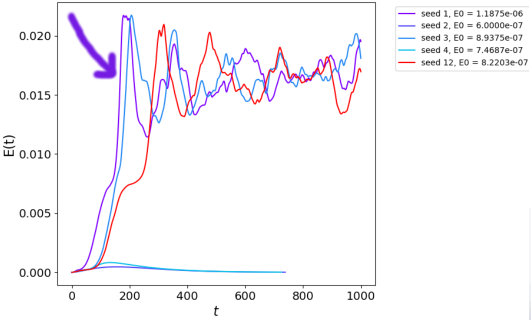


Minimal Seed for a small channel

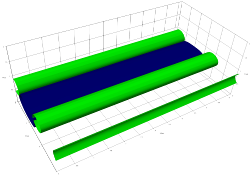
- Computational domain: $2\pi \times 2 \times \pi$
- Target time $T = 200$
- $E_0 = 1.1875 \times 10^{-6}$

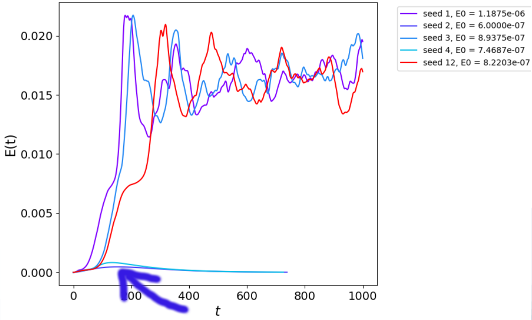




Minimal Seed for a small channel

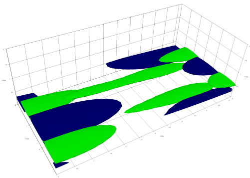
- Computational domain: $2\pi \times 2 \times \pi$
- Target time $T = 200$
- $E_0 = 6.0000 \times 10^{-7}$

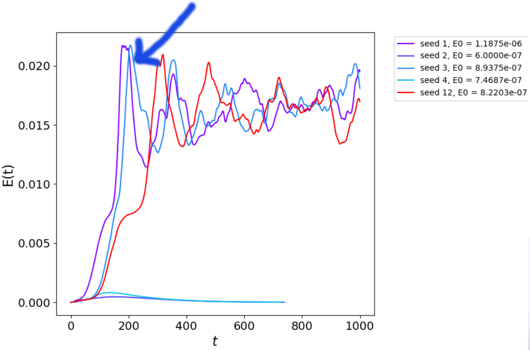




Minimal Seed for a small channel

- Computational domain: $2\pi \times 2 \times \pi$
- Target time $T = 200$
- $E_0 = 8.9375 \times 10^{-7}$





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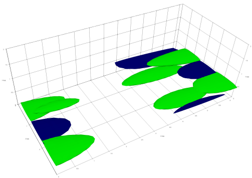
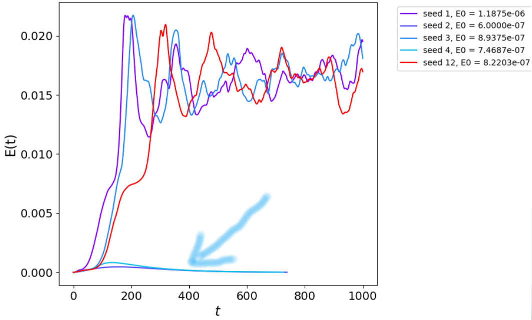
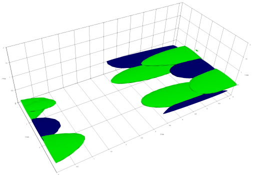
- Computational domain: $2\pi \times 2 \times \pi$
- Target time $T = 200$
- $E_0 = 7.4687 \times 10^{-7}$

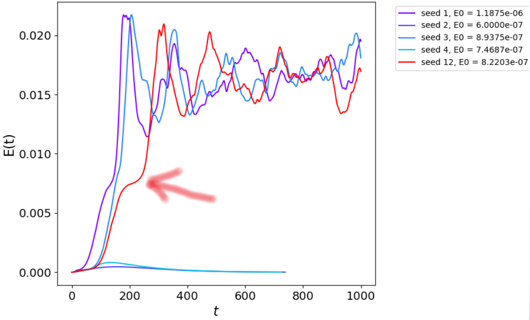


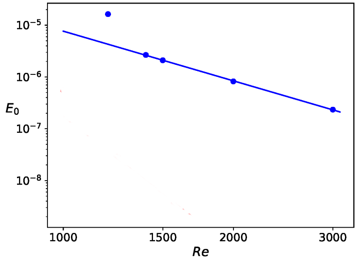


Minimal Seed for a small channel

- Computational domain: $2\pi \times 2 \times \pi$
- Target time $T = 200$
- $E_0 = 8.2203 \times 10^{-7}$

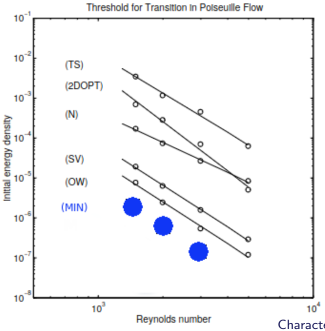


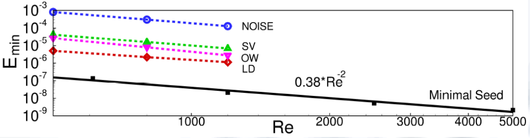




Minimal energy w.r.t Re

- E_{min} is much smaller than the literature values!
- $E_{0_{min}} \propto Re^{-3.2}$ (compared to $Re^{\approx -2}$ of other scenarios (Reddy et al 1998)
- Plane Couette flow: $E_{0_{min}} \propto Re^{-2.7}$ (Duguet et al 2013)
- Asymptotic suction boundary layer $E_{0_{min}} \propto Re^{-2}$ (Cherubini et al 2015)





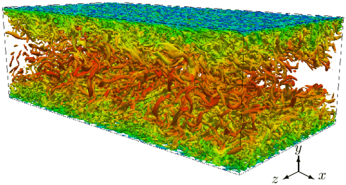
Summary

- Problem formulation
- Results

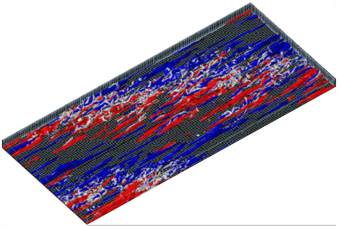
1. **Mapping the edge of chaos**
   - Minimal seeds for fully-developed turbulence
   - **Minimal seeds for localised turbulence**
2. Investigating the turbulent saddle
   - Modeling recurrent coherent structures in turbulence
   - Explaining extreme dissipation events

Localised turbulence

Fully-developed turbulence



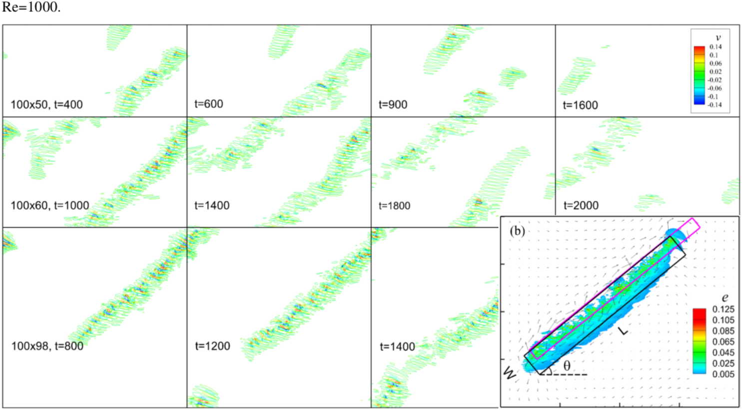
Localised turbulence



- Observed experimentally (Carlson 1982) and numerically (Tsukahara 2005).
- Range of existence:
  - $Re > 660$: transient bands (Xiong 2015)
  - $Re > 960 - 1000$: sustained bands (Gome 2020)
  - $Re > 3900$: fully developed turbulence or laminar state
  - $Re > 5772$: linear instability threshold!!

Flow features

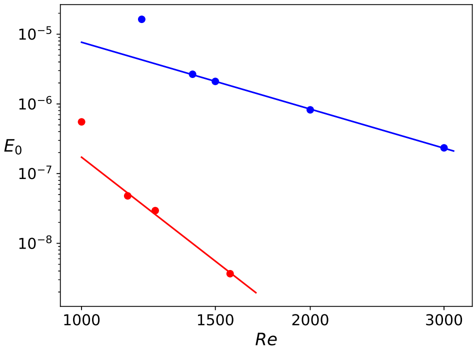
- small scale flow → streaks and vortices
- large scale flow → responsible for the obliqueness (Duguet & Schlatter (2013))



Minimal seed for different domains

Large-domain: $L_x \times L_y \times L_z = 250 \times 2 \times 125$ (Shimizu & Manneville (2019),
Number of grid points: $N_x \times N_y \times N_z = 1024 \times 65 \times 1024$

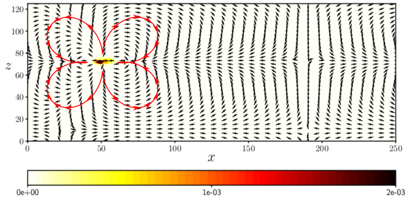
- Small domain:
  $E_{0min} \propto Re^{-3.2}$
- Large domain:
  $E_{0min} \propto Re^{-8.5}$!
- The scaling law changes depending of the domain!



Minimal seed structure

Parameters

$Re = 1150$, $U_{bulk} = 2/3$, $T = 100$,
$E_{0_{min}} = 4.7 \times 10^{-8}$



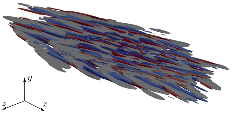
- Isocontour of cross-flow energy $E_{cf} = (1/2) \int (v^2 + w^2)$
- Large scale flow $\rightarrow$ $\overline{u_i} = \int u_i dy$

Two scales:

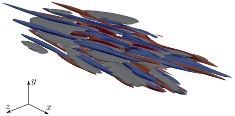
- Small-scale flow inside the spot
- Large scale flow $\rightarrow$ quadrupolar vortices (Lemoult et al. (2014))

Structure for different Re

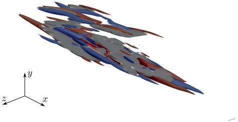
$Re = 1000,\ E_{0_{min}} = 5.5 \times 10^{-7},\ t = 0$



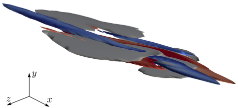
$Re = 1150,\ E_{0_{min}} = 4.7 \times 10^{-8},\ t = 0$



$Re = 1250,\ E_{0_{min}} = 2.9 \times 10^{-8},\ t = 0$



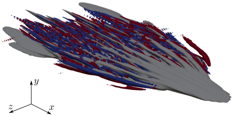
$Re = 1568,\ E_{0_{min}} = 3.6 \times 10^{-9},\ t = 0$



- Alternating streaks with elongated vortices
- $\uparrow\ Re\ \uparrow\ localization$
- Wave-packets with an arrow-shaped structure

Structure for different Re : evolution at $t = T$

$Re = 1000,\ E_{0_{min}} = 5.5 \times 10^{-7},\ t = 100$



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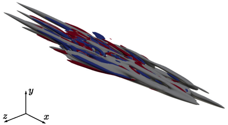
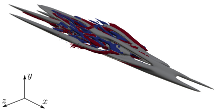
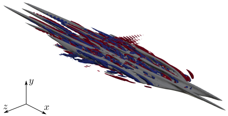


$Re = 1250,\ E_{0_{min}} = 2.9 \times 10^{-8},\ t = 100$



$Re = 1568,\ E_{0_{min}} = 3.6 \times 10^{-9},\ t = 150$

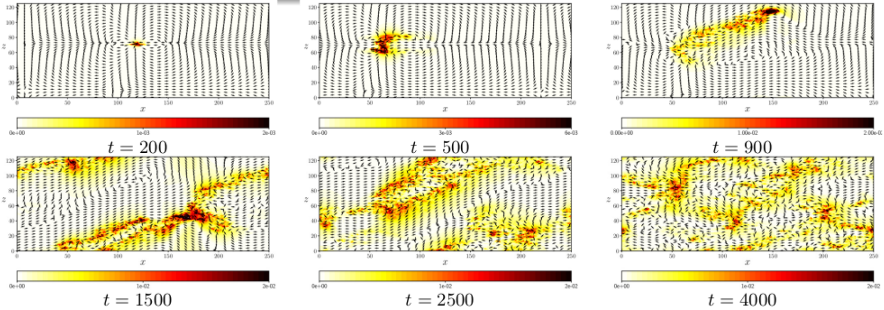


- Alternating streaks with elongated vortices
- $\uparrow\ Re\ \uparrow\ localization$
- Wave-packets with an arrow-shaped structure

Results: Minimal seed evolution in time

Parameters

$Re = 1150,\ U_{bulk} = 2/3,\ T = 100,\ E_{0_{min}} = 4.7 \times 10^{-8}$



- $t = [0, 500]$: rapid symmetry breaking
- $t = 900$: oblique turbulent band with $\alpha = 28^\circ$
- $t > 1500$: band self-interaction

Similar evolution injecting a "seed" of the turbulent bands (Tao & Xiong (2013); Xiong et al.(2015))

Results: Minimal seed evolution in time

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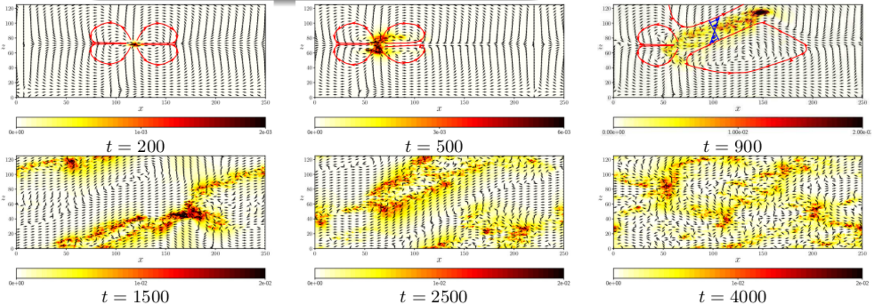

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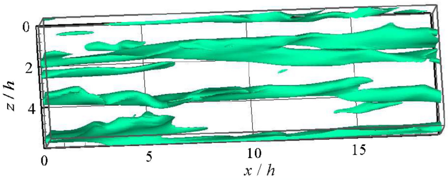
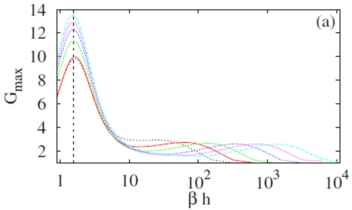
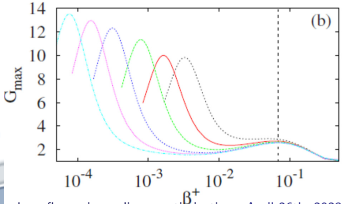
Turbulent flows: the self sustained process

- Turbulent wall-bounded flows are characterised by **coherent** streaky structures connected to the self-sustained process.
- This dynamics is present at inner (wall-close) and outer (center-channel) scales (Hwang & Cossu)







Streaky structures can be explained by a **linear local optimization**: two peaks in the energy gain curves varying the spanwise wave length [Pujals et al. 2009].

Nonlinear optimal perturbation

- On the other hand, bursting events are not part of the SSP process, but they are associated with a sudden energy growth.
- Can a nonlinear optimization in a turbulent framework provide an explanation for the presence of such energetic events?

Optimization problem

Maximize the energy growth of the perturbations around the mean flow $E(t) = \int_V \left(\widetilde{u}^2 + \widetilde{v}^2 + \widetilde{w}^2\right) dV$ in a given time horizon, for a finite initial energy → Lagrange multipliers method / Direct adjoint loop [Cherubini et al. 2010, 2011].

$$\mathfrak{L} = \frac{E(T_{opt})}{E(0)} + \int_T \int_V \left(\mathbf{u}^\dagger \cdot \widetilde{NS}\right) dV\, dt - \int_T \int_V \mathrm{p}^\dagger \left(\nabla \cdot \tilde{\mathbf{u}}\right) dV\, dt - \lambda \left(\frac{E(0)}{E_0} - 1\right)$$

Reynolds-averaged Navier-Stokes framework

Introducing in the NS equations $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{\nabla^2 \mathbf{u}}{Re}$ the decomposition

$$\mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}} \qquad p = \overline{p} + \tilde{p} \tag{1}$$

and time-averaging, we obtain the RANS equations for the mean flow field $\overline{\mathbf{u}}$, $\overline{p}$:

$$\overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} = -\nabla \overline{p} + \frac{1}{Re} \nabla^2 \overline{\mathbf{u}} - \nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}} \tilde{\mathbf{u}}} \right)}_{\tau},$$

Introducing again the initial decomposition and subtracting the above equation, we obtain the perturbative RANS equations

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \mathbf{U} \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{U} = -\nabla \tilde{\mathrm{p}} + \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} + \nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}} \tilde{\mathbf{u}}} \right)}_{\tau},$$

Olga Ladyzhenskaya proposed a model for the stress tensor τ, prescribing that it depends on the symmetric part $Du = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u}^T)$ of the gradient of the velocity in a nonlinear polynomial way, with p-rate of growth, (with $p = 3$, one gets the famous Smagorinsky model)

Turbulent optimization problem

Maximizing the energy growth of the perturbations **around the mean flow** $E(t) = \int_V \left(\widetilde{u}^2 + \widetilde{v}^2 + \widetilde{w}^2\right) dV$ at a given T, for a finite initial energy

$$\mathfrak{L} = \frac{E(T_{opt})}{E(0)} + \int_T \int_V \left(\mathbf{u}^\dagger \cdot \widetilde{NS}\right) dV\,dt - \int_T \int_V \mathrm{p}^\dagger \left(\nabla \cdot \tilde{\mathbf{u}}\right) dV\,dt - \lambda \left(\frac{E(0)}{E_0} - 1\right)$$

The Reynolds averaged Navier-Stokes equations are considered as a constraint:

$$\mathbf{u} = \mathbf{U} + \tilde{\mathbf{u}}, \qquad \mathrm{p} = \mathrm{P} + \tilde{\mathrm{p}}, \tag{2}$$

- $\mathbf{U}$, P → mean flow field (long time averaged $\overline{\bullet}$);

$$\mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \mathrm{P} + \frac{1}{Re} \nabla^2 \mathbf{U} - \nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}}\tilde{\mathbf{u}}}\right)}_{\tau},$$

- $\tilde{\mathbf{u}}$, $\tilde{\mathrm{p}}$ → perturbations;

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \mathbf{U} \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{U} = -\nabla \tilde{\mathrm{p}} + \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} + \nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}}\tilde{\mathbf{u}}}\right)}_{\tau},$$

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- $\mathbf{U}$, P → mean flow field (long time averaged $\overline{\bullet}$);

$$\nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}}\tilde{\mathbf{u}}}\right)}_{\tau} = -\mathbf{U} \cdot \nabla \mathbf{U} - \nabla \mathrm{P} + \frac{1}{Re}\nabla^2 \mathbf{U} = f(\mathbf{U}, Re),$$

- $\tilde{\mathbf{u}}$, $\tilde{\mathrm{p}}$ → perturbations;

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \mathbf{U} \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{U} = -\nabla \tilde{\mathrm{p}} + \frac{1}{Re}\nabla^2 \tilde{\mathbf{u}} + \nabla \cdot \underbrace{\left(\overline{\tilde{\mathbf{u}}\tilde{\mathbf{u}}}\right)}_{\tau},$$

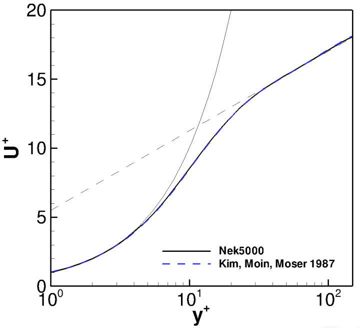
- Reynolds stress tensor τ, computed from a DNS.
- No approximation and closure model for τ.
- **Fully nonlinear problem**: Not suitable for linear analysis ($\tilde{\mathbf{u}} \approx 0$).

Nonlinear optimization

Test case

Turbulent channel flow for $Re_\tau = 180$ in a domain $Lx \times Ly \times Lz = 4\pi h \times 2h \times 2\pi h$ [J. Kim et al., 1987]

Target time → lifetime of the coherent structures:



Two time scales depending to the eddy turnover time (T_e)

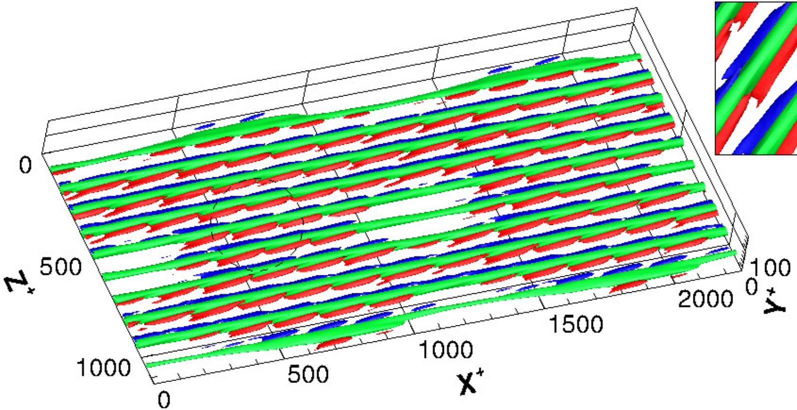
- $T_e^+(y^+ \approx 19) = 80$ [K.M. Butler & B. Farrel, 1993] (Inner scales) corresponding to $T_{in} = 8.16$
- $T_e(y \approx h) = T_{out} = 31.12$ (Outer scales)

where $^+$ indicates variables expressed in *inner units*, non-dimensionalized by the viscous length scale $\delta_v = \mu/(\rho u_\tau)$ with $u_\tau = \sqrt{\frac{\mu}{\rho}\frac{\partial u}{\partial u}|_{y=0}}$ being the friction velocity and $Re_\tau = \rho u_\tau h/\mu$

Inner scales

Validation

Elongated streaks flanked by counter rotating vortices → SSP



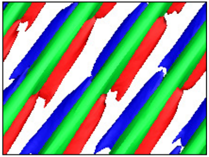


Figure: Shape of the optimal perturbation for T_{inner} and $E_0 = 10^{-2}$. Isosurface of the negative streamwise velocity (green) and Q-criterion coloured by contours of streamwise vorticity (positive blue, negative red).

Outer scale

Highly deformed streaks with hairpin vortices → **Burst?**

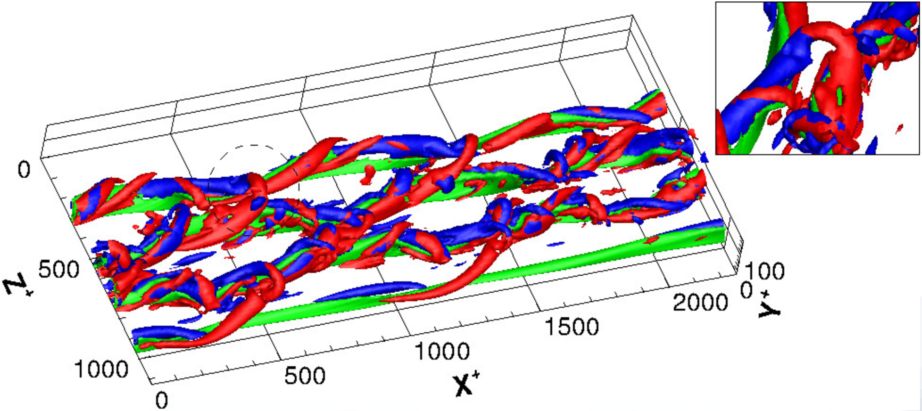
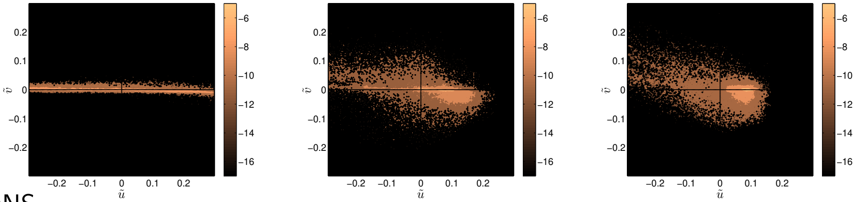


Figure: Shape of the optimal perturbation for T_{outer} and $E_0 = 10^{-2}$. Isosurface of the negative streamwise velocity (green) and Q-criterion coloured by contours of streamwise vorticity (positive blue, negative red).

Probability density function

How is a burst event characterized? → **Ejections and sweeps**

OPT



DNS

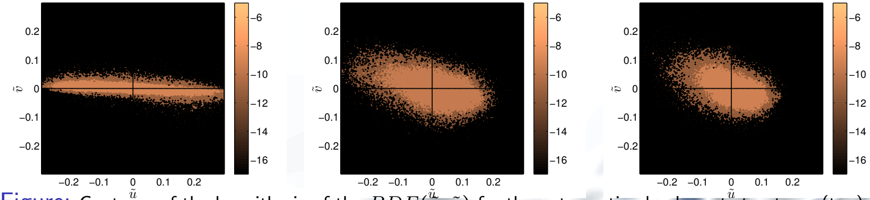


Figure: Contours of the logarithmic of the $PDF(\tilde{u}, \tilde{v})$ for the outer optimal coherent structures (top) and DNS (bottom) for $y^+ = 10, 50, 100$ from left to right.

Comparison with the DNS

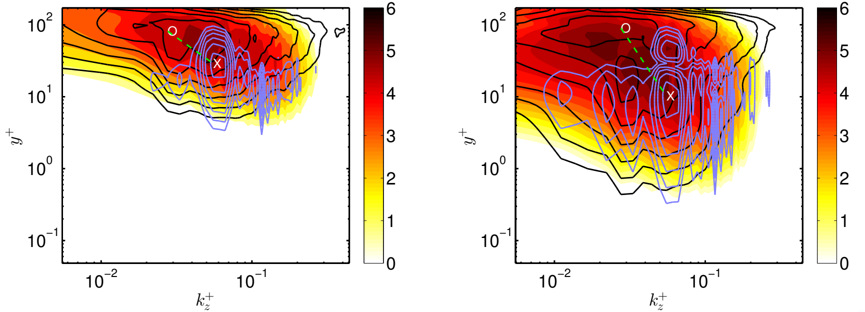


Figure: Logarithm of the premultiplied power energy spectra in the spanwise direction plotted as a function of the wall normal distance y^+ (from left to right: wall-normal and spanwise velocity component).

- Inner optimal: only a small portion of the broadband range of wavenumbers found by the DNS.
- Outer optimal: almost overlapped to that extracted from the DNS → *it represents a good model of the coherent turbulent dynamics!*

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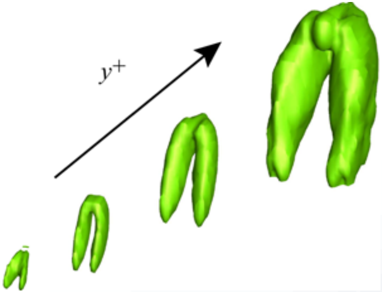
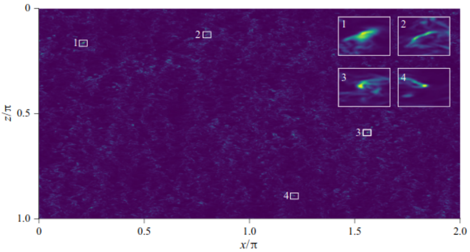
Turbulent flows: extreme dissipation events

- Turbulent wall-bounded flows experience extreme dissipation events, where the velocity gradients locally increase by orders of magnitude
- The 1% most dissipative events, analysed with conditional spacetime proper orthogonal decomposition, provides symmetric "hairpin" structures



From Hack & Schmidt 2020



Can this extreme mechanism be recovered by a nonlinear optimization of a turbulent flow field?

Formulation

Goal

Find the perturbation $\tilde{\mathbf{u}}_0$ to a given turbulent snapshot at time t_0 having initial energy E_0 and providing the largest turbulent mean dissipation in the time interval $(t_0, t_0 + T)$, .

- **Objective function**: the integral turbulent dissipation:

$$\mathcal{J} = \frac{1}{TV} \int_{t_0}^{t_0+T} \int_V \left(\frac{1}{Re} \nabla \mathbf{u}' : \nabla \mathbf{u}' \right) dV\, dt = \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{Re} \left\{ \nabla \mathbf{u}' : \nabla \mathbf{u}' \right\} dt,$$

  where $\mathbf{u}'$ is the velocity fluctuation.
- **Flow decomposition**: at each instant $t \geq t_0$ the flow can be written as the superposition of the unperturbed flow (computed through standard channel DNS) and the time-varying perturbation:

$$\mathbf{u}_p(\mathbf{x}, t) = \mathbf{u}_u(\mathbf{x}, t) + \tilde{\mathbf{u}}(\mathbf{x}, t).$$

- **Constraints**: 1) The 3D nonlinear NS equations constrain the perturbed flow $\mathbf{u}_p(\mathbf{x}, t)$; 2) The initial energy E_0 constrains $\tilde{\mathbf{u}}_0$:

$$\mathfrak{L} = \mathcal{J} - \int_0^T \left\{ \mathbf{u}^\dagger \cdot \left(\frac{\partial \mathbf{u}_p}{\partial t} + \mathbf{u}_p \cdot \nabla \mathbf{u}_p + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u}_p \right) \right\} dt - \int_0^T \left\{ p^\dagger \nabla \cdot \mathbf{u}_p \right\} dt - \lambda \left(\frac{E_0}{E(t_0)} - 1 \right).$$

Formulation

- **First variation of $\mathfrak{L}$ is set to zero →**

  Constraints: direct NS equations and initial energy for $\tilde{\mathbf{u}}$

  $$\frac{\delta \mathfrak{L}}{\delta \tilde{p}} = \frac{\partial u_i^\dagger}{\partial x_i} = 0 \quad \text{Adjoint equations}$$

  $$\frac{\delta \mathfrak{L}}{\delta \tilde{u}_k} = \frac{\partial u_k^\dagger}{\partial t} + \frac{\partial \left(u_k^\dagger u_{p,j}\right)}{\partial x_j} - u_i^\dagger \frac{\partial u_{p,i}}{\partial x_k} + \frac{\partial p^\dagger}{\partial x_k} + \frac{1}{Re}\frac{\partial^2 u_k^\dagger}{\partial x_j^2} - \frac{1}{ReT}\frac{\partial^2 u_k'}{\partial x_j^2} = 0$$

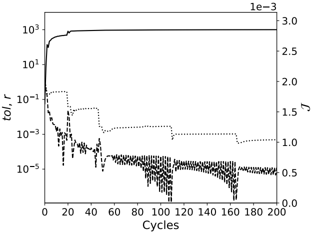
  $$\frac{\delta \mathfrak{L}}{\delta \tilde{u}_k(t_0+T)} = u_k^\dagger(t_0+T) = 0 \quad \text{Compatibility condition}$$

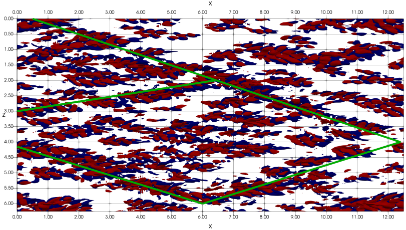
  $$\frac{\delta \mathfrak{L}}{\delta \tilde{u}_k(t_0)} = u_k^\dagger(t_0) - \lambda \tilde{u}_k(t_0) = 0 \quad \text{Gradient w.r.t. the initial perturbation}$$

- $\delta \mathfrak{L}/\delta \tilde{\mathbf{u}}_0$ is iteratively nullified by means of the gradient rotation algorithm [Foures et al. 2013].

Results: initial perturbation

- Computation at $Re_\tau = 180$ with $T = 2$ and $E_0 = 1.0 \cdot 10^{-4}$ using the channeflow code.
- Computational domain: $L_x \times L_y \times L_z = 4\pi \times 2 \times 2\pi$.

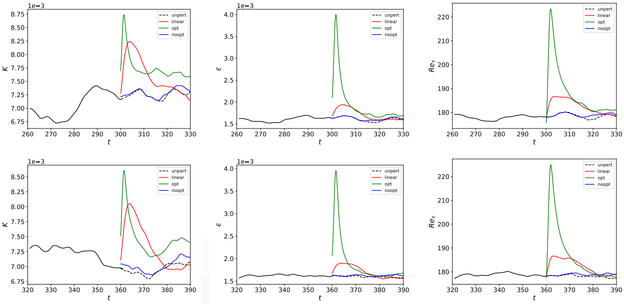




- Convergence comparable to previous nonlinear optimization, even if the flow is fully turbulent! (thanks to the short target time)
- Initial perturbation qualitatively unchanged using different flow realizations as $\mathbf{u}_u(t_0)$.

Time evolution

- **Simulation of the perturbed flow:** a transient is triggered by the perturbation, during which the turbulence intensity (measured by turbulent kinetic energy, turbulent dissipation and wall shear stress) experience a strong peak. After a certain relaxation time ($\Delta T_R \approx 15$) the statistically steady dynamics is recovered.

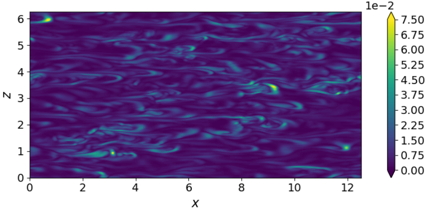


- This behaviour is the same for different realizations.

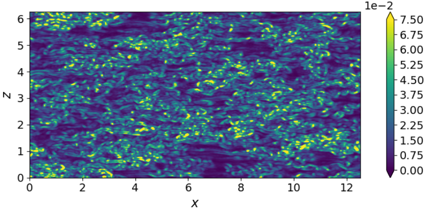
Extreme events generation

- The pre-existing structures break up and the flow becomes populated by **extreme dissipation events**:

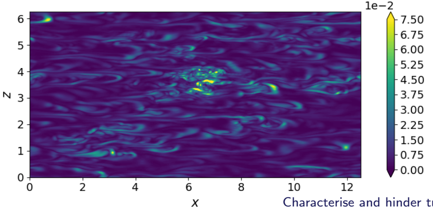
unperturbed flow



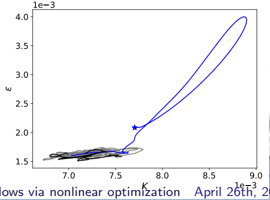
globally perturbed flow



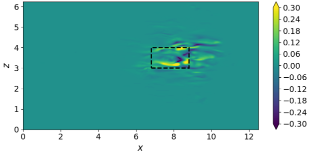
locally perturbed flow



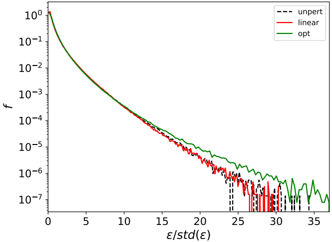
- Strong burst in the phase space.



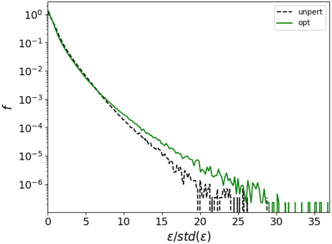
Extreme events generation - p.d.f. tail



globally perturbed flow



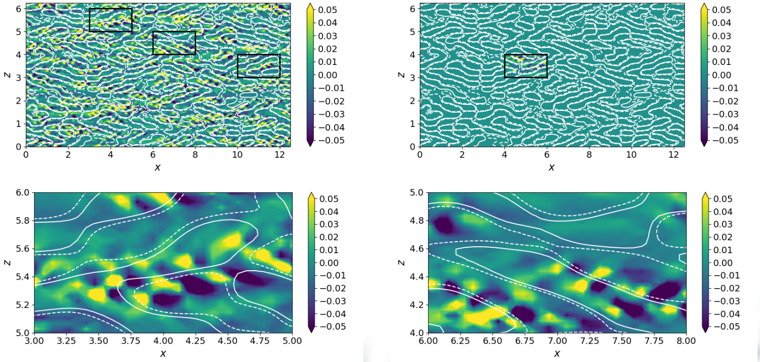
locally perturbed flow



- The statistical *mark* of extreme events is the **heavier tail in the p.d.f.** of dissipation [Sapsis, 2021].
- *Using various realizations of the optimization, the p.d.f. of the turbulent dissipation is computed up to* 30 *times the standard deviation.*
- When the flow is **globally or locally perturbed** with the nonlinear optimal perturbation the tail is visibly heavier w.r.t. the unperturbed flow.

Alignment with pre-existing streaks

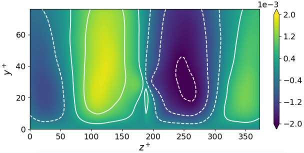
- **Qualitative information on the initial perturbation:**
  - inclined upstream → Orr's mechanism
  - aligned with the pre-existing velocity structures → streaks instability



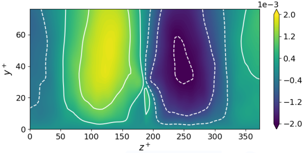
Comparison with Conditional POD

- The extreme events generated artificially by the optimal peturbation are structurally equivalent to the ones naturally found in the turbulent flow.
- A conditional POD [Hack & Schmidt, 2021] performed both on the unperturbed and on the perturbed flows can ne used to show this point:

Antisymmetric modes



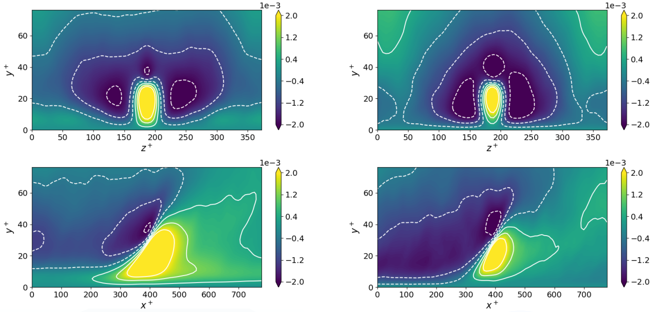
unperturbed flow



perturbed flow

Comparison with Conditional POD

Symmetric modes



unperturbed flow **perturbed flow**

- This result proves that **the optimal perturbation captures physical mechanisms!**

Conclusions

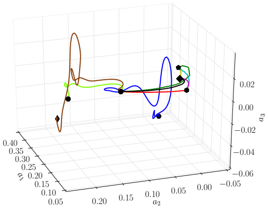
The nonlinear optimization framework allows to tackle several problems in transition and turbulence:

1. Mapping the laminar-turbulent separatrix
   - Its global energy minimum provides **minimal transition thresholds**
   - These thresholds have different scaling laws depending on the domain used → *the minimal flow unit is not sufficient for determining them accurately!*
2. Investigating the dynamics within the chaotic saddle
   - Nonlinear optimal **hairpin-like structures** in turbulent channel flow at low Re_τ have been found, inducing strong bursting events
   - The recurrence of extreme dissipation events is due to an **optimal growth mechanism of pre-existing streaks** on a time-varying flow

Perspectives

- Minimal seed:
  1. Experimentally reproducible perturbations of minimal energy: determine the **critical amplitude for inducing transition in a typical experimental setup**.
- Turbulent optimization:
  1. Increasing Re_τ to investigate optimal **Very Large Scale Motions**
  2. The set of equations proposed here might be also used for computing **invariant solutions in a turbulent framework**

- Chaotic trajectories:
  1. Challenge the simple scenario where trajectories in the phase space are attracted along the stable manifold and ejected by the unstable one → **link bursts with heteroclinic connections or instabilities of periodic orbits!**

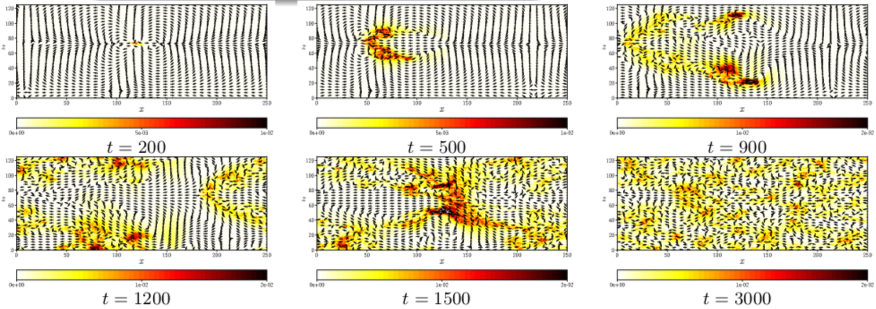


Thank you for your attention!

Results: Minimal seed evolution in time

Parameters

$Re = 1250,\ U_{bulk} = 2/3,\ T = 100,\ E_{0_{min}} = 2.9 \times 10^{-8}$



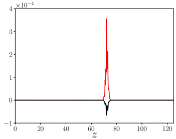
- $t = [0, 500]$: spot formation
- $t = 500$: V-shape structure
- $t = 900$: two oblique turbulent bands with $\alpha = \pm 45°$
- $t > 1200$: bands interaction

Similar evolution injecting a spot (Carlson et al.(1982), Henningson & Kim (1991), Aida et al.(2010))

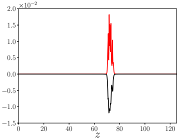
Results: Minimal seed evolution in time

Parameters

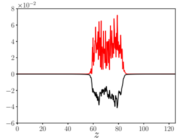
$Re = 1150$,
$U_{bulk} = 2/3$,
$T = 100$,
$E_{0_{min}} =$
4.7×10^{-8}



$t = 0$



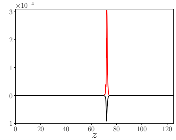
$t = 100$



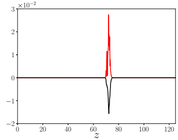
$t = 500$

Parameters

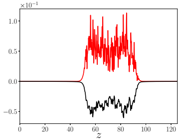
$Re = 1250$,
$U_{bulk} = 2/3$,
$T = 100$,
$E_{0_{min}} =$
2.9×10^{-8}



$t = 0$



$t = 100$



$t = 500$

$$P = -u_i' u_j' \frac{\partial U_i}{\partial x_j}, \qquad \epsilon = \frac{2}{Re} s_{ij}' s_{ij}' \quad \text{with} \quad s_{ij}' = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

The clipped perturbation is obtained multiplying the original global perturbation by a Gaussian function:

$$w(x,z) = exp\left\{-\left[\left(\frac{x-x_c}{\ell_x/2}\right)^n + \left(\frac{z-z_c}{\ell_z/2}\right)^n\right]\right\}, \qquad (3)$$

and subsequently projecting it on a divergence-free field. In the above equation, $\ell_x = 2$ and $\ell_z = 1$ are the streamwise and spanwise dimensions of the localized perturbation, respectively; x_c and z_c are the coordinates of the centroid of the perturbation, for which several values have been chosen; the integer n is set equal to 30.