STABLE INTERFACES: FROM NONLOCAL TO LOCAL

Workshop in honor of Alessio Figalli's DHC from UPC
21st November 2019

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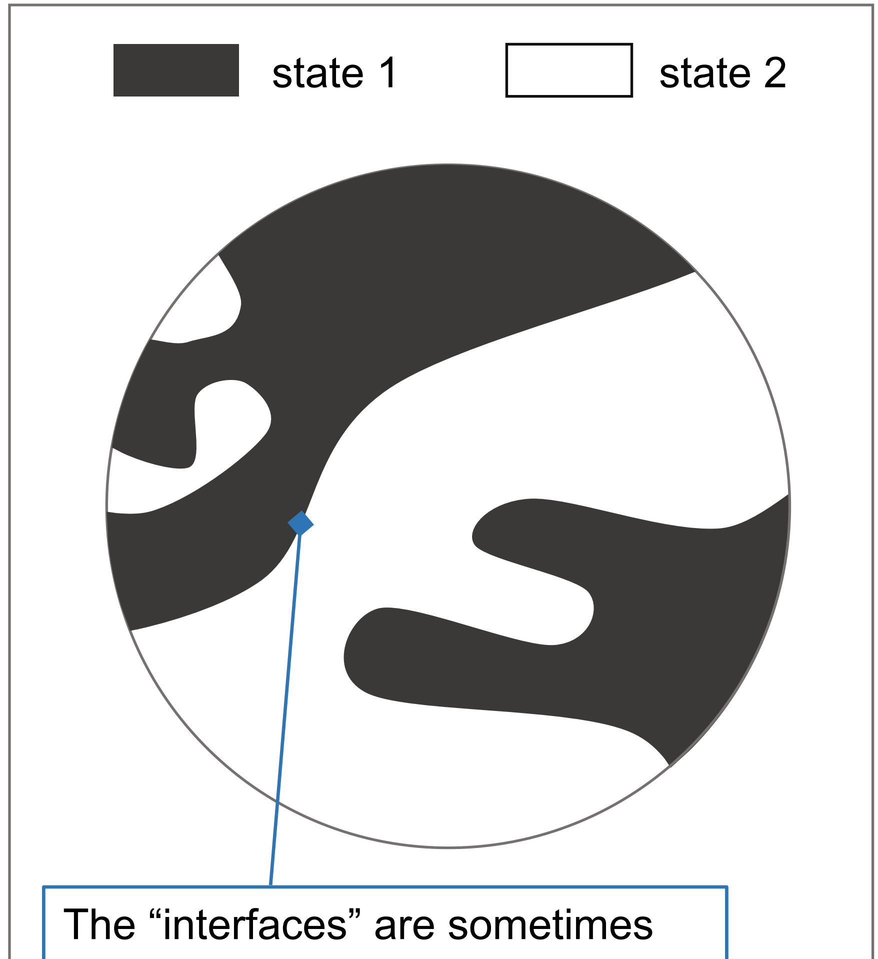
Outline of the talk

1. **On physical models for “interfaces”**
2. Two classical models for phase transitions
3. De Giorgi’s conjecture and stability conjectures
4. From nonlocal to local: how we stablished the stability conjecture for Peierls-Nabarro in three dimensions

Several important models in mathematical physics concern "interfaces", that is, surfaces separating two "competing states"



The "interfaces" are sometimes called phase transitions, minimal surfaces, free boundaries, etc., depending on the concrete model

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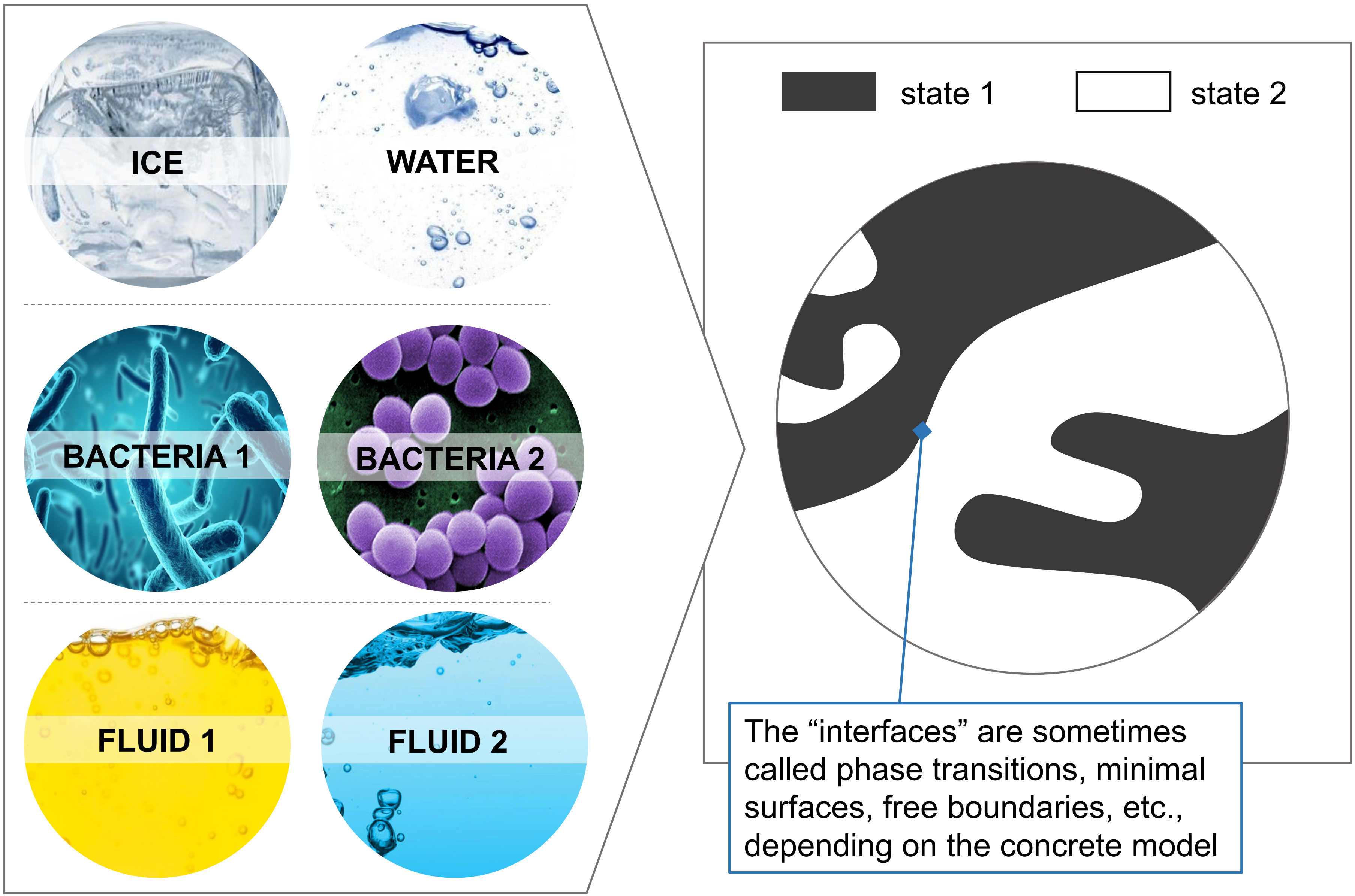

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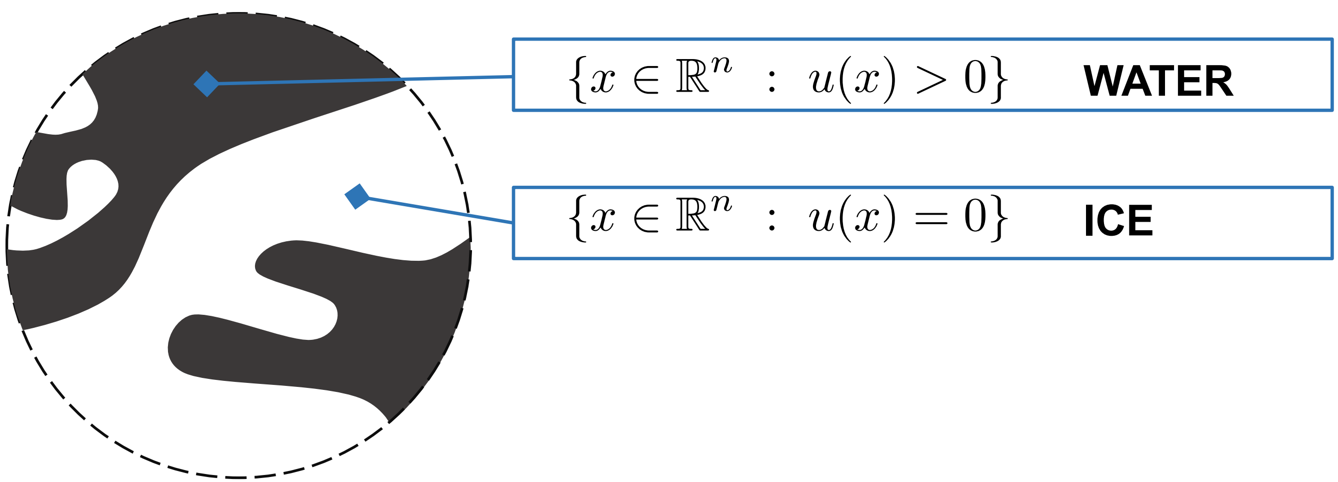
Example 1. **Stefan problem** (ice-water interface)

Energy functional $$J(u) := \int_{\mathbb{R}^n} \left(|\nabla u|^2 + \max(u, 0)\right) dx$$

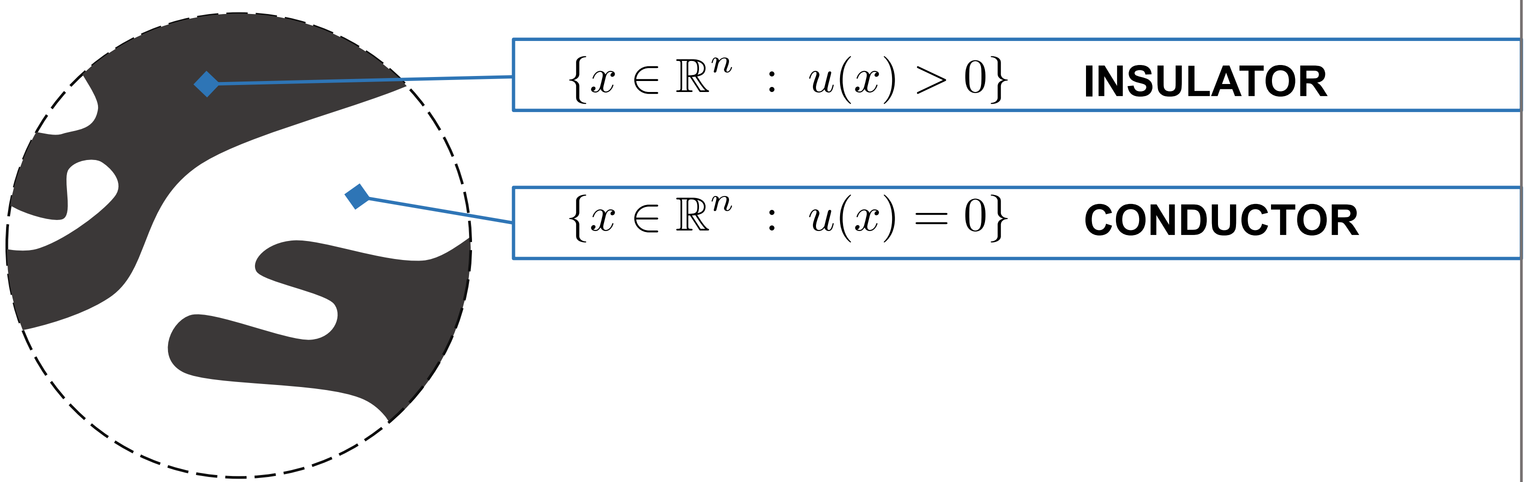


The interfaces are often found as level sets of a certain scalar functions that minimize some energy functional or Lagrangian

- Interfaces are often found as a level set of a function $u : \mathbb{R}^n \to \mathbb{R}$ that solves a certain nonlinear PDE
- Typically solutions to the PDE can be found minimizing some **energy functional** or **Lagrangian**

Example 2: **Bernoulli problem** (optimal design of insulator)

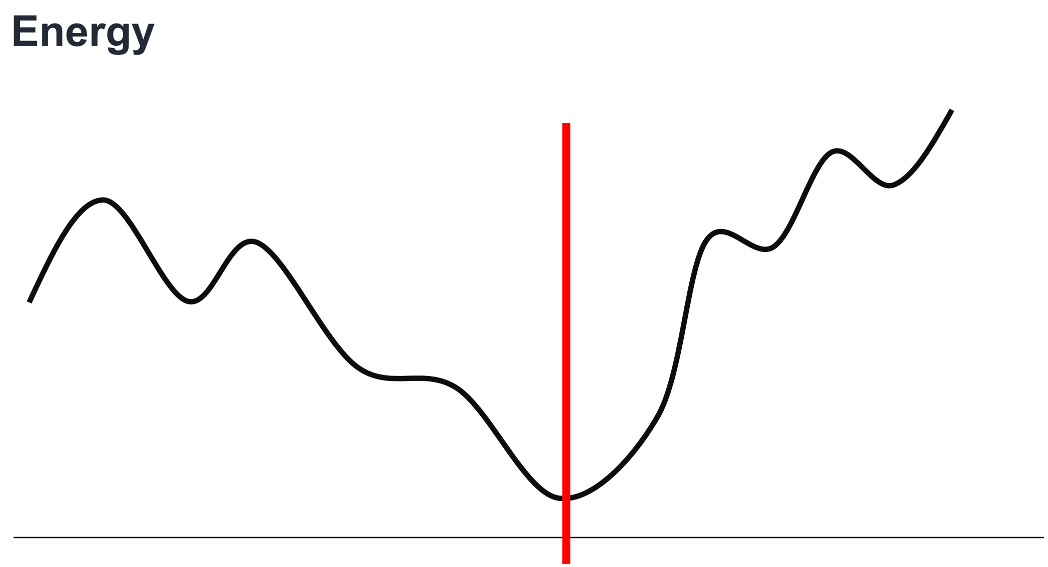
Energy functional $$J(u) := \int_{\mathbb{R}^n} (|\nabla u|^2 + \chi_{\{u>0\}})dx$$



In the same way that real functions may have absolute minima, local minima, and critical points, the same happens for energy functionals

$u : \mathbb{R}^n \to \mathbb{R}$ is called…

- **Minimizer** if $J(u + t\varphi) \geq J(u) \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n) \quad \forall t \in \mathbb{R}$



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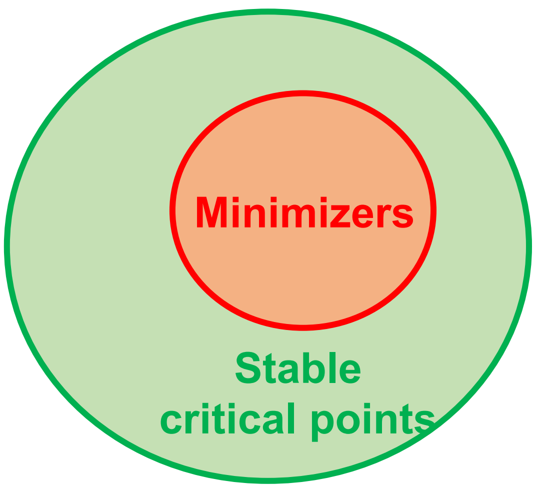
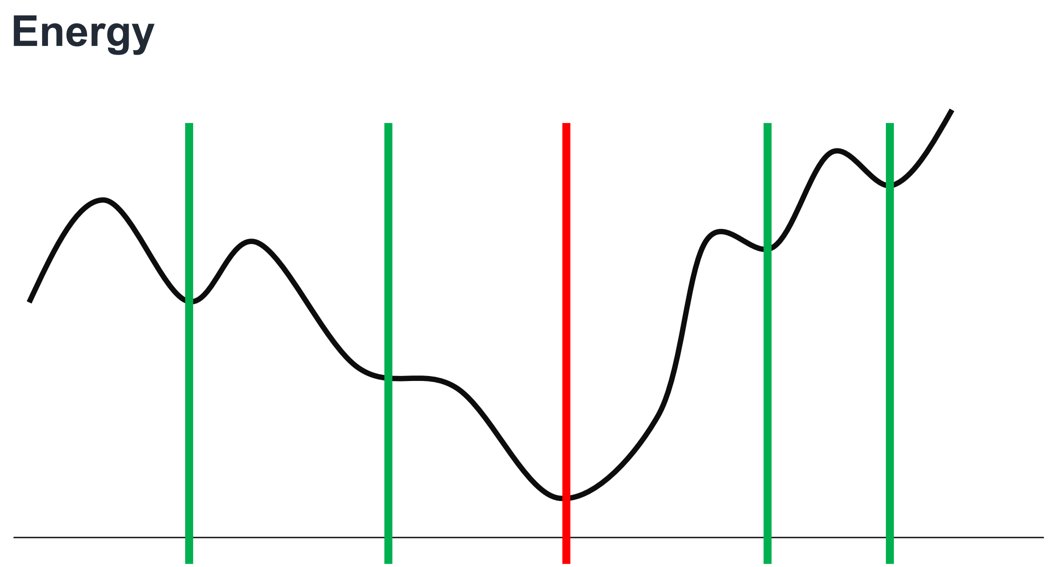
- **Stable critical point** if $J(u + t\varphi) \geq J(u) + o(t^2)$ as $t \downarrow 0$





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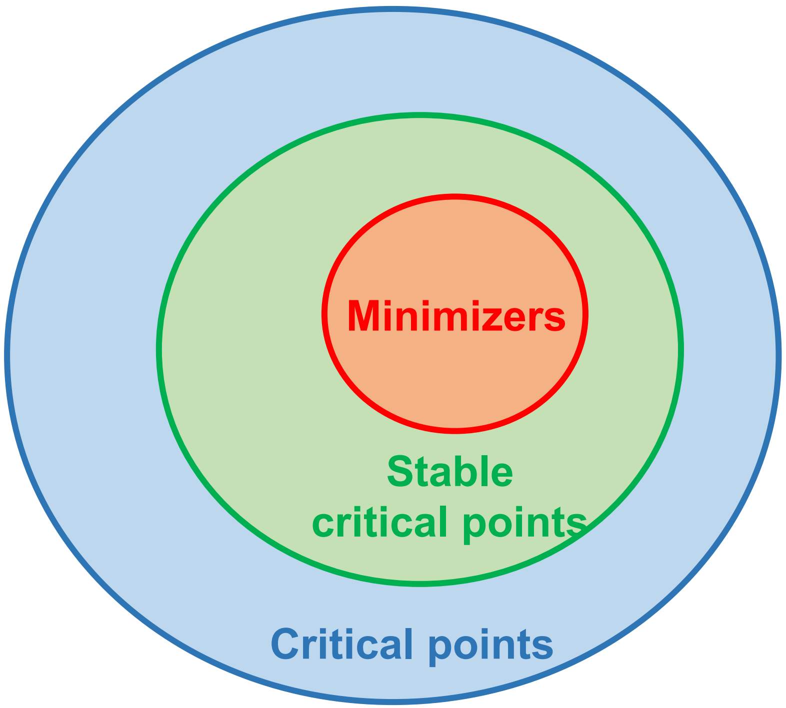

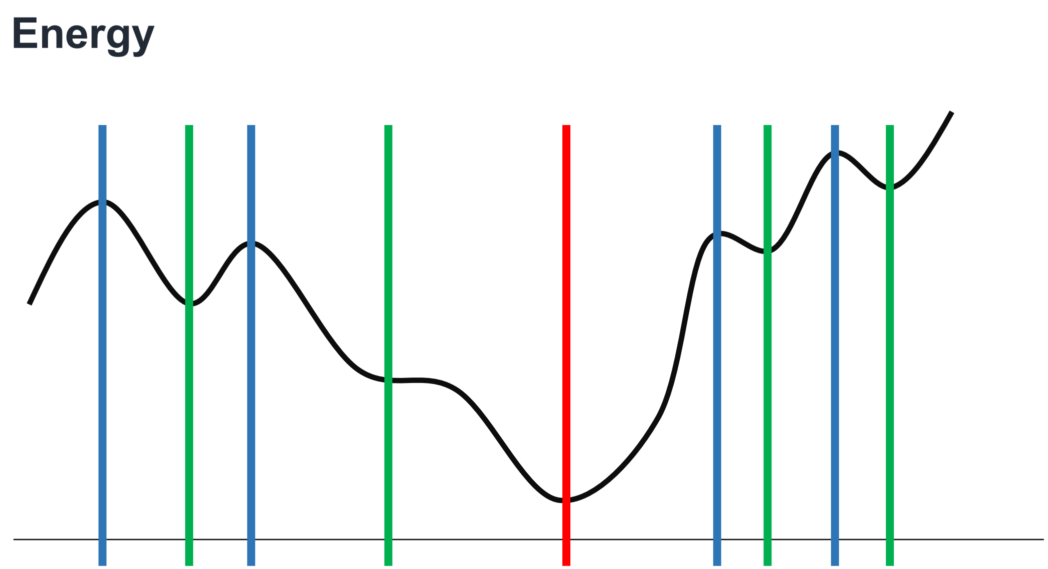

The three notions coincide if the functional is convex

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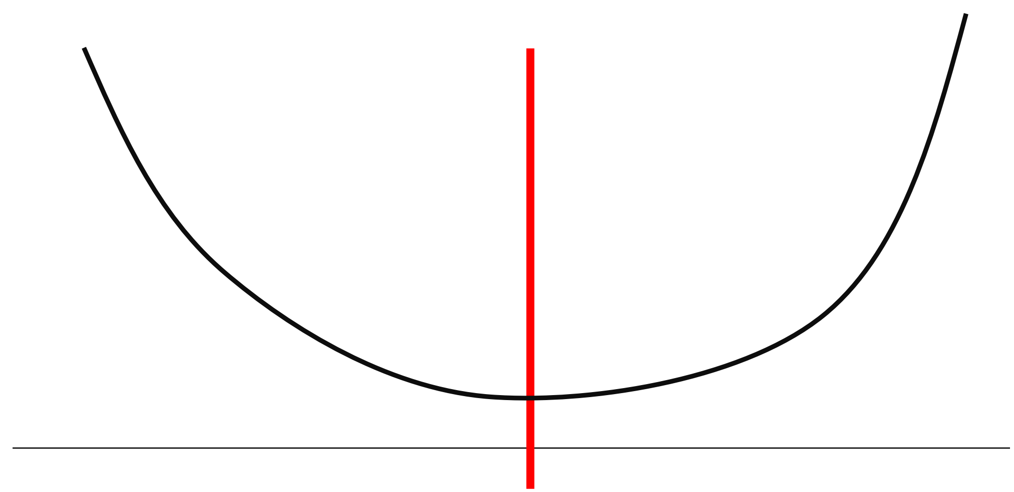
Convex energy

THE THREE NOTIONS COINCIDE !



A classical example in the Calculus of Variations: minimal graphs

Given a **convex** and **bounded domain** $\Omega \subset \mathbb{R}^2$ and a **smooth function** $v : \Omega \to \mathbb{R}$

The piece of **graphical surface** $S := \{(x_1, x_2, v(x_1, x_2)) \ : v(x_1, x_2) \in \Omega\}$

is a **critical point of** the **area** functional $F(v) := \int_\Omega \sqrt{1 + |\nabla v|^2}$

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That is

$$\frac{d}{dt}\bigg|_{t=0} \int_\Omega \sqrt{1 + |\nabla (v + t\varphi)|^2} = \int_\Omega \frac{d}{dt}\bigg|_{t=0} \sqrt{1 + |\nabla (v + t\varphi)|^2}$$

$$= \int_\Omega \frac{\nabla v \cdot \nabla \varphi}{\sqrt{1 + |\nabla v|^2}} = 0$$

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Integrating by parts we find

$$0 = \int_\Omega \frac{\nabla v \cdot \nabla \varphi}{\sqrt{1+|\nabla v|^2}} = -\int_\Omega \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\varphi \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

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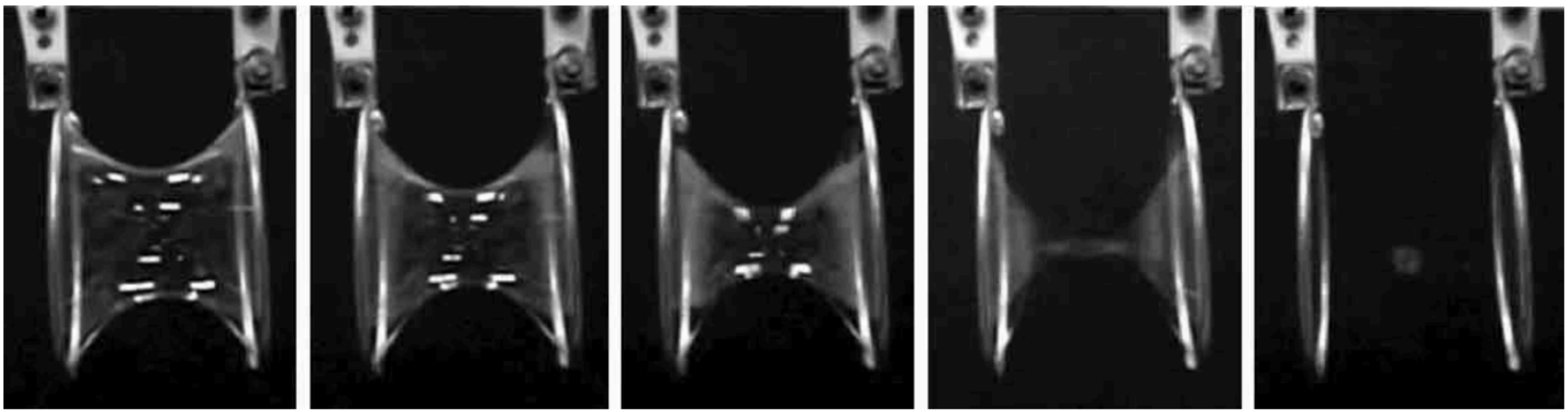
Important remark: Since the **area functional** is **convex for graphs** every critical point as above is a minimizer. However, this is not true for non-graphical minimal surfaces!

Stable critical points (not only minimizers) can be observed in nature, so we would like to understand their structure

Some stable minimal surfaces (non-graphical) are not area minimizers, as we can see for instance by inspecting catenoids

Example. **Soap film with the shape of a catenoid**



SOURCE: M. Ito and T. Sato in European Journal of Physics 2010

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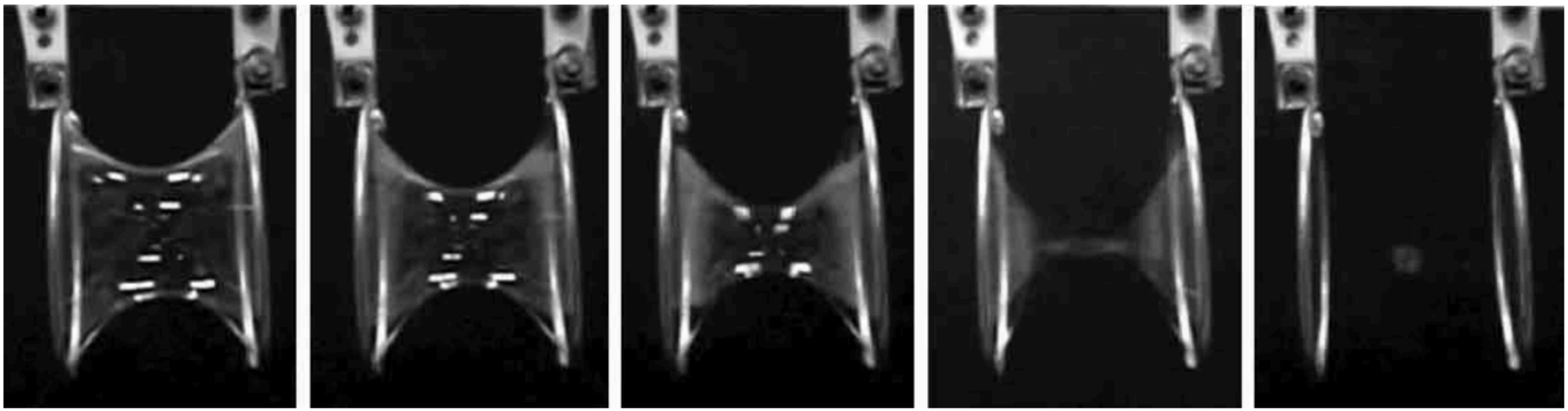

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Stable configurations (i.e. essentially local minima) are the **ones observable in nature**, since noise makes **unstable configurations decay** towards stable ones

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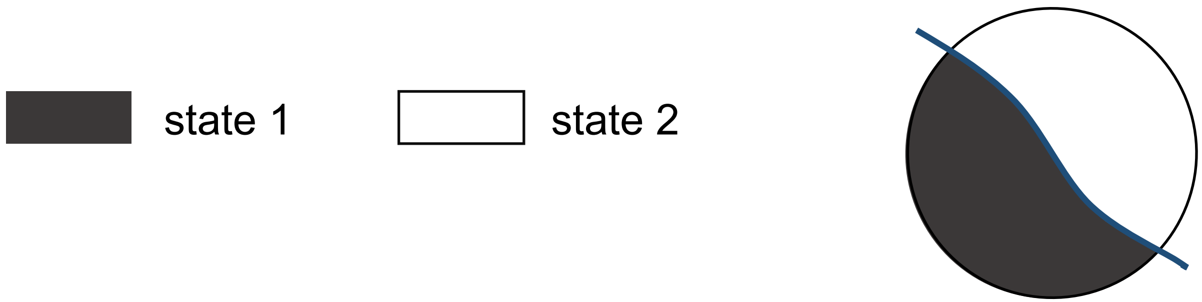
The so-called phase field models describe interfaces as zero level sets of critical points of certain functionals depending on a small parameter

Often interfaces in phase transitions are not described by a “black-or-white” model …



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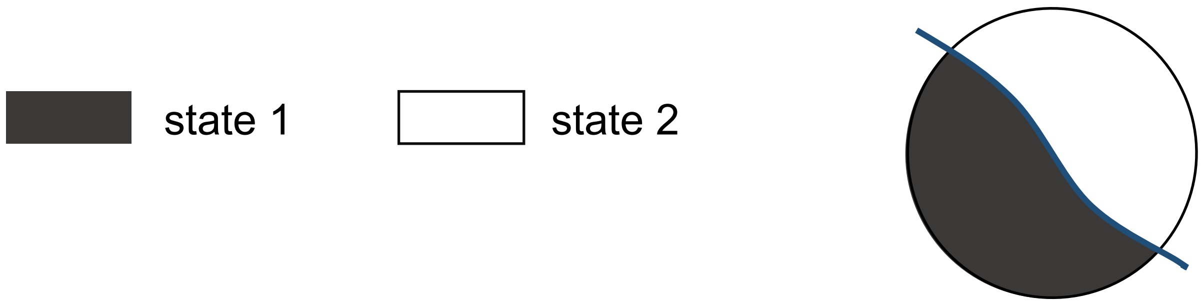
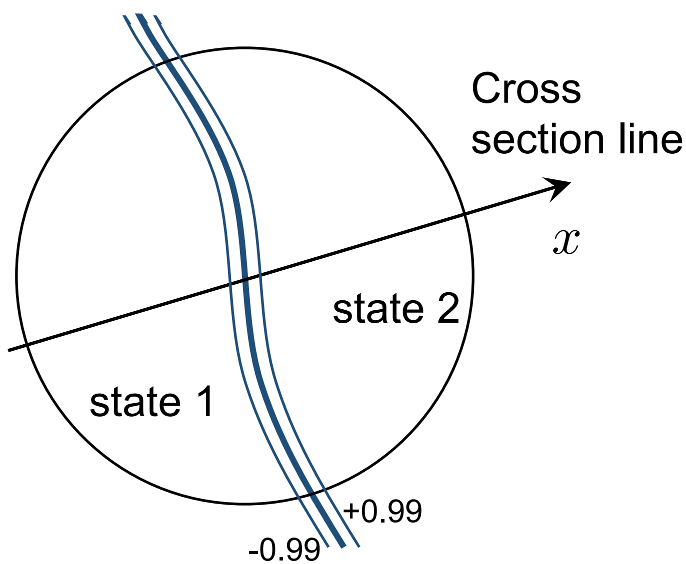
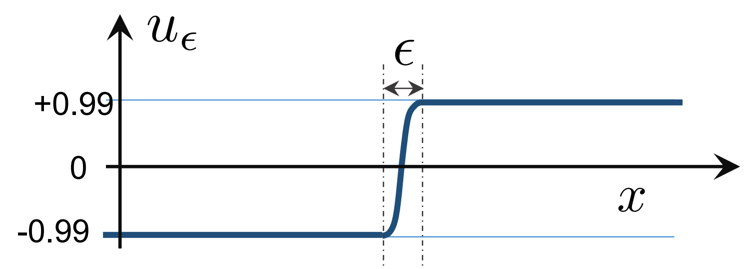


… but as the **0-level set** of **solutions** to some **variational nonlinear PDE** with a small parameter
These solutions (or critical points) make a steep transition between -1 and +1



$u_\epsilon : \mathbb{R}^n \to (-1, 1)$ $\quad \epsilon > 0$ parameter



Allen-Chan and Peierls-Nabarro are two similar phase-field models for phase transitions

ALLEN-CAHN (1950's)

Fluid-fluid interface / metal alloys / …

$$I_\epsilon(u) := [u]^2_{H^1(\mathbb{R}^n)} + \epsilon^{-2} \int_{\mathbb{R}^n} W(u) dx \qquad u : \mathbb{R}^n \to (-1,1)$$

$$[u]^2_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 d\xi$$

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PEIERLS-NABARRO (1940's)

Crystal dislocations / phase transition with line tension effect / …

$$J_\epsilon(u) := [u]^2_{H^{1/2}(\mathbb{R}^n)} + \epsilon^{-1}\int_{\mathbb{R}^n} W(u)dx \qquad u : \mathbb{R}^n \to (-1,1)$$

$$[u]^2_{H^{1/2}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{n+1}} dxdy = \int_{\mathbb{R}^n} |\xi||\hat{u}(\xi)|^2 d\xi$$

where in both models $W(u) := (1-u^2)^2$ **is a so-called double-well potential**

The potential forces minimizers of AC and PN to take values close to +1 and -1 everywhere, except on some “fat surfaces” of transition

Double-well potential

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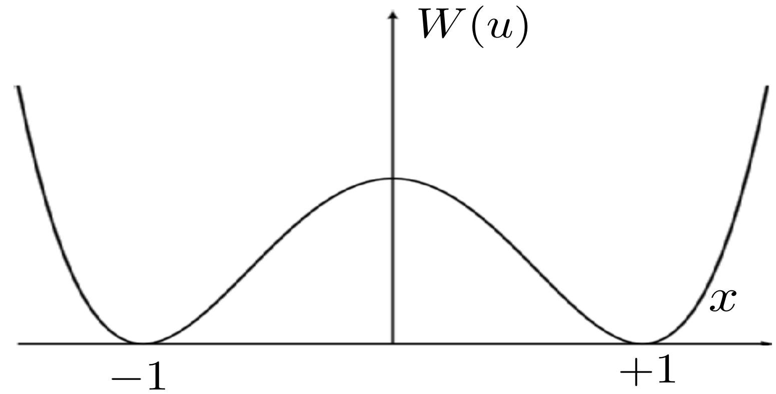

- The potential term (multiplied by a huge constant) forces **minimizers take values very close to +1 or -1** at most points, **except near “fat surfaces” of thickness epsilon** where transitions occur

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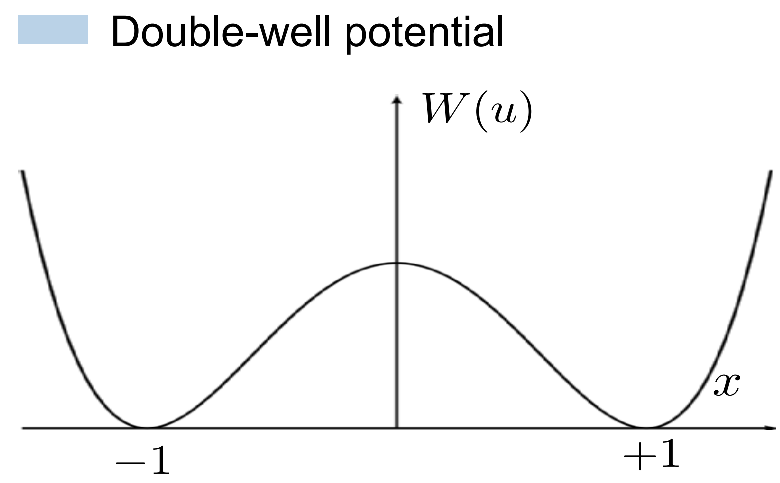

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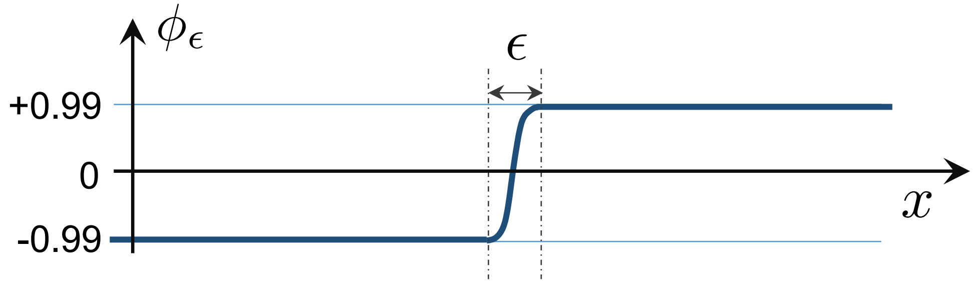
- In 1 dimension the Euler Lagrange equation $-\phi'' = \frac{1}{\epsilon^2}(\phi - \phi^3)$ has a solution as follows



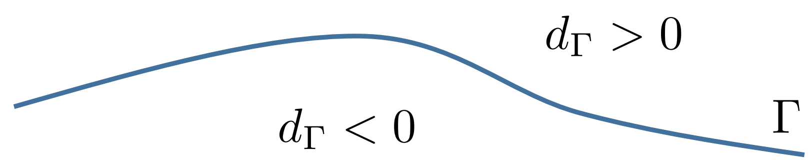
The “Dirichlet” term of the functionals penalizes oscillations, penalizing transitions proportionally to their "area”

$$I_\epsilon(u) := [u]^2_{H^1(\mathbb{R}^n)} + \epsilon^{-2}\int_{\mathbb{R}^n} W(u)dx \qquad J_\epsilon(u) := [u]^2_{H^{1/2}(\mathbb{R}^n)} + \epsilon^{-1}\int_{\mathbb{R}^n} W(u)dx$$

Insightful computation

Γ Piece of smooth surface

d_Γ Signed distance to Γ



Define the Ansatz $U_{\Gamma,\epsilon} := \phi_\epsilon \circ d_\Gamma$

We have

$$\epsilon\, I_\epsilon(U_{\Gamma,\epsilon}) = \text{Area}(\Gamma) + O(\epsilon)$$

$$\frac{1}{|\log\epsilon|} J_\epsilon(U_{\Gamma,\epsilon}) = \text{Area}(\Gamma) + O\left(\frac{\epsilon^2}{|\log\epsilon|}\right)$$

For infinitesimal epsilon phase transitions become minimal surfaces

Theorem (Modica, Mortola 1977; Alberti, Bouchitte, Seppecher 1998)

u_{ϵ_k} minimizers of either I_{ϵ_k} *or* J_{ϵ_k}

with $\epsilon_k \downarrow 0$

➡

$u_{\epsilon_k} \xrightarrow{L^1_{\mathrm{loc}}} \chi_E - \chi_{\mathbb{R}^n \setminus E}$

$E \subset \mathbb{R}^n$ is a **minimizer of perimeter**

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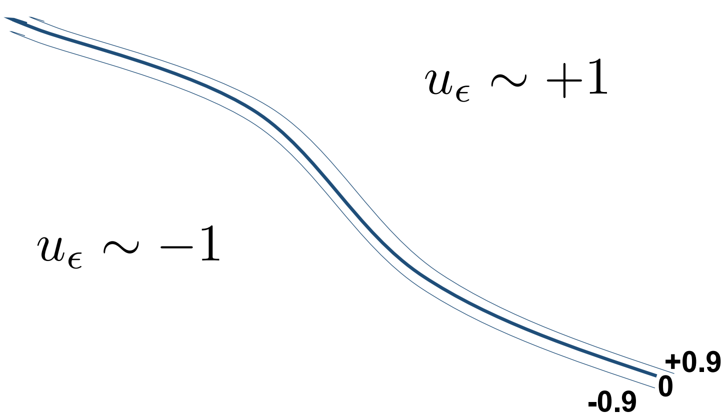
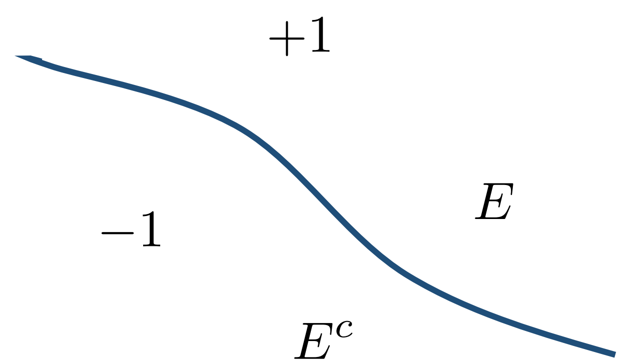
ALLEN-CAHN
PEIERLS-NABARRO

$\xrightarrow{\epsilon \downarrow 0}$ MINIMAL SURFACES





Thanks to this connection, ideas from minimal surfaces theory can be used for the analysis phase transitions and vice versa

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- However, the **flow of ideas is nowadays bidirectional***, as it is sometimes convenient to study questions in minimal surfaces by using methods from phase transitions

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* A good example is found in the paper

O. Chodosh, C. Mantoulidis, *Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates,* Ann Math., to apper

where the authors **construct minimal surfaces in 3-dimensional manifolds** —in relation to Yau's conjecture— **via approximations by** min-max **solutions of Allen-Cahn**

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Bernstein's problem in minimal surfaces …

BERNSTEIN PROBLEM (1914)

Must any entire minimal graph[1] in $\mathbb{R}^n$ be a hyperplane?

1 Solution to the minimal graph eq'n, than is $g : \mathbb{R}^{n-1} \to \mathbb{R}$ $\quad \mathrm{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) = 0$

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- **Positive results**

n = 3	Bernstein ~1915 (also Fleming 1962)
n = 4	De Giorgi 1965
n = 5	Almgren 1966
n = 6,7,8	Simons 1968

- **(Counter)example for n ≥ 9** Bombieri, De Giorgi, Giusti 1969

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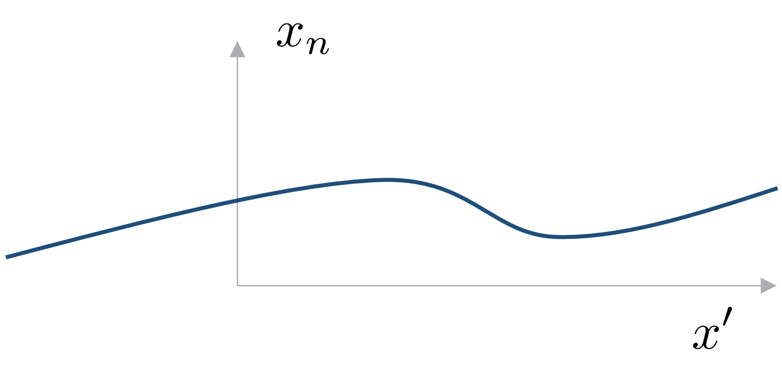
… motivates De Giorgi's conjecture on the Allen-Cahn equation

CONJECTURE (DE GIORGI 1978)

$u : \mathbb{R}^n \to (-1,1)$ critical point of Allen-Cahn satisfying $\partial_{x_n} u > 0$

Then, it must have 1D symmetry (i.e. its level sets must be hyperplanes) if n ≤ 8

Flatness of minimal graphs …



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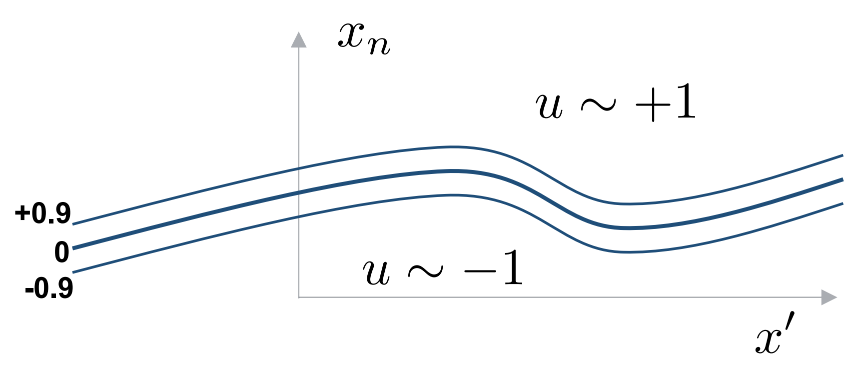

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- **Positive results**

n = 2	Ghoussoub, Gui (Math. Ann., 1998)
n = 3	Ambrosio, Cabre (J. Amer. Math. Soc., 2000)
n = 4,5,6,7,8 *	Savin (Ann. Math., 2009)

*** Under extra assumption that solutions are minimizers**

- **(Counter)example for n ≥ 9** Del Pino, Kowalczyc, Wei (Ann. Math., 2011)

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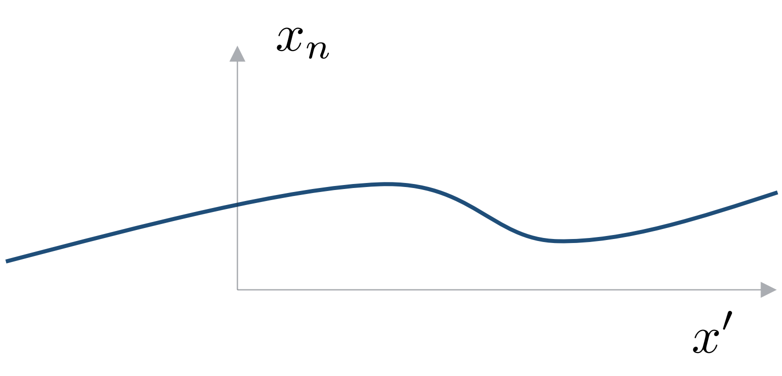

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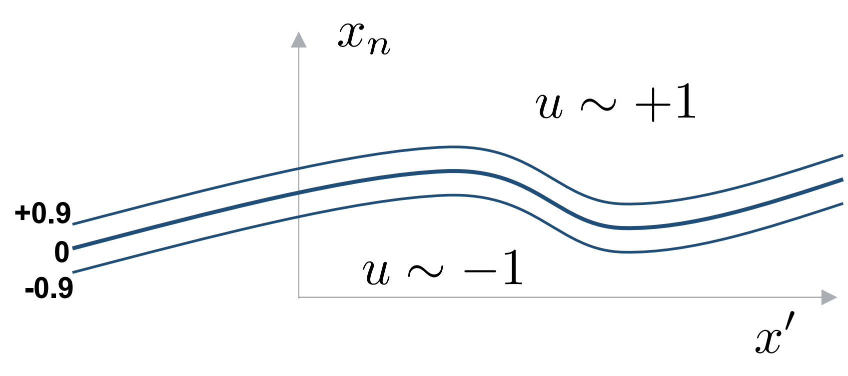

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A *Hyperplanes are the only complete, connected, imbedded, stable minimal hypersurfaces in ambient dimensions* $n \leq 7$

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Remarks

- Both statements are **known to be true** if we replace “Stable solutions” by “Energy **minimizers**” (Simons 1968, Savin 2009)

- **Conjecture A** is only known to be true for n = 3 (Fischer-Colbrie, Scheon / Do Carmo, Peng, 1970’s) and is open for other dimensions (for n = 4 it is equivalent to **Schoen’s conjecture**)

- Using Savin’s results, **stability conjecture** for Allen-Cahn [resp. Peierls-Nabarro] in dimension n-1 **implies the full De Giorgi conjecture** in dimension *n* (without extra assumption

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3. De Giorgi’s conjecture and stability conjectures

4. **From nonlocal to local: how we stablished the stability conjecture for Peierls-Nabarro in three dimensions**

In a recent work with Alessio Figalli we could establish the three dimensional case of the stability conjecture for Peierls-Nabarro

Figalli, S - Invent. Math. 2019

Theorem. Assume that $u : \mathbb{R}^3 \to (a,b)$, with $-\infty < a < b < +\infty$, is a stable critical point of

$$J(u) := [u]^2_{H^{1/2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} F(u)$$

for some potential. Then there exists $\phi : \mathbb{R} \to (a,b)$ and $e \in \mathbb{S}^2$ such that

$$u(x) = \phi(e \cdot x)$$

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Consequences (both still open for Allen-Cahn):

- The stability conjecture for Peierls-Nabarro is true for n=3
- The analogue of De Giorgi's conjecture for Peierls-Nabarro is true for n=4

COMMENT 1 - This theorem builds on crucial previous results that have been developed in the last decades at UPC

- The paper **Cabre and Sola-Morales – Comm. Pure Appl. Math 2005** was pioneer in studying the problem and relating it to the theory for Allen-Cahn. I

- The same paper by Cabre and Sola-Morales already stablished the stability conjecture for Peierls-Nabarro in two dimensions

- Later, in 2010, Cabre and Cinti established the analogous of De Giorgi conjecture for Peierls-Nabarro in three dimensions

COMMENT 2 - Somewhat surprisingly, our proof applies methods from the theory of nonlocal minimal surfaces to our functional, even if our interfaces behave like standard (local) minimal surfaces

- Note that the 1-parameter family of functionals

$$\mathcal{J}_{\epsilon,s}(u) := [u]^2_{H^s(\mathbb{R}^n)} + \epsilon^{-2s} \int_{\mathbb{R}^n} W(u)\, dx \quad \text{where} \quad [u]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi$$

interpolates Peierls-Nabarro and Allen-Cahn:

$$\mathcal{J}_{\frac{1}{2},\epsilon} = J_\epsilon \qquad \mathcal{J}_{1,\epsilon} = I_\epsilon$$

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- But it also make sense to define these functionals for $s \in (0, 1/2)$, and the study of these generalizations led Caffarelli, Roquejoffre and Savin (in Comm. Pure Appl. Math, 2010) to introducing the so-called **nonlocal minimal surfaces**

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- Several important contributions to the theory of Nonlocal Minimal Surfaces and the corresponding CMC have been developed at UPC by various researchers (X. Cabre, M. Cozzi, G. Csato , A. Mas, J. Sola-Morales)

THANK YOU FOR YOUR ATTENTION