Kolmogorov: la K del teorema KAM

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UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Childhood

- Andrey Nikolaevich Kolmogorov was born in April 25 of 1903 at Tambov and died in 1887 at Moscou.

- His father Nikolai Kataev was exiled and died in fighting in 1919. Kolmogorov's mother also died at Kolmogorov's birth. His mother's sister, Vera Yakovlena, brought Kolmogorov up and he got the surnammes of his grandfather, a member of the nobility called Yakov Stepánovich Kolmogórov, who has a house in Yaroslav.
- He spends his first 7 years in Yaroslav.
- In 1910 they move to Moscou and he attends a private school Repman, very prestigious.
- When he was 7, he realizes that: $1^2 = 1$, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$ etc and publish it in a journal of the school...

University

- After Kolmogorov left school he worked for a while as a conductor on the railway. In his spare time he wrote a treatise on Newton's laws of mechanics.
- In 1920, Kolmogorov entered Moscow State University
- For some time Kolmogorov was interested in metallurgic and also in Russian history as well as mathematics. He did serious scientific research on XV-XVI century manuscripts concerning agrarian relations in ancient Novgorod.
- There is an anecdote told by D G Kendall regarding this thesis, his teacher saying: "You have supplied one proof of your thesis, and in the mathematics that you study this would perhaps suffice, but we historians prefer to have at least ten proofs"
- After this he studies mathematics!

University

• The students could choose between exams of original papers and he did not perform any exam!

• When he was 19, in 1922, he found the first example of function $f : [0, 2\pi] \to \mathbb{R}$ such that: $\int_0^{2\pi} |f(t)|\, dt < \infty$ and its Fourier series

$$\sum_{n \in \mathbb{Z}} \hat{f}_n e^{2\pi int}, \; t \in [0, 2\pi] \text{ where } \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-2\pi int}\, dt,$$

does not converge for almost any $t \in [0, 2\pi]$.

This was wholly unexpected by the experts and Kolmogorov's name began to be known around the world.

• B V Gnedenko writes about him:

Almost simultaneously Kolmogorov exhibited his interest in a number of other areas of classical analysis: in problems of differentiation and integration, in measures of sets etc. In every one of his papers, dealing with such a variety of topics, he introduced an element of originality, a breadth of approach, and a depth of thought.

Probability theory

- At the University he contact the Lusitania' group founded by D.F. Egorov and N.N. Luzin.
- He worked in several subjects, like mathematical logic, with very substantial new results, and the first systematic research in the world on intuitionistic logic.
- Around 1924 he became interested in probability theory with A.Y. Khinchine.
- In 1929 he wrote A general theory of measure and the calculus of probabilities, where he began an axiom system for probability theory based on the theory of measure and the theory of functions of a real variable. Such a theory had been first suggested by E. Borel in 1909, was further developed by Lomnicki in 1923. Kolmogorov gave it a successful final form in 1933 in the book Foundations of the Calculus of Probabilities in which he built the axiomatic approach to probability theory.

Probability theory

- The work of Komogorov made mathematicians interested in probability that was considered just an applied (and therefore secondary) subject.
- Gnedenko says:
  In the history of probability theory it is difficult to find other works that changed the established points of view and basic trends in research work in such a decisive way. In fact, this work could be considered as the beginning of a new stage in the development of the whole theory.

Academic career

• In 1929, Kolmogorov earned his Doctor of Philosophy (Ph.D.) degree, from Moscow State University.

• From 1930 – 1940 Kolmogorov published more than sixty papers on probability theory, projective geometry, mathematical statistics, the theory of functions of a real variable, topology, mathematical logic, mathematical biology, philosophy and the history of mathematics.

• In 1931 (28 years old) Kolmogorov was made professor at Moscow University

• 1933 – 1939 he became Director of the Scientific Research Institute of Mathematics at the Moscow State University.

• In 1937 held the chair of theory of probability

• In 1939, he was elected a full member (academician) of the USSR Academy of Sciences.

• During World War II Kolmogorov contributed to the Russian war effort by applying statistical theory to artillery fire, developing a scheme of stochastic distribution of barrage balloons intended to help protect

Turbulence

- In the 30ts he became interested in the theory of turbulence.
  My interest in the study about turbulent flows appeared at the end of the 30ts. I clearly saw that the main mathematical tool should came from the theory of several random variables, just born
- To understand turbulence, one needs to study the velocities of each position of the fluid. The velocity $u(t, x)$ measures the velocity of the articles which are in the time t at position x and $u(t, x)$ satisfies the famous "Navier Stokes equations" which he already knew.
- He saw that Probablility Theory was needed to understand turbulence.

Turbulence

What Kolmogorov realized, after looking at lots of experimental works, was that two laws were always satisfied:

- When the Reynols number increases to infinity (the viscosity goes to 0), the disipation of energy $\varepsilon = \frac{d}{dt}\int |u(t,x)|^2 dx$ has a stricly positive limit. This law is not proved now a days.
- Kolmogorov's law of "2/3":
  In each developed turbulent flow the mean square difference of the velocities at two points is proportional to the 2/3rd power of their distance

$$< (u(x+\ell) - u(x))^2 > \simeq (\varepsilon\ell)^{2/3}.$$

  Again, there is no a mathematical proof.

Turbulence

• With great physical intuition, in two short papers in 1941, Kolmogorov posited in concise mathematical form ideas about the structure of the small-scale components of turbulent motion of fluids and gasses.

• He makes some hypotheses that the fluid has to satisfy and from them he proves the laws.

He deals with the velocities as random variables which satisfy somme statistical properties.

- The fluid is homogeneus for big Reynold numbers, and the statistical properties at small scales (ℓ small) are determined by ε and ν.
- When $\nu \to 0$ the stadistical properties are independent of ν.

• With these hypotheses he proves the two laws, and he also proves:

$$< (\delta u(\ell)) >^3 = -\frac{4}{5}\varepsilon\ell$$

Its relation with P. Alexandrov

• In 1929 we meets P. Alexandrov, and they have a long relation.
• Says Kolmogorov: for me these 53 years of close and indissoluble friendship were the reason why all my life was on the whole full of happiness, and the basis of that happiness was the unceasing thoughtfulness on the part of Aleksandrov.
• Kolmmogorov organized in the summer of 1929 a trip starting from Yaroslavl, they went by boat down the Volga then across the Caucasus mountains to Lake Sevan in Armenia for several weeks.
• After more than 1300Km they set up residence in an unused cell of a monastery on a small peninsula in Lake Savan
• They swim and sunbathe but they also did lots of maths: Kolmogorov in the shadows on integration theory and analytic description of Markov processes in continuous time, and Aleksandrov on his Topology book with Hopf. Kolmogorov's results were published in 1931 and mark the beginning of diffusion theory.

Its relation with P. Alexandrov

• In the summer of 1931 Kolmogorov and Aleksandrov made another long trip. They visited Berlin, Göttingen, Munich, and Paris where Kolmogorov spent many hours in deep discussions with Paul Lévy.

• In his trips, Kolmogorov had contacts with R. Courant on limit theorems, with H. Weyl on intuitionistic logic, and with E. Landau on function theory.

• In this period he turned to topology. Simultaneously with the U.S. topologist J.W. Alexander and independently of him, Kolmogorov discovered the notion of cohomology and founded the theory of cohomological operations. The work of Kolmogorov and his school on the deep connections between topology, the theory of ordinary differential equations, celestial mechanics and the theory of dynamical systems, determined to a considerable extent its present state.

• 1935: They buy an old manor house at Komarovka that becomes a meeting place for mathematicians and students. there were discussions on maths, but also literature, they did lots of sport...

Teaching

• Since the beginning of his career he considered education very important.
• The number of Kolmogorov's research students who have obtained their Ph.D. are 82 acording the Math Genealogy Project.
• When offering a subject for research to a graduate or a research student, the supervisor must not think only about the objective importance, or urgency of the subject, but also whether the work on the subject will stimulate the development of the young scientist, and whether it is within his powers to carry out, and at the same time demand the maximal effort of which he is capable.
• His way of working was to give ideas to solve problems but not to write the technicalities.

Teaching

• Funds a Kolmogorov boarding school for young people who do not live in a university city. In fact 4: Moscu, Kiev, San Petersburg, Novosibirsk. In this way, they can have a better education. He also goes around the country to help talented students and send good teachers to the schools.
• In 1963 he obtains the Bolzman prize and he spentt part of the money to create a and preserve the library devoted to probability, stochastic processes and stadistics at Moscou U.
• In 1970, jointly wit the phisicist I. Kikoin, they create a journal in phisics and mathematics for students and teachers in primary and secondary school.

Other results

- It seems that until the 50 there was a period of silence of Kolmogorov. But then there are some of his more important results.
- 1956 With V. Arnold solves a problem proposed by Hilbert: Any continuous function of $n \geq 3$ variables can be represented as a composition of functions of 3 variables.
  Later Arnolf proved the same with only 2 variables.
  Finally, in 1957 Kolmogorov proves that any continuous function of $n \geq 2$ variables can be expressed as sums and compositions of functions of 1 variable.
- 1958 He works with the concept of entropy.
- 1960 He introduces "Kolmogorov complexity"
- And finally, **in 1954 in the International Conference of Mathematics, he talks about invariant tori, stability of the solar system...and for this reason he is the K of the KAM theorem.**

Dynamical Systems

- Roughly speaking, a Dynamical System is a mathematical means of describing something (a "system") that changes over time.
- We need a suitable phase space, where each point of which gives all relevant information about the system.
    - To describe a population of animals, the phase space is $\mathbb{N}$. We say the phase space has dimension one.
      If we have two populations—say, a population of predators and a population of prey—the phase space is of dimension two, etc
    - To describe the motion of a satellite, we need its three coordinates in space and the three components of its velocity. The phase space has points described by six numbers, so the its dimension is 6.
- The phase space can be $\mathbb{R}^N$ or a manifold M of dimension N.
- We also need a mathematical rule of evolution. In some physical problems this law is given by an ordinary differential equation.
- The goal of Dynamical SystemS: Given a point in phase space, can we predict its motion?

Dynamical Systems

• The universal example is a particle $q = q(t) = (q_1(t), q_2(t), q_3(t)) \in \mathbb{R}^3$ where $t \in \mathbb{R}$ is time, moving through the action of a force which depends on its position $F(q)$, then, the second Newton law (1687) gives:

$$m\ddot{q}(t) = F(q(t)), \text{ where } \dot{q} = \frac{dq}{dt}$$

and we have used that the acceleration $a(t) = \ddot{q}(t)$ and we have called m the mass of the particle.

• To write this equation as a first orden system of equations, a classical trick is to call $p = \dot{q} = \frac{dq}{dt}$ and then we have:

$$\dot{q} = p$$
$$\dot{p} = \frac{1}{m}F(q)$$

Which has the form, calling $x = (q, p)$:

$$\dot{x} = \mathcal{F}(x), \; x \in \mathbb{R}^6,$$

Dynamical Systems

$$\dot{x} = \mathcal{F}(x), \ x \in \mathbb{R}^N$$

• Given $x_0 \in \mathbb{R}^N$, it is known that, under quite mild conditions, there is only a solution $x(t)$ such that $x(0) = x_0$.

• We will also assume, during this talk, that solutions exist for all times.

• A more difficult question is to find the previous solution $x(t)$ explicitely or, at least, by quadratures!!

• A smooth function $G : \mathbb{R}^N \to \mathbb{R}$ is a conserved quantity (first integral) if $G(x(t)) = G(x(0))$ for all $t \in \mathbb{R}$.

• Conserved quantities play an important role in integrability. The level hypersurfaces $G = ct$ confine the solutions.

• Integrable systems are those that can be integrated by quadratures. Equivalently, they have enough first integrals. The orbits are at the intersections of the level hypersurfaces.

Hamiltonian Systems

We don't want to go deeply in Hamiltonian systems, even if they are the core of our story. We just say:

• The name comes from the Irish mathematician W.R. Hamilton (1805–1865), who obtained those equations by applying Newton's laws.

• A H.S of n degrees of freedom is a system of differential equations of a special form:

$$\left.\begin{array}{rcl} \dot{q}_i & = & \frac{\partial H}{\partial p_i}(q,p) \\ \dot{p}_i & = & -\frac{\partial H}{\partial q_i}(q,p) \end{array}\right\} \quad i = 1, \dots n$$

The Hamiltonian $H(q,p) = H(q_1, \dots, q_n, p_1, \dots, p_n)$ is a smooth function.

• A compact form, calling $x = (q_1, \dots, q_n, p_1, \dots, p_n)$ is:

$$\dot{x} = \mathcal{J}\nabla H(x), \; \mathcal{J} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$$

• The Hamiltonian H is a conserved quantity of the system.

Hamiltonian Systems: an easy example

A Hamiltonian system oof 1 degree of freedom is:

$$\left.\begin{array}{rcl} \dot{q} & = & \omega p = \frac{\partial H}{\partial p}(q,p) \\ \dot{p} & = & -\omega q = -\frac{\partial H}{\partial q}(q,p) \end{array}\right\}$$

where $H = \frac{\omega}{2}(q^2 + p^2)$.
Observe that it is equivalent to the second order equation:

$$\ddot{q} = -\omega^2 q$$

which is the equation of a spring, and is integrable:

$$q(t) = q_0 \cos \omega t + p_0 \sin \omega t, \;\; p(t) = -q_0 \sin \omega t + p_0 \cos \omega t$$

Integrable Hamiltonian Systems

To define what is an integrable hamiltonian system we need to recall the definition of the Poisson bracket of two funtions $f(q,p)$ and $g(q,p)$:

$$\{f,g\} = \sum_{k=0}^{n}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k}\right) = Df\mathcal{J}\nabla g$$

and give two definitions:

• A finite set of functions $\{f_1, \ldots, f_n\}$ on M is independent on $K \subset M$, if their gradient vectors $\{\nabla f_1, \ldots \nabla f_n\}$ are linearly independent at almost every point of K.

• Two smooth functions f, g on M are in involution if $\{f,g\} = 0$ (i.e. their Poisson bracket vanishes on M). A set of more than two functions is in involution if each pair of functions in the set is in involution.

Integrable Hamiltonian Systems

Definition A Hamiltonian system H with n degrees of freedom is completely integrable if it has n constants of motion $\{f_1, \ldots, f_n\}$ which are independent and in involution.
The reason to call these systems integrable is the following theorem, (Liouville- Mineur-Arnold-Jost)
Theorem
There exist a change of variables: $(q, p) \to (\theta, I)$
such that in the new variables, called action-angle variables, the system is again Hamiltonian and the hamiltonian in the new variables takes the form:

$$K = K(I)$$

Observe that now the equations are simple!

$$\dot{\theta} = \omega(I), \quad \dot{I} = 0, \quad \omega(I) = \nabla K(I).$$

Now we can obtain the trajectories of the system!

$$I(t) = I_0, \quad \theta(t) = \theta_0 + \omega(I_0)t$$

Hamiltonian Systems: an easy example

In our example:

$$\left.\begin{array}{rcl} \dot{q} & = & \omega p \\ \dot{p} & = & -\omega q \end{array}\right\}$$

with $H = \frac{\omega}{2}(q^2 + p^2)$.
Use the change $q = \sqrt{2I}\cos\theta$, $p = \sqrt{2I}\sin\theta$ and the system becomes:

$$\left.\begin{array}{rcl} \dot{\theta} & = & \omega \\ \dot{I} & = & 0 \end{array}\right\}$$

which is Hamiltonian with Hamiltonian $K(I) = \omega I$.
The solutions are periodic orbits: $I = I_0$, $\theta = \theta_0 + \omega t$.

Integrable Hamiltonian Systems: resonant and non-resonant tori

The solutions of the integrable hamiltonian system $K(I)$ are confined in n-dimensional tori:

$$\mathbb{T}^n = \mathbb{T}^n_{I_0} = \{(\theta, I), I = I_0,\ \theta \in \mathbb{T}^n\} \subset \mathbb{T}^n \times \mathbb{R}^n$$

and all the orbits in the torus $\mathbb{T}^n_{I_0}$ move with a motion of the same frequency $\omega = \omega(I_0)$:

$$\theta(t) = \theta_0 + \omega t$$

To clarify ideas take $n = 2$:

$$\theta^1(t) = \theta^1_0 + \omega^1 t,\ \theta^2(t) = \theta^2_0 + \omega^2 t$$

and it is easy to see that:

- If $\frac{\omega^1}{\omega^2} = \frac{p}{q}$ is a rational number, then all the solutions in $\mathbb{T}_{I_0}$ are are periodic.
- If $\frac{\omega^1}{\omega^2}$ is an irrational number, then all the solutions in $\mathbb{T}_{I_0}$ are dense.

Integrable Hamiltonian Systems: resonant and non-resonant tori

This phenomenon is true in any dimension: The orbits in the torus $\mathbb{T}^n_{I_0}$ move with a motion of the same frequency $\omega = \omega(I)$:

$$\theta(t) = \theta_0 + \omega t$$

- We call a torus resonant when its frequency $\omega(I)$ is resonant, that is, there exist $k_1, \dots k_n$ entire numbers not all zero such that

$$k_1\omega^1 + \dots k_n\omega^n = 0.$$

- If $k_1\omega^1 + \dots k_n\omega^n = 0$ only happens when $k_1 = \dots = k_n = 0$ then the torus is non-resonant and the motion is quasiperiodic.

Integrable/ non-integrable Systems

We understand the phase space of Integrable Hamiiltonian Systems.

- All the pase space is folliated by tori.
- Any solution is confined in a torus.
- The angles of all solutions in a given torus move with the same frequency.
- All solutions are bounded.

Now the questions are:

- what happens if the system is not integrable?
- Are integrable systems abundant?

A paradigmatic example: the N body problem

• Understanding the motion of celestial bodies:
Copernicus, Kepler, Galileo, Leibniz, several of the Bernoullis, Euler, Lagrange, Clairaut, Laplace, Liouville, Hamilton, Jacobi, Gauss, Kowalevski, and many others....**Poincaré**.

• The "mathematical treatment" of the problem is own to **Newton: Philosophiae Naturalis Principia Mathematica, 1687**.

• What is the n body problem?
Is the dynamical system that describes the motions of n given masses interacting by mutual attraction according to Newton's law of gravity.
It is the basic mathematical model for a planetary system such as our solar system (where $n-1$ is the number of planets and the sun)

• First question: Is the n body problem integrable?

A paradigmatic example: the n body problem, $n = 2$

If one considers $n = 2$ two bodies:

$$\begin{aligned}
x_1' &= v_1 && (3 equations)\\
m_1 v_1'(t) &= G\frac{m_2 m_1 (x_2(t) - x_1(t))}{||x_2(t) - x_1(t)||^3} && (3 equations)\\
x_2' &= v_2 && (3 equations)\\
m_2 v_2'(t) &= G\frac{m_1 m_2 (x_1(t) - x_2(t))}{||x_1(t) - x_2(t)||^3} && (3 equations)
\end{aligned}$$

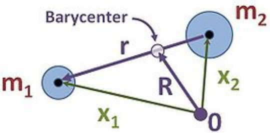
$$x_i = x_i(t) = (x_i^1(t), x_i^2(t), x_i^3(t)), \quad v_i = v_i(t) = (v_i^1(t), v_i^2(t), v_i^3(t))$$

$$||x_2 - x_1|| = \left((x_2^1 - x_1^1)^2 + (x_2^2 - x_1^2)^2 + (x_2^3 - x_1^3)^2\right)^{1/2}$$

The model for the 2 body problem gives 12 non-linear differential equations!!

The 2 body problem is INTEGRABLE

- Adding and substracting the previous equations the equations decouple and can be solved independently.
- Adding them we obtain the equations of the center of mass $R(t) = \frac{x_1(t)m_1 + x_2(t)m_2}{m_1 + m_2}$ one easily obtains $\ddot{R} = 0$.
- Substracting them one obtain the equations of the vector $r(t) = x_1(t) - x_2(t)$ which gives the relative positions of a body respect the other. These equations are the “Kepler problem” (a body moving around the sun) and can be integrated.
- Once we know $R(t)$ y $r(t)$ we can recover the trajectories of the 2 bodies $x_1(t)$ and $x_2(t)$.



Integrability of the n body problem for $n > 2$

- Using the language of integrable systems, we can say that the 2 body problem has enough constants of motion to be integrated.
- In fact, Ch.E. Delaunay ($1816 - 1872$) found the action-angle variables for the 2-body problem.
- Newton tried to "integrate" the 3-body problem (18 equations!) to calculate the motions of the earth-moon-sun system but he fail.
- Lots of matematicians tried to integrate it.
- Until $19th$ century, there was the idea that most of mechanical systems where integrable.
- Lapace (using a work of Lagrange) "proved" that the n body problem was integrable!
  Its mistake was to use a "simplifyed model" ignoring the interactions between the planets.

Integrability versus Stability

A related question to the integrability is the question of Stability.

- Is the solar system stable?

That is, even if its motions aren't periodic, or completely predictable, can we determine whether it will 'hang together,' or suffer calamity (fly apart or suffer internal collision) over some given long interval of time (say, the next few billion years)?

This question—the problem of the stability of the solar system—was present in some form in astronomers minds long before the mathematization of astronomy by way of Newton's laws.

After Kepler, Galileo, and above all Newton, the ancient problem of planetary motion was recast mathematically as the n body problem, and although initial hopes of solving it in the traditional sense withered, the stability problem remained and gained importance as it too was reformulated with more mathematical precision.

Nearly integrable hamiltonian systems and Hamiltonian perturbation theory

Recall that a hamiltonian system has a hamiltonian only depending on actions: $h = h(I)$, $I \in \mathbb{R}^n$.
We consider now a Hamiltonian of the form:

$$H(\theta, I; \varepsilon) = h(I) + \varepsilon f(\theta, I; \varepsilon)$$

- we call this Hamiltonian a perturbed Hamiltonian system
- $\varepsilon f(\theta, I; \varepsilon)$ is the perturbation
- ε is a "parameter" (a number) which measures the "strength" of the perturbation.
- When ε is small we say that we have a nearly integrable Hamiltonian System
- Hamiltonian perturbation theory studies the dynamics of nearly integrable hamiltonian systems.

An example of nearly integrable system: the n body problem

- Newton already used perturbation theory to study the 3-body problem.

$$
\begin{aligned}
x_1'' &= G\frac{m_2(x_2-x_1)}{||x_2-x_1||^3} + G\frac{m_3(x_3-x_1)}{||x_3-x_1||^3}\\
x_2'' &= G\frac{m_1(x_1-x_2)}{||x_1-x_2||^3} + G\frac{m_3(x_3-x_2)}{||x_3-x_2||^3}\\
x_3'' &= G\frac{m_1(x_1-x_3)}{||x_1-x_3||^3} + G\frac{m_2(x_2-x_3)}{||x_2-x_3||^3}
\end{aligned}
$$

- If m_2 and m_3 are small compared with m_1, the second and third equations are small pertubations of two different two body problems!
- If we ignore the interaction between the small bodies we have an integrable system!
- If we only consider the Sun and one planet and we ignore the interactions with the other planets we have a two body problem. The interactions between the planets are a small perturbation of the two body problem.

So, to understand the solar system we need to understand nearly integrable Hamiltonian systems.

An example of nearly integrable system: the n body problem

• As the n-body problem has the form of a nearly integrable system $H(\theta, I; \varepsilon) = h(I) + \varepsilon f(\theta, I; \varepsilon)$ several mathematicians and astronomers tried to find the solutions using power series.
• Karl Weierstrass (1815-1897) calculates formal series solutions (now a days known as Lindstedt series) for the n body problem.
He though they were convergent, but had difficulties to prove it. In fact, in 1858, one year before his death, Dirichlet said to L.Kronecker that he had shown the convergence.
• His friend Mittag-Leffler convinces the King of Norway (Oscar II) to put a price (in a scientific competition for his 60 birthday) for the proof of the convergence of these series. In the competition they used the expression: On the stability of our planetary system.
But then appears Henri Poincaré.

Hamiltonian perturbation theory

Poincaré (1854-1912) called **The fundamental problem of dynamics** to understand the dynamics of nearly integrable systems.
What can we do tho know if $H(\theta, I; \varepsilon) = h(I) + \varepsilon f(\theta, I)$ is integrable?
Jacobi (1804-1851) method: look for a change of variables:

$$(\theta, I) = T(\phi, J; \varepsilon)$$

Such that:

- The new system is again Hamiltonian (these changes are called Canonical or Symplectic)
- The new Hamiltonian only depends on the new actions J.
- We expect the change to be of the form $\mathrm{Id} + \mathcal{O}(\varepsilon)$

How can we find the change of variables T?

Hamiltonian perturbation theory

A way to obtain a symplectic change of variables $(\theta, I) = T(\phi, J; \varepsilon)$ is to take $T(\phi, J; \varepsilon) = \varphi(\varepsilon, \phi, J)$ where $\varphi(t, x)$, $x = (\phi, J)$ is the flow (the general solution) of "another" hamiltonian system of Hamiltonian χ:

$$\frac{d}{d\,t}\varphi(t,x) = \mathcal{J}\nabla\chi(\varphi(t,x)), \quad \varphi(0,x) = x$$

Using Taylor's formula:

$$T(x;\varepsilon) = \varphi(\varepsilon,x) = \varphi(0,x) + \frac{d}{dt}\varphi(0,x)\varepsilon + \mathcal{O}(\varepsilon^2) = x + \mathcal{J}\nabla\chi(x)\varepsilon + \mathcal{O}(\varepsilon^2)$$

Therefore the change has the form:

$$(\theta, I) = T(\phi, J; \varepsilon) = (\phi, J) + \varepsilon\mathcal{J}\nabla\chi(\phi, J) + \mathcal{O}(\varepsilon^2)$$

And the new hamiltonian will be $K(\phi, J; \varepsilon) = H(T(\phi, J; \varepsilon); \varepsilon)$. If we use Taylor again:

$$K(\phi, J; \varepsilon) = H(\phi, J; \varepsilon) + \varepsilon DH(\phi, J; \varepsilon)\mathcal{J}\nabla\chi(\phi, J) + \mathcal{O}(\varepsilon^2)$$

Hamiltonian perturbation theory

We have then:

$$K(\phi, J; \varepsilon) = h(J) + \varepsilon f(\phi, J) + \varepsilon\{\chi, h\}(\phi, J) + \mathcal{O}(\varepsilon^2)$$

So, to eliminate the angle ϕ from the $\mathcal{O}(\varepsilon)$ terms we need to find a (Hamiltonian) function χ that solves:

$$0 = f(\phi, J) + \{\chi, h\}(\phi, J) = f(\phi, J) + \sum_{k\in\mathbb{Z}^n} \frac{\partial\chi}{\partial\phi_k}\frac{\partial h}{\partial J_k}$$

As all the functions we look for are periodic in ϕ, to solve this equation we expand by Fourier series the previous functions:

$$f(\phi, J) = \sum_{k\in\mathbb{Z}^n} \hat{f}_k(J)e^{2\pi ik\dot{\phi}}, \quad \chi(\phi, J) = \sum_{k\in\mathbb{Z}^n} \hat{\chi}_k(J)e^{2\pi ik\dot{\phi}}$$

and the previous equation becomes

Hamiltonian perturbation theory

$$\sum_{k\in\mathbb{Z}^n} \hat{f}_k(J)e^{2\pi ik\dot{\phi}} + 2\pi i \sum_{k\in\mathbb{Z}^n} \hat{\chi}_k(J)k\cdot\omega(J)e^{2\pi ik\dot{\phi}}$$

recall $\omega(J) = \nabla h(J)$. Which gives:

$$2\pi ik\cdot\omega(J)\hat{\chi}_k(J) = -\hat{f}_k(J)$$

Observation: For $k = 0$ we can not solve this equation, therefore we can not eliminate $\hat{f}_0(J)$ but this is not problem because this term does not depend on ϕ. If we can solve the rest of equations for $k \neq 0$, the new hamiltonian will be $K(J) = h(J) + \varepsilon\hat{f}_0(J) + \mathcal{O}(\varepsilon^2)$ and we are happy!!
Then, apparently, we can solve it:

$$\hat{\chi}_k(J) = \frac{-\hat{f}_k(J)}{2\pi ik\cdot\omega(J)}, \quad k \neq 0$$

Now that we have χ that eliminates the terms of order $\mathcal{O}(\varepsilon)$ we can proceed analogously and eliminate the terms of order ε^2 etc.

The small divisors problem

Recall that to see that the nearly integrable Hamiltonian $H(\theta, I; \varepsilon) = h(I) + \varepsilon f(\theta, I)$ is integrable up to order $\mathcal{O}(\varepsilon)$ we need to find a change related with the function χ whose Fourier coefficients are given by:

$$\hat{\chi}_k(I) = \frac{-1}{2\pi i} \frac{\hat{f}_k(I)}{k \cdot \omega(I)}, \quad k \neq 0$$

• If $n = 1$, then $k \in \mathbb{Z}$ and everything is ok. Hamiltonian systems of 1-degree of freedom are integrable.

• But for $n \geq 2$, $k \in \mathbb{Z}^n$, and we already saw that there will be I such that their frequency $\omega(I)$ will be resonant, and therefore $k \cdot \omega(I) = 0$ for some $k \in \mathbb{Z}^n$ with $|k| \neq 0$, and the change will not be defined for these values of the actions!!

• The portion of phase space where the divisors vanish has Lebesgue measure 0, but is dense. Therefore, even if we remove this set, the divisors still become arbitrarily small on the remaining part of phase space.

The small divisors problem: Poincaré

• It seems impossible for most of perturbations εf that the series converge and the new system is integrable in a nice open domain.
• Poincaré saw the problem of convergence of these series in any open set and therefore that most systems are not integrable. Poincaré non-existence theorem. Moreover

- He studied more deeply what hapens with the unperturbed resonant tori. The resonant tori split under the perturbation and only some periodic orbits survive the perturbation.
- The majority of the orbits follow a complicated dynamics that he called a sort of trellis.
- This complicated orbits can not exist in an integrable system!

The small divisors problem: Poincaré

When one tries to depict the figure formed by these two curves and their infinity of intersections, each corresponding to a doubly asymptotic solution, these intersections form a sort of trellis, web, or infinitely tight mesh; neither of the two curves can ever intersect itself, but must fold back on itself in a very complex way in order to intersect all the links of the mesh infinitely many times. One is struck by the complexity of this figure that I shall not even attempt to draw. Nothing is better suited to give us an idea of the complexity of the three body problem and all of the problems of dynamics in general where there is no uniform integral and Bohlin's series diverge.

In modern language: What Poincaré discovered is Chaos: the stable and unstable manifolds of these periodic orbits intersect giving rise to chaotic behaviour.

The post-Poincaré era; ergodicity

- Ergodicity is a stronger form of non-integrabilty:
- An ergodic system has trajectories that do not confine themselves to separate portions of phase space; instead, trajectories pass through almost all points in phase space that are energetically accessible.
- Mathematically in an ergodic system the only invariant sets have measure 0 or 1.
- Boltzmann was one of the fundators of stadistical mechanics (lots of partcles in interaction) and in 1871 stablished the so-called ergodig hypothesis of Boltzmann which assumes that most dynamical systems are ergodic.
- It was clear that ergodicity implies non-integrability, but after Poincaré there were lots of researchers thinking that non-integrability also implied ergodicity and that most of Hamiltonian systems were ergodic.
- This was the situation until Kolmogorov anounced his theorem in the International Mathematical Congress of 1954 in Amsterdam.

Kolmogorov solves the problem of small divisors

Recall that to find the change which eliminates the ε terms in the Hamiltonian $H(\theta, I; \varepsilon) = h(I) + \varepsilon f(\theta, I)$ we need to find the function χ whose Fourier coefficients are given by:

$$\hat{\chi}_k(I) = \frac{-1}{2\pi i} \frac{\hat{f}_k(I)}{k \cdot \omega(I)}, \quad k \neq 0$$

and this function is not defined in the dense set of resonant tori.
Kolmogorov has a different idea:
What if we don't look for a classical transformation defined on an open set and we just look for some nonempty set on which we can define the transformation?
Using this idea he was able to prove that the set of quasiperiodic tori which survive the perturbation had positive measure.
Therefore, the ergodic hypothesis was false!

The Siegel problem

The idea of Kolmogorov is inspired by a problem that Carl Ludwig Siegel (1896-1981) solved before in 1942: the linearization of a center.
$F : \mathbb{C} \to \mathbb{C}$ and local holomorphic map, $F(0) = 0$, $F'(0) = \lambda$.
Therefore $F(z) = \lambda z + f(z)$, where $f(0) = f'(0) = 0$, analogously $f(z) = \mathcal{O}(z^2)$.
The question is: there exists Φ biholomorphic such that the change $w = \Phi(z)$ transforms the map F to the linear map $L(z) = \lambda z$?
One can easily see that Φ has to satisfy:

$$\Phi(F(z)) = \lambda\Phi(z), \quad \Phi(\lambda z + f(z)) = \lambda\Phi(z).$$

If we write this equation in power series: $f(z) = \sum_{j\geq 2} f_j z^j$, and we look for $\Phi(z) = z + \sum_{j\geq 2} \Phi_j z^j$ we obtain the set of equations:

$$\lambda(1-\lambda)\Phi_2 = f_2$$
$$\lambda(1-\lambda^2)\Phi_3 = f_3 + 2\lambda f_2\Phi_2$$
$$\lambda(1-\lambda^{n-1})\Phi_n = f_n + \ldots$$

The Siegel problem

The general equation to solve for ϕ_n is:

$$\lambda(1-\lambda^{n-1})\Phi_n = f_n + \dots$$

From these equations we have:

- If the fix point is hyperbolic: $|\lambda| \neq 1$, we can solve the equations and obtain a formal series which solves the problem. In fact, one can prove that the series for Φ has positive radius of convergence. This was proven by Poincaré using a slightly different method.
- In the elliptic case $|\lambda| = 1$, it is clear that the equations can not be solved if λ is a root of unity.
- Even if λ is not a root of unity, its powers may accumulate to 1. This would give small divisors in the formal series of ϕ which casts doubt on its convergence. We face an analogous problem to the one in near integrable systems!

The Siegel problem

To solve this problem C.L. Siegel, writing $\lambda = e^{2\pi i\alpha}$, introduced the following Diophantine conditions. For some $\gamma > 0$ and $\tau > 0$ we require that:

$$|\alpha - \frac{p}{q}| \geq \frac{\gamma}{q^{\tau}}$$

It turns out that if this condition is satisfied he was able to prove the convergence of the formal solution $\Phi(z)$.

• One can see that, if $\tau > 2$ and $\gamma > 0$ is small enough, the values of λ satisfying the diophantine conditions form a set satisfying:

- It is a Cantor set in $\mathbb{T}^1$ (nowhere dens, compact and perfect).
- The measure of its complementary is $\mathcal{O}(\gamma)$, as $\gamma \to 0$.

Now we will see how Kolmogorov used these ideas to prove his theorem about the persistence of tori.

Kolmogorov solves the problem of small divisors

Let's now go back to the nearly-integrable Hamiltonian and the transformation χ, which needs to satisfy:

$\hat{\chi}_k(J) = \frac{-1}{2\pi i}\frac{\hat{f}_k(J)}{k\cdot\omega(J)}$, $k \neq 0$

Choose a closed ball $B \subset \mathbb{R}^n$. The main observations are:

- If f is a smooth function, we know that $||\hat{f}_k||_B \leq \frac{C}{|k|^b}$ for some $b > 1$ (or $||\hat{f}_k||_B \leq Ce^{-k\rho}$ if it is analytic, which is the case Kolmogorov considered).
- If $\gamma > 0$ is small enough and $\tau > n-1$, the set of diophantine frequecies $\mathcal{D}^\omega = \{\omega \in \mathbb{R}^n,\ |k\cdot\omega| \geq \frac{\gamma}{|k|^\tau}\}$ is not empty.
- If the unperturbed hamiltonian h is non-degenerate ($\omega(J) = \nabla(J)$ is a difeo), then the corresponding set of J- values: $\mathcal{D}^J = \{J \in \mathbb{R}^n,\ \omega(J) \in \mathcal{D}^\omega\}$ is not empty.

Then, the idea of Kolmogorov is to adapt the method of Siegel to define the function χ in this Cantor set.

Kolmogorov solves the problem of small divisors

• If we take the values of J in the set $\mathcal{S} = \mathcal{D}^J$ we obtain the following bounds for the Fourier coefficients of χ:

$$||\hat{\chi}_k(J)|| = ||\frac{-\hat{f}_k(J)}{2\pi i k \cdot \omega(J)}|| \leq \frac{C}{2\pi\gamma|k|^{b-\tau}}$$

then, if $b - \tau > n$ one can see that the Fourier series for χ is absolutely convergent.

• Remember that χ was only the first step in the procedure! it was constructed to eliminate the terms of order ε in the Hamiltonian!

• We are involved in an iterated procedure (or a power series) and we have to face a problem that Siegel did not had 12 years before, in all the steps of the iteration or the series in Siegel problem, the λ was the same value. This will be different here.

Kolmogorov solves the problem of small divisors

- We take a $J \in D^J$ the good set of actions such that $\omega(J) = \nabla h(J)$ is a good diophantine frequency.
- After the first step the new Hamiltonian will be

$$h(J) + \varepsilon \hat{f}_0(J) + \varepsilon^2 R(\phi, J; \varepsilon),$$

the integrable part is $h_1(J; \varepsilon) = h(J) + \varepsilon \hat{f}_0(J)$ and the frequency has changed $\omega_\varepsilon(J) = \nabla h(J) + \varepsilon \nabla \hat{f}_0(J)$.

- The frequency changes too much and can go closer to the resonant set.
- In the general step of our procedure we would have a Hamiltonian

$$K_n(\phi, J; \varepsilon) = h_n(J; \varepsilon) + \varepsilon^{n+1} R_n(\phi, J; \varepsilon).$$

At this step we have the same problem: the ω changes too much and can go too close to be resonant and this makes the iterative scheme not convergent as $n \to \infty$.

Kolmogorov solves the problem of small divisors

- Kolmogorov new how to overcome this problem.
- His idea was to keep ω fixed and a good diophantine frequency.
- Then look for a process that finds the good action iteratively, not necessarily the same as in the unperturbed case.
- But how to solve the problem that everything changes too much at any iteration? how to control this?
- He needed a method wich converges faster than the classical iterative methods.
- His idea was: Don't use a clasical iterative method but a different method wich convergest more fast, in fact quadratically.

A fast overview of iterative methods to solve $G(x) = 0$

• To solve $G(x) = 0$ we transform this equation in another one: $x = M(x)$, then we apply the iteration method: $x_{n+1} = M(x_n)$ and we choose a good approximation x_0 to begin the iteration.
If M is a contractive function (has a small Lipchitz constant $0 < L < 1$), we will have that the error decreases linearly, if we cal x^* to the solution:

$$e_{n+1} = ||x_{n+1} - x^*|| = ||M(x_n) - M(x^*)|| \leq Le_n$$

In our problem, due to the presence of small divisors, the "natural" function M is not a contraction and the method does not converge.
• The alternative is to use the classical Newton method: consider $M(x) = x + DG(x)^{-1}G(x)$, equivalently: $x_{n+1} = x_n + DG(x_n)^{-1}G(x_n)$
• This method satisfies
$e_{n+1} \leq Ke_n^2$,
therefore the error decreases faster!

Kolmogorov solves the problem of small divisors

- The idea of Kolmogorov was to use a Newton method in a functional space (L.V. Kantorovich generalized Newton method to functional spaces) to find the "change" of variables $T(J, \phi; \varepsilon)$.
- The rapid convergence of the Newton method combined with the diophantine conditions of the fixed frequency gives the existence of a transformation of the Hamiltonian to the integrable form $K_\infty = h_\infty(J; \varepsilon)$ fot any given J in the Cantor set $\mathcal{D}^J$.
- In this way Kolmogorov proof gives the existence of an invariant torus in the original system correponding to the value $J^* \in \mathcal{D}^J$ with frequency ω.
- The torus in the original variables is given by $(\theta, I) = T(\phi, J^*, \varepsilon) = (\phi, J^*) + \dots$.

Kolmogorov solves the problem of small divisors

- The discovery of kolmogorov was the begining of KAM theory.
- The set of surviving tori has large Lebesgue measure if $\gamma > 0$ is small.
- It can be shown that there are non-countably many invariant tori.
- Kolmogorov gave the ideas of the proof in the analytic case in a short paper in 1954
- in 1962 J. Moser proved the k-differentiable case for $k \geq 333$ using ideas of J. Nash.
- In 1963 V. Arnold gave a complete proof of Kolmogorov result.

Moser also proved that in these tori the Lindsted series were convergent, and therefore Weierstrass was right; the series converge for a large set of initial conditions!
But also Poincaré was right, when he said that they could not converge in open sets!

The n body problem and the problem of stability

- The non-degeneracy condition of Kolmogorov is $\det \frac{\partial^2 h}{\partial I^2} \neq 0$.
- This condition does not work in the n- body problem.
- Arnold introduced the isoenergetic condition which applies to fixed energy levels.
- Arnold also reformulated the KAM theorem to apply it to certain versions of the n body problem. (Unfortunately, a mistake in this application was later discovered, then repaired, first by M. Herman and J. Féjoz, then by L. Chierchia and coworkers F. Pusateri and G. Pinzari)
- Even if the theorem can be applied, this guaranties a large set where the motion is stable, but can we say something of the rest of orbits?

This would need another talk!!!

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