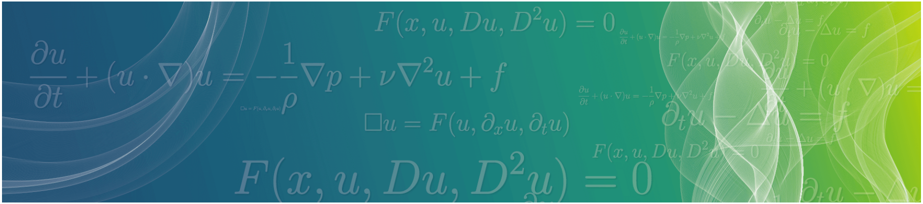
Generic regularity in free boundary problems

Xavier Ros Oton

Universität Zürich

Barcelona, November 2019



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- De Giorgi - Nash (1956-1957): YES, u is always C^1 ! (and hence C^∞)

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- OPEN PROBLEM: What happens in $\mathbb{R}^3$ and $\mathbb{R}^4$?

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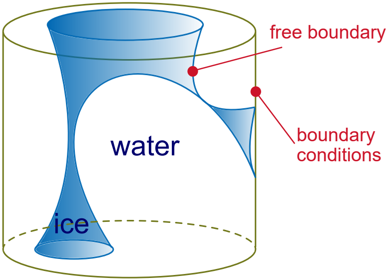

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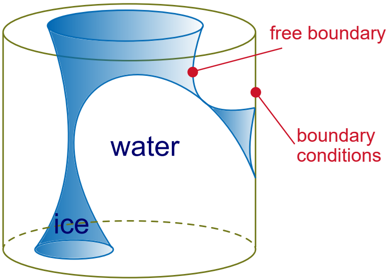

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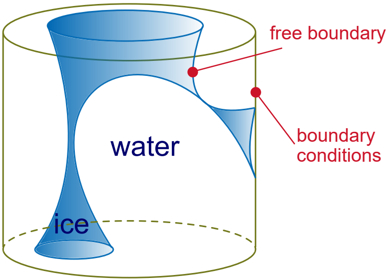

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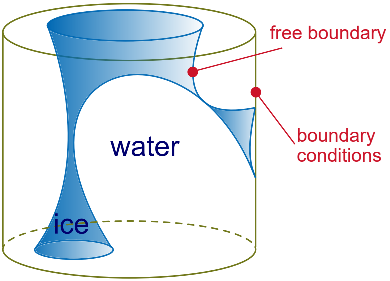
  - $u := \int_0^t \theta \geq 0$ solves:

$$\boxed{u_t - \Delta u = -\chi_{\{u>0\}}}$$



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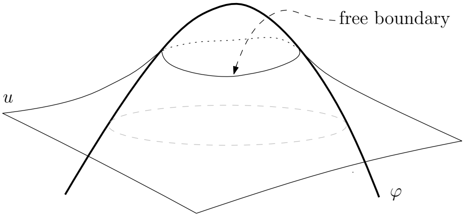
Given $\varphi \in C^\infty$, minimize

$$\mathcal{E}(v) = \int_\Omega |\nabla v|^2 dx$$

with the constraint $\boxed{v \geq \varphi}$



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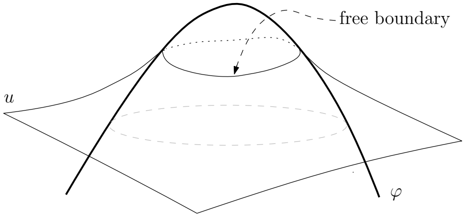

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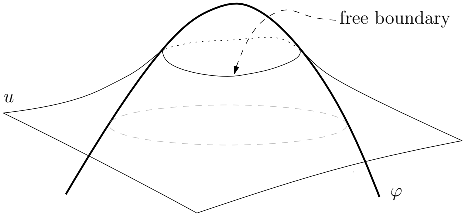

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Taking $u = v - \varphi$, we get...

$$\begin{cases} u \geq 0 & \text{in} \quad \Omega, \\ \Delta u = 1 & \text{in} \quad \{x \in \Omega : u > 0\} \\ \nabla u = 0 & \text{on} \quad \partial\{u > 0\}. \end{cases} \quad \longleftrightarrow \quad \begin{array}{c} u \geq 0 \quad \text{in } \Omega \\ \Delta u = \chi_{\{u>0\}} \quad \text{in } \Omega \end{array}$$

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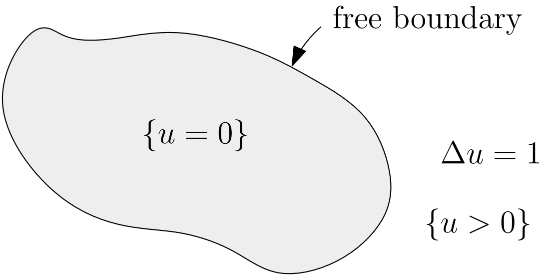
The free boundary (FB) is the boundary $\partial\{u > 0\}$



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- Random matrices...

All these examples give rise to the obstacle problem!

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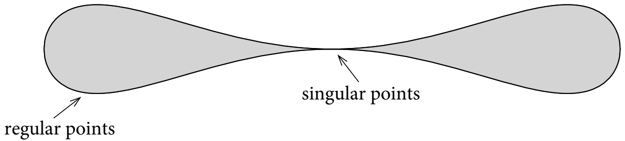

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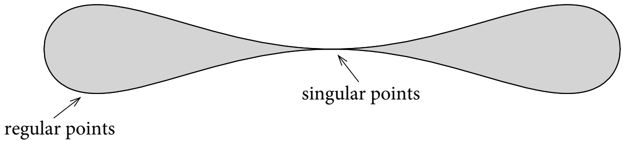

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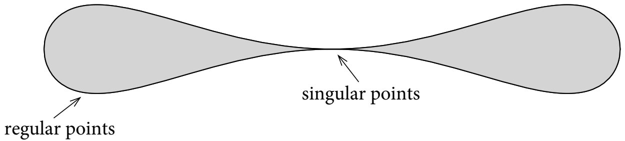

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SHAW PRIZE '18!

Shaw Prize 2018: Luis Caffarelli

"For his groundbreaking work on PDEs, including creating a theory of regularity for nonlinear equations and free boundary problems such as the obstacle problem, work that has influenced a whole generation of researchers in the field."

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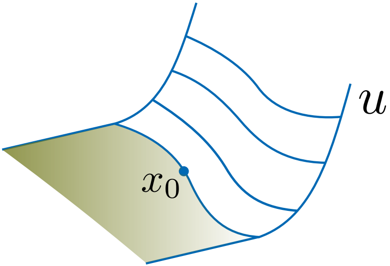

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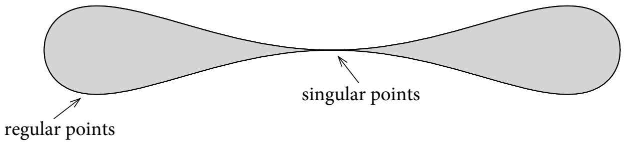

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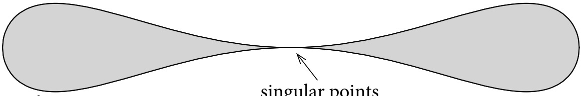

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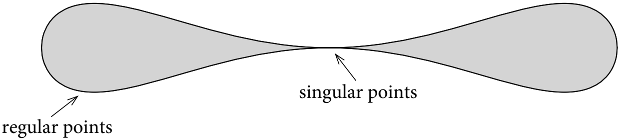

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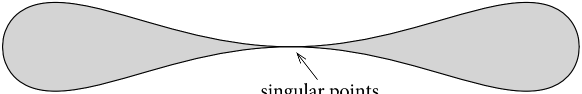

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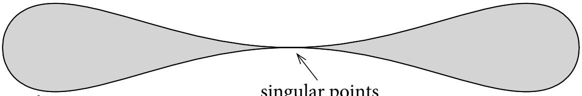

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- Figalli-Serra (2017): Outside a small set of dimension $n-3$, singular points are contained in a $C^{1,1}$ manifold.

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- Nothing known in higher dimensions!

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- In other words: Generically, the singular set is very small!

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- In a future paper, we will apply these techniques to the Stefan problem.

Thank you!