Regularity of stable solutions to semilinear elliptic equations

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Workshop in honor of Alessio Figalli's Doctor Honoris Causa at UPC

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- The Laplacian.

$$-\Delta u = -(u_{x_1x_1} + \dots + u_{x_nx_n}) \quad ; \quad u = u(x),\ x \in \Omega \subset \mathbb{R}^n$$

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- The Laplacian is an isomorphism between the spaces:

$$-\Delta u = -(u_{x_1x_1} + \dots + u_{x_nx_n}) \quad ; \quad u = u(x)\,,\ x \in \Omega \subset \mathbb{R}^n$$

$$X := \{u \in C^{2,\alpha}(\overline{\Omega}) : u_{|\partial\Omega} \equiv 0\}$$

$$\Big\downarrow -\Delta$$

$$Y := C^{\alpha}(\overline{\Omega})$$

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and $0<\alpha<1$

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where $\Omega\subset\mathbb{R}^n$ bdd. smooth domain
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That is: $\forall g\in C^{\alpha}(\bar\Omega)=Y$ $(g=g(x))$ $\exists!$ $u\in X$ solution of

$$\begin{cases} -\Delta u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

← The Dirichlet pb. for the Laplacian.

and $\quad c_1\|g\|_{C^{\alpha}(\bar\Omega)} \leq \|u\|_{C^{2,\alpha}(\bar\Omega)} \leq C_2\|g\|_{C^{\alpha}(\bar\Omega)}$

- A nonlinear (reaction-diffusion) PDE:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$\lambda \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ the "nonlinearity"

Take $f(0) > 0$

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$u \equiv 0$ is solution only for $\lambda = 0$.

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Solving the problem for λ small through the Implicit Function Theorem:

$$\phi : X \times \mathbb{R} \longrightarrow Y$$
$$(u, \lambda) \longmapsto \phi(u, \lambda) := -\Delta u - \lambda f(u)$$

$\phi(0,0) = 0$. Want $u = u(\lambda)$ for λ small, through IFT $\longrightarrow$

Linearized operator $= D_u \phi(0,0) = ?$

$$-\Delta(u + \varepsilon \xi) - \lambda f(u + \varepsilon \xi) = -\Delta u - \lambda f(u) + \varepsilon \left(-\Delta \xi - \lambda f'(u) \xi\right) + O(\varepsilon^2)$$

At $(u, \lambda) = (0,0)$,

$D_u \phi(0,0) \cdot \xi = -\Delta \xi$ isomorphism ☺

The nonlinear problem has a Lagrangian or "energy" behind:

$$E(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - \lambda F(v) \quad , \text{ where } F' = f.$$

- 1st variation of energy: $\frac{d}{d\varepsilon} \int_\Omega \frac{1}{2}|\nabla u + \varepsilon \nabla \xi|^2 - \lambda F(u+\varepsilon\xi)$

$$= \int_\Omega (\nabla u + \varepsilon \nabla \xi)\nabla \xi - \lambda f(u+\varepsilon\xi)\xi \rightsquigarrow$$

u critical point of E

$\Updownarrow$

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} = 0$ i.e.,

$$0 = \int_\Omega \nabla u \nabla \xi - \lambda f(u)\xi = \int_\Omega (-\Delta u - \lambda f(u))\xi$$

$$\forall \xi \Rightarrow \boxed{-\Delta u - \lambda f(u) = 0.}$$

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- 2nd variation of energy:

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} = \int_\Omega |\nabla \xi|^2 - \lambda f'(u)\xi^2 = \langle \underbrace{(-\Delta - \lambda f'(u))}_{\text{Linearized operator}}\xi, \xi\rangle_{L^2(\Omega)}$$

$(u,\lambda) = (0,0) \rightarrow$ Lin. op'r $= -\Delta > 0$

POSITIVE OPERATOR

$$\left\{ \langle -\Delta \xi, \xi \rangle_{L^2(\Omega)} = \int_\Omega |\nabla \xi|^2 \geq 0. \right.$$

• **Def'n** A solution of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$f: \mathbb{R} \to \mathbb{R}$ is the "nonlinearity"

is said to be a **stable** solution

if and only if

$-\Delta - f'(u) \geq 0$ as operator, i.e.,

$$\langle -\Delta\xi - f'(u)\xi, \xi \rangle_{L^2(\Omega)} = \underline{\int_\Omega |\nabla\xi|^2 - f'(u)\xi^2 \geq 0} \quad \forall \xi \in \underline{X}$$

$\xi \in C^{2,\alpha}$, $\xi|_{\partial\Omega} \equiv 0$.

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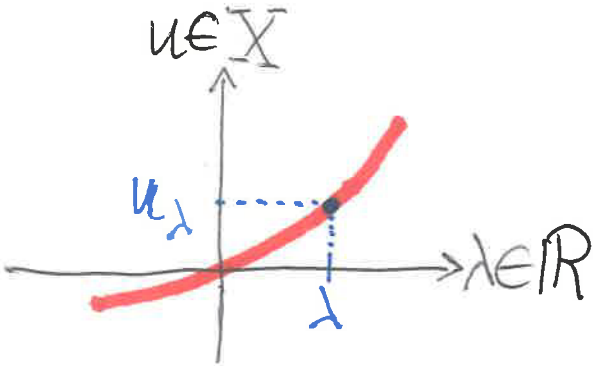
Thus, stable solutions correspond to critical points of the energy which are local minima $\longleftrightarrow$ observable solutions

or solutions that persist under small perturbations

or stationary solutions that are limits as $t\to+\infty$ of nonlinear heat flows after small perturbation of the initial datum.



$$\begin{cases} -\Delta u_\lambda = \lambda f(u_\lambda) & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases}$$

Linearized operator at u_λ:

$-\Delta - \lambda f'(u_\lambda)$

As long as the linearized operator at u_λ is $\boxed{>0}$, it will be an isomorphism and the branch of solutions can be continued through the IFT.

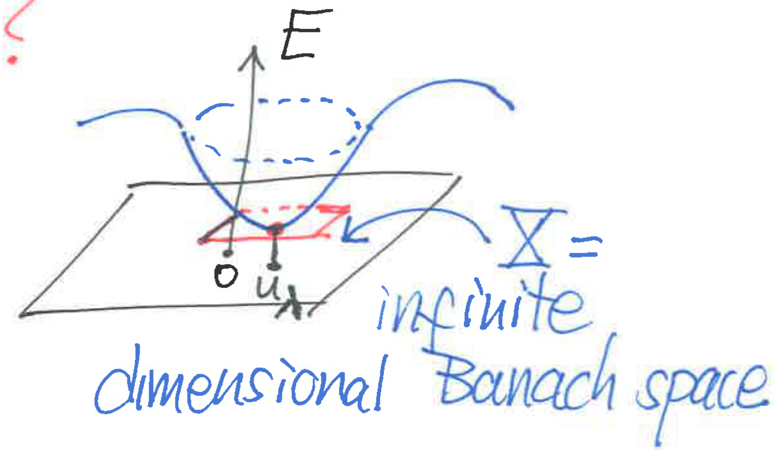
Model case : $f(u)=e^u$

Are there solutions for $\lambda>0$ large ?

"Drawing" the energy :

$v>0$ in Ω $\rightsquigarrow$ $E(tv)=$

$$= \frac{t^2}{2}\int_\Omega |\nabla v|^2 - \lambda \int_\Omega e^{tv}$$



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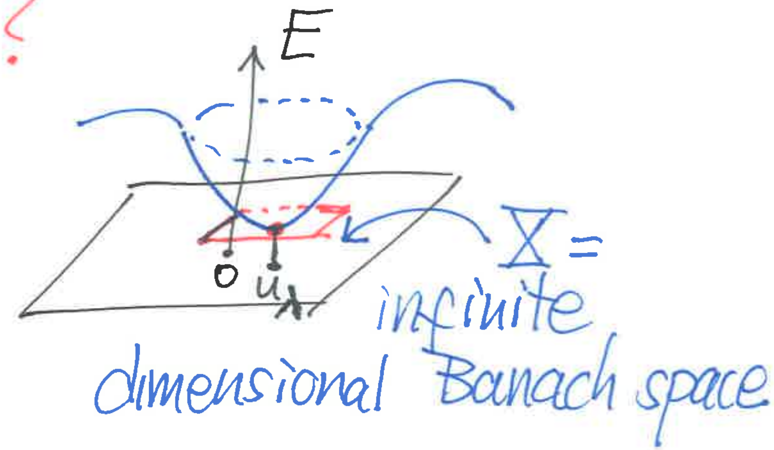
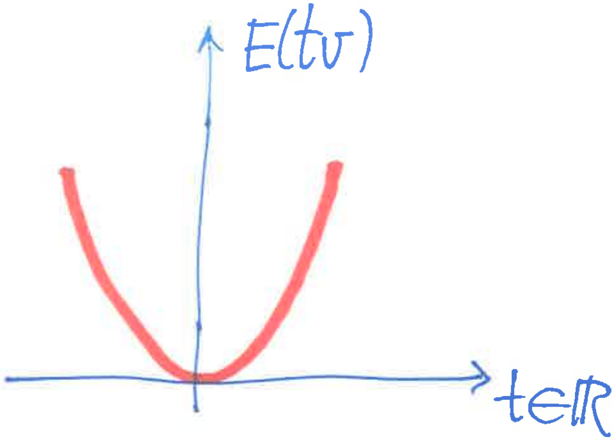
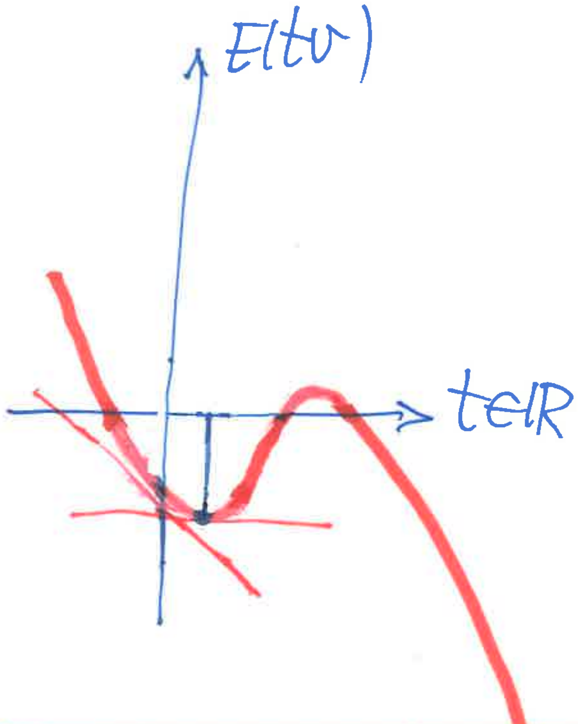


$\lambda=0$



$\lambda>0$ small



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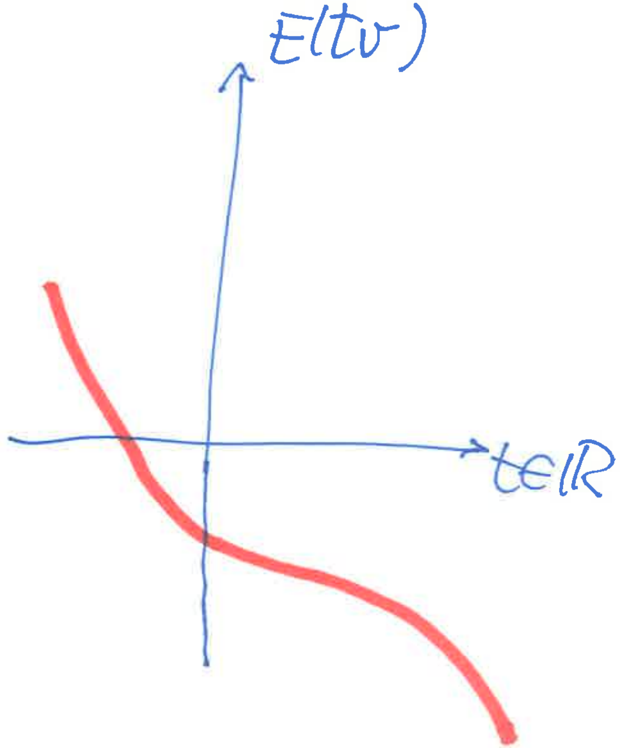

- The Barenblatt-Gelfand problem 1963 :

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > 0$, $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

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with $\lambda > 0$, $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

- Then, $\exists \lambda^* \in (0, +\infty)$ & $0 < \lambda < \lambda^* \Rightarrow \exists u_\lambda > 0$ stable classical (L^∞) sol'n

- $u_\lambda \nearrow u^*$ as $\lambda \nearrow \lambda^*$

  $\hookrightarrow$ $u^* \in L^1(\Omega)$ is a distributional stable solution for $\lambda = \lambda^*$

  u^* = the extremal solution of the pb.

- $\nexists$ solutions for $\lambda > \lambda^*$

Model nonlinearities: $f(u) = e^u$ (combustion theory)

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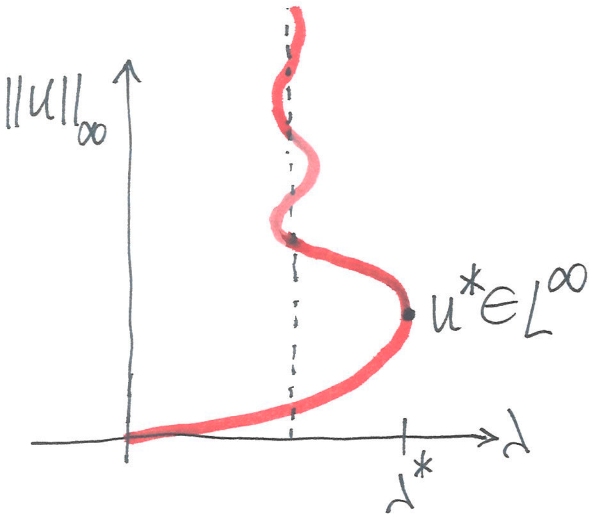
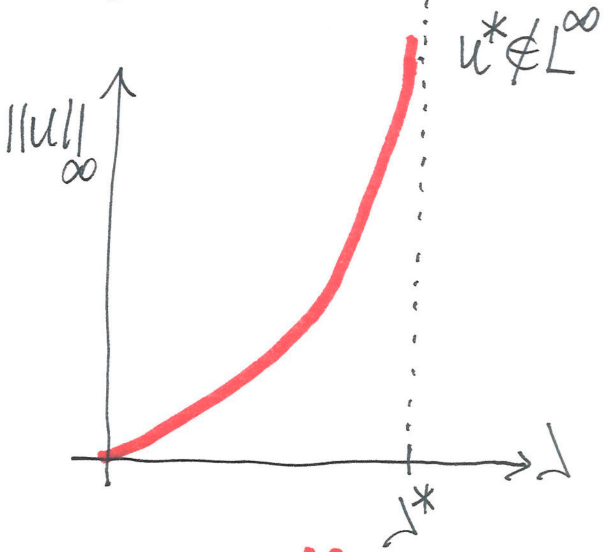
• [Joseph-Lundgren '72] $f(u) = e^u$ & $\Omega = B_1$ (RADIAL case) :



$3 \leq n \leq 9$



$n \geq 10$

▪ ODE techniques

• **Hardy's inequality:**

When is the operator $-\Delta - \frac{c}{|x|^2} \geq 0$ in Ω ?

• **Prop'n** $0 \in \Omega \subset \mathbb{R}^n$, $\xi \in C_c^1(\Omega)$ [compact support in Ω], $n \geq 3 \Rightarrow$

$$\frac{(n-2)^2}{4} \int_\Omega \frac{\xi^2}{|x|^2} \leq \int_\Omega |\nabla \xi|^2$$

& this is the best constant in the inequality

ANSWER: if and only if $c \leq (n-2)^2/4$

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- **Proof:**

$$\int_\Omega \frac{\xi^2}{|x|^2} = \int_{S^{n-1}} d\sigma \int_0^{+\infty} dr\, \xi^2(r\sigma)\, \underbrace{r^{n-3}}_{= \frac{1}{n-2}(r^{n-2})'} = -\frac{1}{n-2} \int_{S^{n-1}} d\sigma \int_0^{+\infty} dr\, 2\xi\, \xi_r\, \underbrace{r^{n-2}}_{= r^{\frac{n-1}{2}} r^{\frac{n-3}{2}}}$$

& Cauchy-Schwarz ∎

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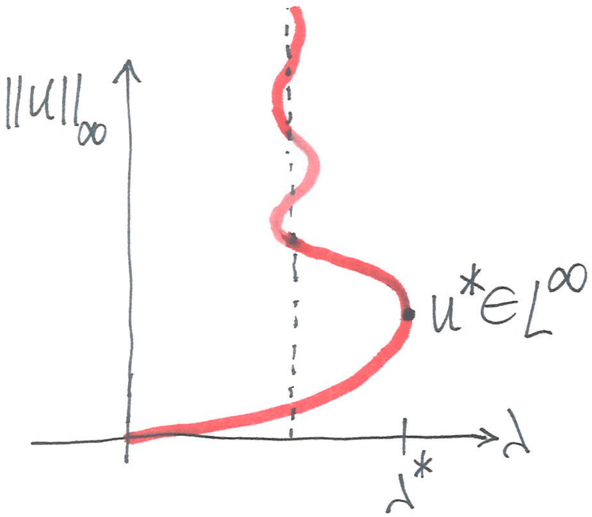

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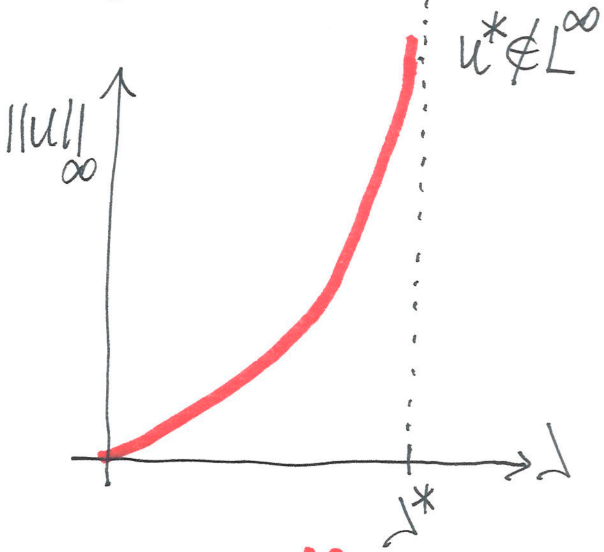

$n\geq 10$

■ ODE techniques

■ Explicit singular solution :

$u(x)=-2\log|x| \in W_0^{1,2}(B_1)$

solves $-\Delta u = 2(n-2)\, e^u$ in B_1, $n\geq 3$

Linearized operator $= -\Delta - 2(n-2)\frac{1}{|x|^2}$

(Hardy's ineq) $\rightarrow$ u stable $\Leftrightarrow$ $2(n-2) \leq \frac{(n-2)^2}{4}$ $\Leftrightarrow$ $n\geq 10$

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  - $\rightarrow$ Not true for $n \geq 8$
  - $\rightarrow$ True for $n=3$ ([Fischer-Colbrie & Schoen '80] [Do Carmo & Peng '79])
  - $\rightarrow$ Open pb for $4 \leq n \leq 7$!!

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■ 1st result $\forall \Omega$: [Crandall-Rabinowitz '75]
$u^* \in L^\infty(\Omega)$ if $n \leq 9$ and $f(u) \sim e^u$ or $f(u) \sim (1+u)^p$

- [Brezis-Vázquez '97] Is it always $u^* \in W_0^{1,2}(\Omega)$?
- [Brezis '03] Is there something "sacred" about dim 10 ?
  Is it possible to construct a singular stable sol'n for $n \leq 9$, in some domain & for some f ?

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  & Interior L^∞ bound if $n \le 4$ $\forall f$

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- [Villegas '13] $u^* \in L^\infty(\Omega)$ if $n \leq 4$; $u^* \in W_0^{1,2}(\Omega)$ if $n \leq 6$

- [Cabré & Ros-Oton '13] L^∞ if $n \leq 7$ & Ω of double revolution
- [Cabré-Sanchón-Spruck '16] L^∞ if $n \leq 5$ & $f'/f^{1+\varepsilon} \leq C(\varepsilon)$ $\forall \varepsilon > 0$

■ [Cabré, Figalli, Ros-Oton, Serra '19]

Thm 1 $u \in C^2(B_1)$ stable sol'n of $-\Delta u = f(u)$ in B_1 & $f \geq 0 \Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C(n) \|u\|_{L^1(B_1)} \quad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(n) \|u\|_{L^1(B_1)}$ $(\alpha = \alpha(n) > 0)$.

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Corol 1 $L^\infty(\Omega)$ estimate for $n \leq 9$ (if $f \geq 0$) and any stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$ if Ω is bdd convex C^1 domain.

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Thm 2 Ω bdd C^3 domain, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$.

$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ $\Rightarrow$

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Corol 2 Ω bdd C^3 domain $\Longrightarrow$ $\begin{cases} u^* \in W_0^{1,2+\gamma}(\Omega) & (\gamma = \gamma(n) > 0) \\ \text{if } n \leq 9,\ u^* \in L^\infty(\Omega). \end{cases}$

Thm 3 Sharp Morrey $M^{p,\beta}(\Omega)$ estimates for stable sol'ns when $n \geq 10$.

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Thm 3 $\quad$ Sharp Morrey $M^{p,\beta}(\Omega)$ estimates for stable sol's when $n \geq 10$.

• PROOFS :

$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \qquad \forall \xi \in C_c^1(\Omega)$$

$$\Big\downarrow \ \xi = c \cdot \eta_{|\partial\Omega} \equiv 0$$

$$\int_\Omega c\,(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2 .$$

• Proofs $\xi = c\cdot\eta_{|\partial\Omega} \equiv 0 \Rightarrow \int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2 |\nabla\eta|^2.$

- [Crandall-Rabinowitz] & [Nedev] : $\underline{\xi = h(u)}$
- [Cabré-Capella] ($\Omega = B_1$) : $\xi = \underbrace{\boxed{r u_r}}_{= c} \cdot \underbrace{r^{-a}\varphi}_{= \eta}$, φ cut-off near ∂B_1

$= x\cdot\nabla u \; |x|^{-a}\cdot\varphi$

- [Cabré '10] : ($n \leq 4$) $\xi = \underbrace{\boxed{|\nabla u|}}_{= c} \cdot \underbrace{g(u)}_{= \eta}$

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For our interior result ($n \leq 9$) we will use both
$c = x \cdot \nabla u$
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$c = x\cdot\nabla u$

& $c = |\nabla u|$

$\underline{(\Delta + f'(u))(x\cdot\nabla u)} = 2\Delta u$

& $\underline{(\Delta + f'(u))|\nabla u|} = \frac{1}{|\nabla u|}\left\{ \sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2 \right\}$

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x)$ $\rightsquigarrow$ $\int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

<u>Lemma 1</u> $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C_c^1(B_1) \implies$

$$\int_{B_1} \underbrace{\{(n-2)\eta + 2x\cdot\nabla\eta\}}_{0} \eta\, \underline{|\nabla u|^2} - 2\,\underline{(x\cdot\nabla u)}\ \underline{\nabla u\cdot\nabla(\eta^2)}_{1}$$
$$- \underbrace{|x\cdot\nabla u|^2}_{2} |\nabla\eta|^2 \leq 0.$$

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

Lemma 1 $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C^1_c(B_1) \Longrightarrow$

$$\boxed{\int_{B_1} \{(n-2)\eta + 2x\cdot\nabla\eta\}\,\eta\,|\nabla u|^2 - 2\,(x\cdot\nabla u)\,\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2\,|\nabla\eta|^2 \leq 0.}$$

(0) (1) (2)

$$\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\,\varphi(x)}_{=\,\eta(x)}$$

(2-n)/2 so that (0) $\{\cdots\} \geq 0$

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x)$ $\rightsquigarrow$ $\int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

Lemma 1 $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta\in C^1_c(B_1)$ $\Longrightarrow$

$$\int_{B_1} \underset{⓪}{\{(n-2)\eta + 2x\cdot\nabla\eta\}}\,\eta\,|\nabla u|^2 - 2\,(x\cdot\nabla u)\,\nabla u\cdot\nabla(\eta^2) \quad ①$$

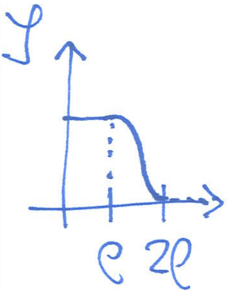
$$\underset{②}{-}\,|x\cdot\nabla u|^2\,|\nabla\eta|^2 \le 0.$$

$$\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\varphi(x)}_{=\eta(x)}$$

so that $\{\cdots\}\geq 0$ ⓪

$$①\,\&\,② \longrightarrow 2(n-2) - \frac{(n-2)^2}{4} = \frac{1}{4}\left\{8(n-2)-(n-2)^2\right\}$$

$$= \frac{1}{4}(n-2)(10-n)$$



$$\frac{1}{4}(n-2)(10-n)\int_{B_\rho}|x|^{2-n}u_r^2 \le C\int_{B_{2\rho}\setminus B_\rho}|x|^{2-n}|\nabla u|^2$$