Elliptic and parabolic equations of fractional nonlocal type. An introduction

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Workshop in honor of Alessio Figalli

DHC at UPC, Univer. Polit. de Catalunya

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Outline

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1. **Diffusion**
2. **The heat equation as a source of problems**
3. **Nonlinear equations. Degenerate or singular equations**
4. **Fractional diffusion**
5. **My special Project: Nonlinear Fractional Diffusion**
   - Two main models
   - Problems in bounded domains
   - Model I. A Porous medium model with fractional diffusion
   - Higher order variant of model
6. **Mathematical Details**
   - Main estimates for this model
7. **Comment**

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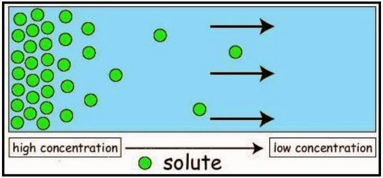
Diffusion

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space.

- Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the momentum of a viscous (Newtonian) fluid diffuses, there is diffusion in the stock market,...



Diffusion of particles in a water solution

- So the question is : what is diffusion for a mathematician? how to analyze diffusion mathematically?
  This question has received two quite different answers in recent history.

The two ways to diffusion

The two answers:

- First direction: Is diffusion more or less related to random walk ? This is a correct answer, and this approach leads to Brownian motion and Stochastic Processes, with the famous Ito equation:

$$dx = b\,dt + \frac{1}{2}\sigma dW.$$

- Second direction: how to explain it with “standard mathematics” based on Analysis? The answer is PDEs of parabolic type, as explained by Kolmogorov in the 1930s. The mother equation is the Heat Equation:

$$\partial_t u = \Delta u.$$

- Understanding this double way has been the source of much effort and the work goes on today.
- Here we will follow the way of Analysis with PDEs, inaugurated by Joseph Fourier (1807, 1822) in an apparently different context, Heat Propagation.

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Expanding the basic model

Some of the problems we face today

- How much of it can be explained with linear models, how much is essentially nonlinear? which are the most relevant mathematical models?
- The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...
- The Laplacian Δ is really the King of Differential Operators. The fractional Laplacian is close family. How strong is the theory and application of the so-called nonlocal or long-range operators that include the fractional Laplacian family?
- Are we able to treat complex systems and describe their behaviour with the combination of the tools we have?

Main tools : Modelling, Analysis, Stochastics, Asymptotics and Numerics.

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- During and after his stay in Texas there appeared many Spanish coauthors.

- **Real Academia de Ciencias** nominated **Alessio** foreign corresponding member, Dec 2018. *Bienvenido a nuestra Academia!*

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Heat Equation

- We begin our presentation with the Heat Equation

$$u_t = \Delta u + f$$

and the analysis proposed by J. Fourier, 1807, 1822 : (Key words: Fourier decomposition, spectrum).

- The mathematical models of heat propagation and diffusion have made great progress both in theory and application,
- It had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Probability and Diff. Geometry.
- It has also had an immense influence in Science and Engineering. The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.

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The heat Equations

- The Heat Equation and the parabolic families of related PDEs

$$u_t = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + cu + f$$

and

$$u_t = \sum_{ij} \partial_i(a_{ij}\partial_j u) + \sum_i \partial_i(b_i u) + cu + f$$

(where (a_{ij}) is a positive definite matrix, possible variable with space and time) are a powerful tool in advanced mathematics.

- The HE and he Parabolic Equation Models have produced a huge number of concepts, techniques and connections for pure and applied science. Today mathematically educated people talk often and casually about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, functional inequalities, positive operators in Banach spaces, entropy dissipation, ...

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The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

$$u_t = \Delta u, \qquad u(x,0) = f(x) \tag{1}$$

which is solved by convolution with the evolution version of the Gaussian function

$$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right). \tag{2}$$

Note that G has very nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss). (G is the Fundamental Solutions. This is a key idea that we would like to copy, they are different in stationary and evolution problems. The concept is problematic in some nonlinear PDEs and very useful in some of them. G is self-similar).

- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot,t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.

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Regularity and asymptotics

- **Regularity**. Solutions in the standard class are unique, exist globally in time and they are C^∞ smooth in space and time. For nonnegative data they are strictly positive.

- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian**. Under very mild conditions on u_0 it is proved that

$$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\,G(x,t)) = 0 \qquad (3)$$

uniformly, if $M = \int u_0(x)\,dx$. For convergence in L^p less is needed. Thus,

$$\lim_{t\to\infty} \|(u(x,t) - M\,G(x,t)\|_1 = 0 \qquad (4)$$

This is the famous Central Limit Theorem in its continuous form (Probability).

(we will try to repeat those questions over and over; the answers vary with the models)

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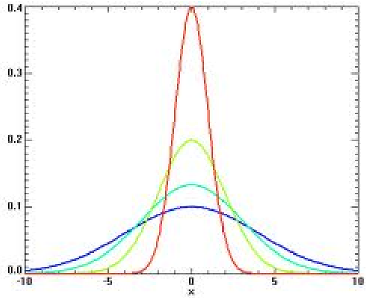
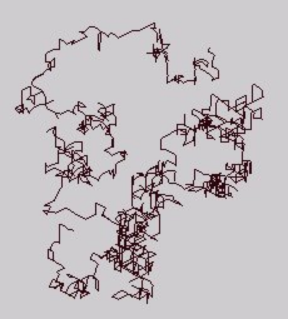
Heat equation graphs. Conflicting views

- The comparison of ordered dissipation vs underlying chaos





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Right, a sample of random walk, origin of Brownian motion

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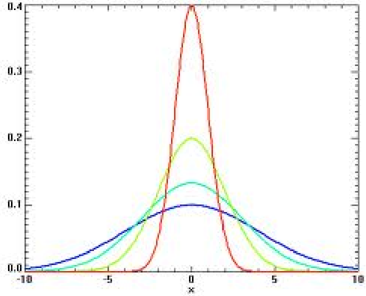

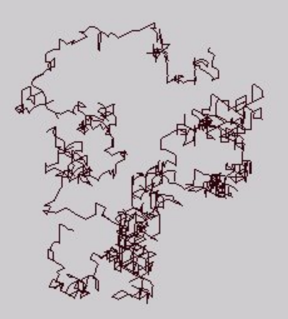

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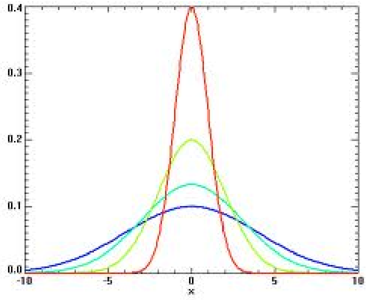

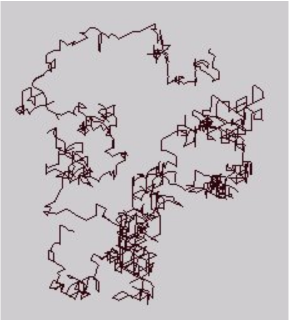

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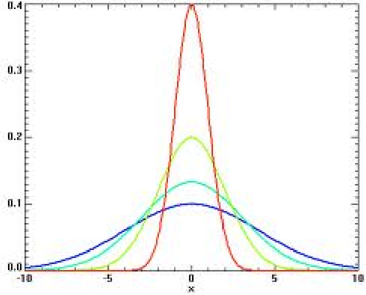

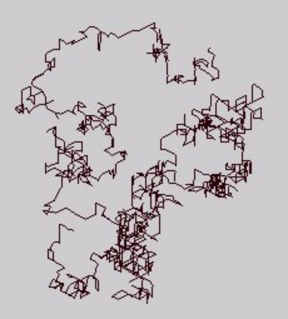

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A detailed text expanding on the topic of the talk

J. L. Vázquez. *The mathematical theories of diffusion. Nonlinear and fractional diffusion*,

♠ ArXiv 1804.08398v1.

♣ appeared as Chapter 5 of Springer Lecture Notes in Mathematics 2186, “Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions”, CIME Summer Course in Cetraro, Italy, 2016; pp. 205–278. Volume Authors: J.A. Carrillo, M. del Pino, **A. Figalli**, G. Mingione, JLV. Editors: M. Bonforte, G. Grillo.

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Nonlinear equations. *Los senderos que bifurcan*

- Let us take a step forward and expand the family of diffusive models in a difficult direction, that of including nonlinearities.
- Indeed, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very non-linear. A general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

  with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)
- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.

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Nonlinear heat flows

- Many specific examples, now considered the "classical nonlinear diffusion models", have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins.
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water),
  Hele-Shaw Problem (potential flow in a thin layer between solid plates),
  Porous Medium Equation: $u_t = \Delta(u^m)$,
  Evolution p-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$. Also diffusion+ absorption $u_t = \Delta u - u^p$.
  The novel phenomena are blow-up and extinction. *A huge community working on that. I spent part of my life with them.*
- The systems are very important and the models are quite different. The chemotaxis system by Keller and Segel is very popular.
- Finally, recall that "elliptic and parabolic problems go together well".

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Un sendero reciente. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

- The Basic Stationary and basic evolution equations

$$\boxed{(-\Delta)^s u = f(x,u)} \qquad \boxed{u_t + (-\Delta)^s u = 0}$$

- Intense work in Stochastic Processes for some decades,
  and the fractional Laplacian was known in Harmonic Analysis
  but research in Analysis of PDEs did not start in force until less than two decades ago. A new from the work done by and around Prof. Caffarelli in Texas, in particular his seminal work with L. Silvestre to be mentioned soon.

Un sendero reciente. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

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The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator.**
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$

  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$. This one has kernel $C_1|x-y|^{n-2s}$, which is not integrable.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where P_{ik} denotes the transition function which has a . tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator A as the infinitesimal generator of a Lévy process: if X_t is the isotropic α-stable Lévy process we have

$$Au(x) = \lim_{h \to 0} \mathbb{E}(u(x) - u(x + X_h))$$

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The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha} \nabla v) = 0 \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x, 0) = u(x), \quad x \in \mathbb{R}^n.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y \to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre, 2007.

This construction is generalized to other differential operators, like the harmonic oscillator, by Stinga and Torrea, Comm. PDEs, 2010.

- The semigroup formula in terms of the heat flow generated by Δ:

$$(-\Delta)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x) \right) \frac{dt}{t^{1+s}}.$$

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Nonlocal elliptic problems. A short note

- The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as a large number works on elliptic equations, mainly of the linear or semilinear type (Caffarelli school; Bass, Kassmann, and others)
- Elliptic problems are usually posed on bounded domains of the space. While in $\mathbb{R}^n$ all the previous versions of fractional Laplacian are equivalent, in a bounded domain $\Omega \subset \mathbb{R}^n$ we have to re-examine all of them. Two main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at/outside the boundary or what is the domain of the energy functionals. There are several alternatives. You will hear more.
- There are many works on the subject. Here is a good basic reference to fractional elliptic work by

  Xavier Ros-Otón. *Nonlocal elliptic equations in bounded domains: a survey*, arXiv:1504.04099.

  For a very recent reference to the topic by me and collaborators

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Mathematical theory of the Fractional Heat Equation

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We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site x_j of the mesh to another site x_k has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$. The power we take is $c\,|x_k - x_j|^{-n-2s}$. The range is $0 < s < 1$. The limit from random walk to the continuous equation is done by E. Valdinoci, in *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33-44.

- The solution of the linear equation can be obtained in $\mathbb{R}^n$ by means of convolution with the fractional heat kernel

$$u(x,t) = \int u_0(y) P_t(x-y)\,dy,$$

and people in probability (like Blumental and Getoor) proved in the 1960s that

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(n+2s)/2}} \qquad \Rightarrow \text{look at the fat tail.}$$

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- The paper B. Barrios, I. Peral, F. Soria, E. Valdinoci. *"A Widder's type theorem for the heat equation with nonlocal diffusion"* Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629-650, studies the theory in classes of (maybe) large functions and studies the question: is every solution representable by the convolution formula. The answer is yes if the solutions are 'nice' strong solutions and the growth in x is no more that $u(x,t) \le (1+|x|)^a$ with $a < 2s$.

- Our recent paper M. Bonforte, Y. Sire, J. L. Vázquez. *"Optimal Existence and Uniqueness Theory for the Fractional Heat Equation"*, Arxiv:1606.00873v1

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Outline

The first nonlinear model

- Since 2007 I have been involved in the analysis of several models of nonlinear diffusion equations driven by fractional Laplacians and other nonlocal integro-differential operators.
- The first model: called Porous Medium Equation with Fractional Pressure

$$\boxed{u_t = \nabla\cdot(u\,\nabla p), \quad p = \mathcal{K}(u).} \tag{6}$$

where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear integral operator $\mathcal{K}$.

$$u_t = \nabla\cdot(u\,\nabla(-\Delta)^{-s}u)$$

- We will explain below this model, worked in collaboration with L. Caffarelli since 2007. Main features are:

  -Existence is OK, but no good uniqueness theorem in several dim.

  -It does not obey a maximum principle: This makes life difficult.

  -It has solutions with compact support and free boundaries.

  -The L^1-L^∞-C^α smoothing effect works.

  -Entropy dissipation methods apply beautifully.

Second model

- The second natural model is given by the equation we have called Fractional PME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{7}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.

- A complete analysis of the Cauchy problem done by A. de Pablo, F. Quirós, Ana Rodríguez, and J.L.V., in 4 papers appeared in Advances in Mathematics (2011), Comm. Pure Appl. Math. (2012), J. Math. Pures Appl. (2014), and J. Eur. Math. Soc. (2017).
  In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works, C^α regularity (if m is not near 0),
  Nonnegative solutions have infinite speed of propagation for all m and s $\Rightarrow$ no compact support. Further smoothness for positive solutions.

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Bounded and unbounded domains

- There are a number of other models, like fractional p-Laplacian models, chemotaxis systems with fractional diffusion (lots in Carrillo's talk), KPP propagation models, see the JLV Springer CIME survey paper.
- Much effort has been devoted recently by us to understanding the properties of fractional flows posed on bounded domains with Dirichlet conditions, as a complement to previous work on the whole space, $x \in \Omega^c$.

  The Dirichlet condition is not imposed on the boundary but on the whole exterior of the domain.

  The definition of fractional Laplacian is open to several choices. The behaviour of the solutions differs a lot with the choices.
- We refer to work in collaboration with M. Bonforte, A. Segatti, Y. Sire, D. Stan, B. Volzone, and lately with Alessio Figalli.

Work with Figalli

- Two recent papers appeared in 2018 contain the results of discussions by Matteo B., Alessio F. and myself, started in Austin in 2016, on a basic regularity question: the way solutions of these nonlinear degenerate fractional problems take up the zero Dirichlet boundary conditions.
- It was known that some elliptic problems one version of the Fractional Laplacian tends to satisfy the standard Hopf principle at the boundary, while the most current version (called restricted fract Laplacian) does not, and indeed positive solutions are C^s regular and no more (famous paper by X. Ros Oton + J. Serra 2014).
- So we were busy with the same question for nonnegative solutions for the evolution problem

$$\boxed{\partial_t u + (-\Delta)^s u^m = 0, \quad m > 1.} \tag{8}$$

and its stationary equivalent

$$\boxed{v^p + (-\Delta)^s v = f.} \tag{9}$$

Note $v = u^m$, $p = 1/m$. Thanks to Alessio's magical computations we found a gap between the evolution and the elliptic versions, that happens for some flat initial data of the evolution that refuse to regularize as they should.

Nonlocal nonlinear diffusion model I

- Modeling: the problem arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.
- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  $\Rightarrow$ See JLV PME Book (2007) for these and other applications (around 20!).

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- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  $\Rightarrow$ See JLV PME Book (2007) for these and other applications (around 20!).

Nonlocal nonlinear diffusion model I

- Modeling: the problem arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

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Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel

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- The diffusion model with nonlocal effects is thus given by the system

$$\boxed{u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).} \tag{10}$$

where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$. $u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u)$

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- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L}u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.
- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.

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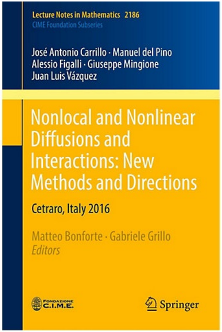
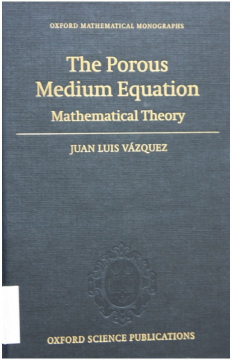
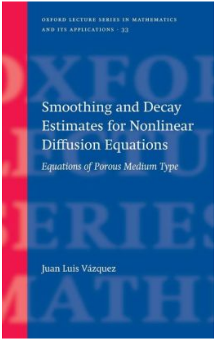
Books







2017 and 2006-2007

Work that we do at this moment

We recall that Model 1 can be written as a system

$$\boxed{u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).} \tag{12}$$

where u is a function of the variables (x, t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$.

- But it is not clear why $\mathcal{K}u = (-\Delta)^{-s}u$. People have suggested other nonlocal integral operators. Much work is going on.

- In a project with Antonio Segatti, started one year ago in Shanghai, we use the choice $\mathcal{K}u = (-\Delta)^{+s}u$. This leads to a higher order equation (differ. order $2+2s$) that **interpolates** between PME ($s=0$) and 4th order nonlinear thin film equation ($s=1$). There are related 1d models by Imbert, Mellet, and Tarhini. Their application is again crack dynamics.

Outline

Main estimates for this model

We recall that the equation of M1 is $\partial_t u = \nabla \cdot (u \nabla K(u))$, posed in the whole space $\mathbb{R}^n$.

We consider $K = (-\Delta)^{-s}$ for some $0 < s < 1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator. We let $H = K^{1/2} = (-\Delta)^{-s/2}$.

We do next formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions. This is to be justified later by approximation.

- Conservation of mass

$$\frac{d}{dt} \int u(x,t)\, dx = 0. \tag{13}$$

- First energy estimate:

$$\frac{d}{dt} \int u(x,t) \log u(x,t)\, dx = -\int |\nabla H u|^2\, dx. \tag{14}$$

- Second energy estimate

$$\frac{d}{dt} \int |Hu(x,t)|^2\, dx = -2 \int u |\nabla K u|^2\, dx. \tag{15}$$

Main estimates

- Conservation of positivity: $u_0 \geq 0$ implies that $u(t) \geq 0$ for all times.
- L^∞ estimate. We prove that the L^∞ norm does not increase in time.

  Proof. At a point of maximum of u at time $t = t_0$, say $x = 0$, we have

  $$u_t = \nabla u \cdot \nabla P + u\,\Delta K(u).$$

  The first term is zero, and for the second we have $-\Delta K = L$ where $L = (-\Delta)_q$ with $q = 1 - s$ so that

  $$\Delta K u(0) = -Lu(0) = -\int \frac{u(0) - u(y)}{|y|^{n+2(1-s)}}\,dy \leq 0.$$

  This concludes the proof.
- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved for this Model. However, good comparison works for Model 2

  $$\partial_t u + (-\Delta)^s u^m = 0,$$

  presented above, actually, it helps produce a very nice theory.
- Finite propagation is true for model M1. Infinite propagation is true for model M2.

Boundedness

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
  For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \qquad (16)$$

  with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.
  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

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  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

Energy and bilinear forms

- **Energy solutions:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. The general energy property is as follows: for any F smooth and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\int F(u(t_2))\,dx - \int F(u(t_1))\,dx = -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt = -\int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt$$

where h is a function satisfying $h'(u) = uf'(u)$. We can write the last integral as a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u), u)$$

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s}\iint \nabla v(x)\frac{1}{|x-y|^{n-2s}}\nabla w(y)\,dx\,dy\,. \qquad (17)$$

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where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x)-v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x)-w(y))\, dx\, dy \tag{18}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant K_n depending only on n.

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- The equation is generalized into $u_t = \nabla \cdot (u^{m-1}\nabla(-\Delta)^{-s}u)$ with $m > 1$. Recent work with D. Stan and F. del Teso shows that finite propagation is true for $m \geq 2$ and propagation is infinite is $m < 2$. This is quite different from the standard porous medium case $s = 0$, where $m = 1$ is the dividing value.
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Outline

Present and future. A celebration

The fractional diffusion field, both in elliptic and parabolic version, has turned out a very productive source of multiple *senderos que se bifurcan*. It is now an occupation for many talented senior and younger people, too many to mention here. Such colleagues spread over all continents (in places we would like to visit).

Fractional diffusion combines nicely with other effects, like aggregation, drift, reaction, fluid dynamics, information and economy models, ... We will hear about free boundaries and change of phase, and maybe about chemotaxis. And so on.

I am just arrived from London where we are doing fractional versions of Newtonian vortex flow equations, work with J.A. Carrillo and D. Gómez-Castro. In the next step we need new methods!

Viatge a Itaca. These mathematical worlds are a magic play of words and symbols, plus so many hours of calculations, to produce fruitful abstractions that can be computed. If you venture into this wonderful field, do remember that you are not alone: in case of doubt or despair, in case the right theorem eludes you, if nothing else works, then

there is always the one and only Alessio Figalli.

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Thank you for your attention
and a very happy future, Alessio and UPC

moltes gràcies / muchas gracias / tante grazie

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