Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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Workshop in honor of Alessio Figalli's Doctor Honoris Causa at UPC
Five talks and a Round Table with Prof. Alessio Figalli
Facultat de Matematiques i Estadistica
UNIVERSITAT POLITECNICA DE CATALUNYA
Barcelona, Spain, November 21, 2019

Outline of the talk

- **Introduction to the Parabolic Problem on Domains**
- **The Classical Porous Medium Equation (PME)**
  - **A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"**
- **The Fractional PME I: Basic theory**
  - **Three Different Fractional Laplacians on Bounded Domains**
  - **Existence, Uniqueness and Boundedness**
- **The Fractional PME II: Sharp Boundary Behaviour**
  - **Positivity Estimates and Infinite Speed of Propagation**
  - **Global Harnack Principles**
  - **Asymptotic Behaviour**
  - **Anomalous Boundary Behaviour and Counterexamples**
  - **Some Numerics**

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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The Classical Porous Medium Equation (PME)

A Brief Summary about the Dirichlet Problem for PME in few “Blackboards”

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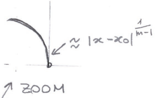
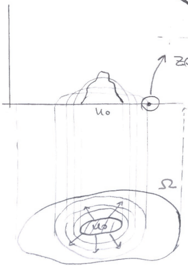
(0) $0 < t < \underline{t}$ INITIAL TIMES

(Barenblatt behaviour)

$$u(t,x) \approx \left(C - \frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} t^\alpha = B(t,x)$$





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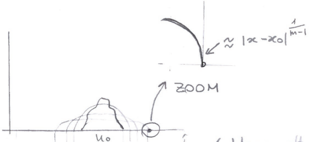

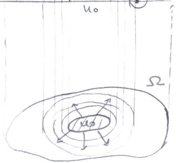

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$t = \underline{t}$

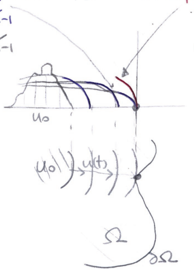
(the first "touching point")

- When the support of $u(t)$ touches $\partial\Omega$ for the first time.
- Transition of boundary behaviour:

"Barenblatt" Beh. VS "Elliptic" Beh.

$|x-x_0|^{\frac{1}{m-1}}$ $\qquad$ $|x-x_0|^{1/m}$

$d(x,\partial\Omega)^{\frac{1}{m-1}}$ $\qquad$ $d(x,\partial\Omega)^{1/m}$



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the solution starts to "inflate"

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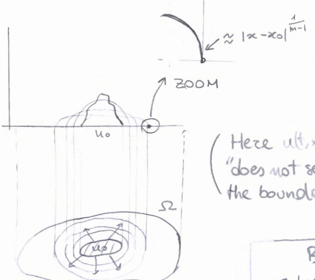

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REGULARITY

- solutions are smooth when positive & bounded.
- Free boundary: delicate issue (CAFFARELLI, VAZQUEZ, WOLANSKI, KOCH, ...)

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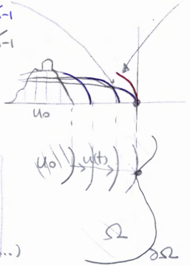

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(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp($u(t)$) touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the Elliptic one:

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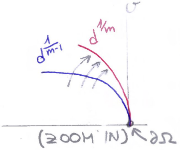

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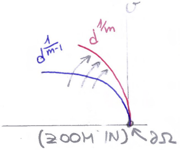

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE & LARGE TIMES)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\operatorname{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1 \frac{\operatorname{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

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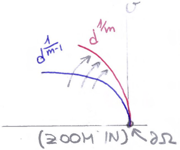

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⟷ (STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \dfrac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

⇓ rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

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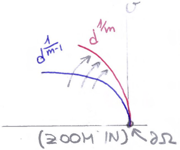

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Semilinear Structure: $S^m = V$, $p = 1/m < 1$

rewritten

(ELLIPTIC THEORY)

$$C_0' \frac{S(x)}{t^{1/m-1}} \le u(t,x) \le C_1' \frac{S(x)}{t^{1/m-1}}$$

$V \asymp \text{dist}(\cdot,\partial\Omega)$

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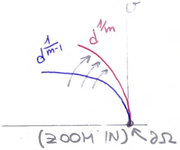

SLOW MOTION DYNAMICS:
(LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \rightsquigarrow \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

(SAME LATERAL B-C)

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

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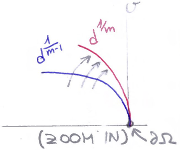

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(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \quad \Big\} \quad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

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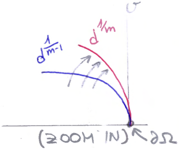


SLOW MOTION DYNAMICS:
(LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \xrightarrow{\text{SAME LATERAL B.C.}} \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \qquad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

SHARP

$$\left|\frac{u(\tau,x)}{\mathcal{U}(t,x)} - 1\right| \le \frac{C}{1+\tau} \qquad \left|\frac{v(t,x)}{S(x)} - 1\right| \le C e^{-t}$$

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES) & LARGE

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}} \le u(t,x) \le C_1 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}}$$

(STATIONARY for RESCALED FLOW)
ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \frac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

Semilinear Structure: $S^m = V$, $p = 1/m < 1$

$$\begin{cases} -\Delta V = \frac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

(ELLIPTIC THEORY)

$V \asymp \text{dist}(\cdot,\partial\Omega)$

$S \asymp \text{dist}(\cdot,\partial\Omega)^{1/m}$

rewritten

$$C_0' \frac{S(x)}{t^{1/m-1}} \le u(t,x) \le C_1' \frac{S(x)}{t^{1/m-1}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{1/m-1}} = \mathcal{U}(t,x)$$

← SEPARATE VARIABLE SOLUTION.

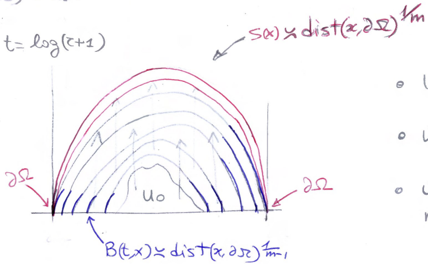
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

(SAME LATERAL B.C.)

SLOW MOTION DYNAMICS

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$



- $v_t = \Delta v^m + \frac{v}{m-1}$
- $v_t \geq 0$ ← (BENILAN-CRANDALL)

  ⇓
- $v(t,x) \nearrow S(x)$ monotonically increases to $S(x)$

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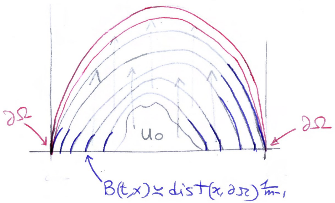

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- $v(t,x) \nearrow S(x)$ monotonically increases to $S(x)$
- $S(x)$ Represents AN ABSOLUTE UPPER BOUND FOR ALL SOLUTIONS !! "FRIENDLY GIANT". (DAHLBERG–KENIG)

$$\begin{cases} u(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}} \\ u(0,x) = +\infty \end{cases} \quad \longleftarrow \text{RESCALING BACK.}$$

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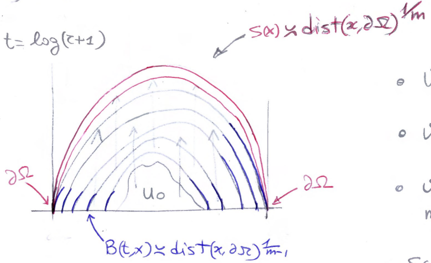

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$$\begin{cases} \mathcal{U}_T(\tau,x) = \dfrac{S(x)}{(T+t)^{\frac{1}{m-1}}} \\ \mathcal{U}_T(0,x) = \dfrac{S(x)}{T^{\frac{1}{m-1}}} \end{cases}$$

$T \to 0$

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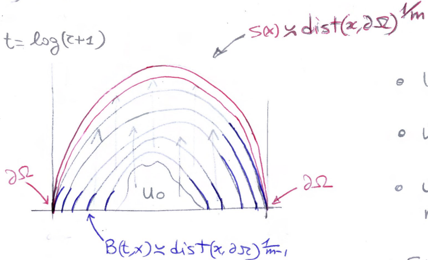

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ARONSON-PELETIER (80s), VAZQUEZ (2000s), MB.-FIGALLI-SIRE-VAZQUEZ (2015-18?)

- **The Fractional PME I: Basic theory**
  - **Three Different Fractional Laplacians on Bounded Domains**
  - **Existence, Uniqueness and Boundedness of solutions**

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

- We have seen what happens when $\mathcal{L} = -\Delta$ is the classical Laplacian
- We now focus our attention to a particular scenario:
  - When $\mathcal{L} = (-\Delta)^s$, with $s \in (0,1)$ is a Fractional Laplacian: there are three different choices of fractional Laplacian on bounded domains.
  - When $F(u) = |u|^{m-1}u$, with $m > 1$ have the classical PME nonlinearity

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Three Different Fractional Laplacians on Bounded Domains

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s} \hat{f}(\xi) .$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, dz,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

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Three Different Fractional Laplacians on Bounded Domains

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are the eigenfunctions of the classical Laplacian Δ_Ω:

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)\,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

The Green function of SFL satisfies, letting $\delta^\gamma(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

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Three Different Fractional Laplacians on Bounded Domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
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Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty)\times(\mathbb{R}^N\setminus\Omega)\,.$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

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Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \mathrm{P.V.} \int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

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Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
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Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

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- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
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Elliptic VS Parabolic: Asymptotic Behaviour

Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$
Let S be the unique solution to the Elliptic Dirichlet Problem for $\mathcal{L}S^m = S$.

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Let $u \ge 0$ be any nonnegative WDS to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\sup_{x\in\Omega} \left| t^{\frac{1}{m-1}} u(t,x) - S(x) \right| \xrightarrow[t\to\infty]{} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$\boxed{u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation.
We shall see that this is not always the case.

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The Fractional PME II

Sharp Boundary Behaviour

- **Positivity Estimates and Infinite Speed of Propagation**
- **Global Harnack Principles**
- **Asymptotic Behaviour**
- **Anomalous Boundary Behaviour and Counterexamples**
- **Some Numerics**

Positivity Estimates and Infinite Speed of Propagation

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $0 < s < 1$ and $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$, such that

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Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

(recall that $\gamma = 1$ for SFL, $\gamma = s$ for the RFL and $\gamma = 2s - 1$ for the CFL)

- Note that, for $t \geq t_*$, the dependence on the initial data disappears

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- But also note that these estimates can not hold for small times when $s = 1$, by the finite speed of propagation that holds in the local case...

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- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
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Theorem. (GHP I) (M.B., A. Figall, X. Ros Oton & J. L. Vázquez)

Let $\mathcal{L}$ be either the RFL ($\gamma = s$) or the CFL ($\gamma = 2s-1$). Let $u \ge 0$ be a weak dual solution to the (CDP). Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$:

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- *Notice that this result* ***does not apply for*** *$s = 1$, is purely nonlocal.*
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As a consequence of GHP with matching powers we get:

Theorem. (Sharp Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that a GHP with matching powers hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \ge t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

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This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier (1981) and Vázquez (2004) with a different proof.

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In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

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Anomalous Boundary Behaviour and Counterexamples

Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear.

We consider the SFL, hence $\gamma = 1$ from now on. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} = \frac{2sm}{m-1} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{1}{2} - \frac{1}{2m}.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \geq \epsilon_0 S \qquad \text{implies} \qquad u(t) \geq \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \qquad \text{for all } 0 \leq u_0 \in L^1_{\Phi_1}(\Omega).$$

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We next show that the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa}\Phi_1^\alpha(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq [A^{1-m} - \tilde{C}t]^{-(m-1)}\,\Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Different boundary behaviour when $\sigma < 1$.

We next show that the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa}\Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

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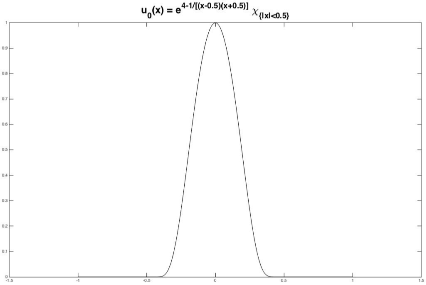
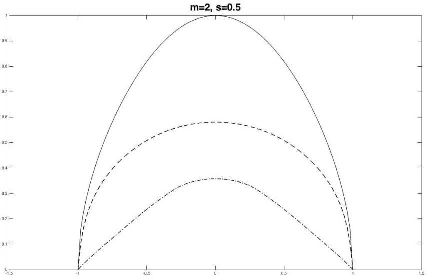
Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, SIAM Num. Anal. (2018)
Graphics and videos: courtesy of F. Del Teso (BCAM, Bilbao, ES)

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





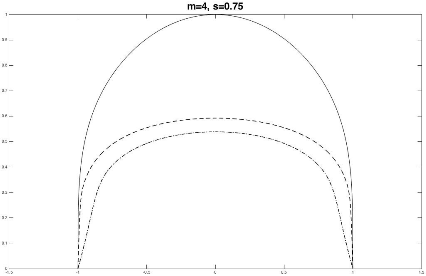
Left: the initial condition $u_0 \le C_0 \Phi_1$

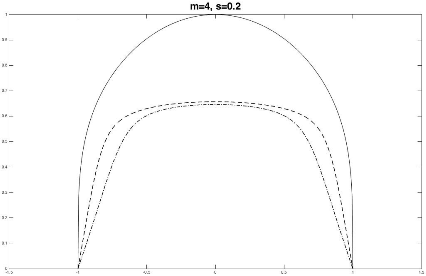
Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0 \Phi_1$, solutions with different parameters





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time,
and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

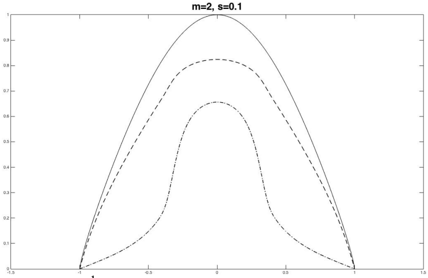
Non-matching: $u(t) \asymp \Phi_1$ even after long time.

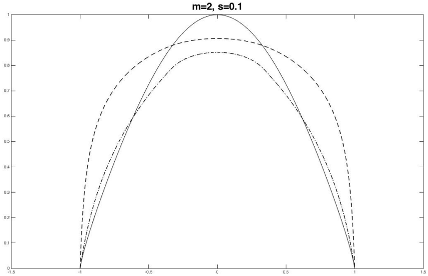
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{\sigma}{m}}} = 0 \qquad \text{for any } t > 0.$$

The End

Muchas Gracias!!!

Moltes Grácies!!!

Thank You!!!

Congratulazioni Alessio!!!!!

The End

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Moltes Grácies!!!

Thank You!!!

Congratulazioni Alessio!!!!!

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