RIEMANN AND PARTIAL DIFFERENTIAL EQUATIONS. A ROAD TO GEOMETRY AND PHYSICS

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1. Introduction

This is the edited text of a talk given at the Facultat de Matemàtiques i Estadística of UPC on February 20th, 2008, in the framework of the "Jornada Riemann".

In preparing the text I have endeavored to avoid or minimize repetitions of material that appears in other contributions. I have also tried to keep the text as simple as possible, in accordance with the style of the talk and the open and lively discussions held during the event.

Before entering the subject proper of this work, let me summarize a few reflections on Riemann's life and work that I find relevant for a deeper understanding of the significance of his legacy.

Left: the scientist as a young man; Right: a classical picture.

Bernhard Riemann died very young, like other geniuses of the 19th Century, but he left an impressive legacy to Mathematics, pure and applied. He had a permanent love for Italy, where he traveled for health reasons, but where he also had friends that continued his deep ideas. Besides Mathematics, Riemann had a sustained interest in Philosophy and in Physics.

An easy source for his mathematics is the collection of his works, [12], which has translations to several languages. His life is described in Detlef Laugwitz's book [7], and I have used M. Monastyrsky's [10]. I have found very useful Ferreirós' "Riemanniana Selecta" [5], in Spanish. Besides that, Internet is a very convenient source of details, and I have consulted Wikipedia, MacTutor, Encyclopaedia Britannica, and other Internet archives.

Riemann had a philosophical formation, and as such he sustains that *the essence of Reality lies in a hidden world*. That is not really new; medieval scholars would say that *Videmus in aenigmata, et per speculum.*

In his century Riemann was not alone in seeing the key to understanding the hidden reality of the World in the *Concepts and Formulas of Abstract Mathematics*, or in the original German, in *die Begriffe und Formeln der Höhere Mathematik.*

So maybe we can see here a new look for old ideas. Indeed, the look is not only new, it will prove to be revolutionary.

In keeping with his philosophical frame of mind, and also because of his short life, his work is deep in concepts and ideas, but rather scarce in details. A century and a half of research has provided answers and details to a large number of the topics treated in his complete works, but not to all: remember the Riemann hypothesis!

2. Mathematics, Physics and PDEs

2.1. **PDEs and the origins of differential calculus.** The Differential World, i.e, the world of derivatives, was invented, or discovered as you may prefer to see it, in the 17th Century, almost at the same time that Modern Science (then called *Natural Philosophy)*, was born. We owe it to the great Founding Fathers: Galileo, Descartes, Leibnitz and Newton (mainly). Motivation came from the desire to understand the World around us, more specifically Motion, Mechanics and Geometry.

Newton formulated Mechanics in terms of ODEs, by concentrating on the movement of particles. The main magic formula is

$$m\frac{d^2\mathbf{x}}{dt^2} = F(t, \mathbf{x}, \frac{d\mathbf{x}}{dt})$$

though he would write derivatives with dots, and not as quotients, which is Leibnitz style. Here are the magical words, to which we are now so used: mass (m), force at a distance (F), and acceleration, and here is where the (second) derivative enters the picture.

Newton thought about fluids, in fact he invented *Newtonian fluids*, and there you need dependence on space and time simultaneously, x as well as t. It involves the partial derivatives, which means that we get Partial Differential Equations (PDEs for short). But his progress was really small if you compare it with the rest of *Principia Mathematica Philosophiae Naturalis*, 1687, and other works of the early Calculus time.

We conclude that there was not much time for PDEs from the Big Bang to 1700 AD.

In the 18th Century, PDEs appear in the work of Jean Le Rond D'Alembert about string oscillations: there a *set of particles* moves together due to *elastic forces*, but every one of the infinitely many *solid elements* has a different motion, $u = u(x, t)$. This is one of the first instances of *continuous collective dynamics*. *PDEs are the mode of expression of such CCD.*

Johann and Daniel Bernoulli and then Leonhard Euler lay the foundations of Ideal Fluid Mechanics (1730 to 1750), in Basel and St Petersburg. This is PDEs of the highest caliber:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \qquad \nabla \cdot \mathbf{u} = 0.$$

The system is *nonlinear*; it does not fit into one of the three main types that we know today (elliptic, parabolic, hyperbolic); the main pure-mathematics problem is still unsolved (existence of classical solutions for good data; Clay Problems, year 2000).

2.2. **Modern times: PDEs in the 19th Century.** The 19th Century confronts revolutions in the concept of heat and energy, electricity and magnetism, and also in the very concept of space. The Newtonian edifice begins to shake. You may add a lesser revolution, *real fluids*.

Mathematically, all of these fields take up the form of PDEs:

(i) Heat leads to the heat equation, $u_t = \Delta u$, and the merit goes primarily to J. Fourier.

(ii) Electricity leads to the Coulomb equation in the Laplace-Poisson form: $-\Delta V = \rho$. Surprisingly, this equation also represents gravitation! (with a difference, for electricity the right-hand side may have two signs).

(iii) Electromagnetic fields are represented by the Maxwell system. The vector potential satisfies a wave equation, the same as D'Alembert's (but now it is vector-valued and in several dimensions).

(iv) Real fluids are represented by the Navier-Stokes equations. Sound waves follow wave equations, but they can create discontinuous solutions called *shocks* (and here Riemann appears as we will see).

2.3. **PDEs continued in the 19th Century.** Geometry was transformed from the Euclid tradition plus Cartesian Algebra to the spirit of PDEs by G. Gauss and B. Riemann. The new spirit is condensed in a number of key words. Space is determined by its *metric* which is a local object that has tensor structure. The connection from point to point is a new object called *covariant derivative*, the *curvature* is a second order operator, a nonlinear relative of the Laplace operator.

After the work of these people, in particular Riemann, *Reality* is seen as mainly continuous, and its essence lies in the *physical law*, that is a law about a *field* or a number of fields. In symbols, we have $\Phi(x, y, z, t)$ and an equation (or system)

$$L\Phi = \mathbf{F},$$

where F is the force field (a tensor).

2.4. **20th Century. Summing Up.** In the 20th Century General Relativity and Quantum Mechanics take this same form. *Space, matter and interactions become fields.*

A main variant from the scheme is Statistical Mechanics, a thread that leads to Brownian motion (Einstein, Smoluchowski), abstract probability (Kolmogorov, Levy, Wiener), stochastic calculus and stochastic differential equations (Itō).

Summing up in a rather succint form: *the main (technical) task of the Mathematician working in Mathematical Physics is to understand the world of Partial Differential Equations, linear and nonlinear.*

The same is true nowadays for geometers (you only need to travel to the Universitat Autònoma of Barcelona and attend the now running CRM semester on Ricci flows!).

The main abstract tool is Functional Analysis. The *combination* of Functional Analysis, PDES and ODEs, Geometry, Physics and Stochastic Calculus is one of the Great Machines of today's research, a child of the 20th Century.

3. Riemmann, complex variables and 2-D fluids

In the sequel we will try to convey some of the mathematics of B. Riemann that had a profound impact on PDEs, with attention to specific concepts and calculations. In other words, let us do some math!

3.1. **Complex Variables (Euler, Cauchy, Gauss, Riemann, Weierstrass).** We start with a function

$$u(x, y) = u(z), \qquad z = x + iy = (x, y)$$

that is supposed to be a good function of two real variables.

• A good function of two real variables means (could mean) $u \in C^1(\Omega)$ for some subdomain Ω of $\mathbb{R}^2$.

• Therefore, u has a gradient: $\nabla u(z_0) = (u_x, u_y)$.

• But, what is a good function of one complex variable?

• First of all, to keep the symmetry, there must be two real functions of two real variables:

$$u = u(x, y), \quad v = v(x, y)$$

which we write as $f = f(z)$ with $f = u + iv$, and $z = x + iy$.

• The question is: Do we ask that $f \in C^1$ and that is all? The answer is no and this is a consequence of algebra.

• Let us explain why: very nice real functions of one variable are polynomials, and very nice complex functions of one complex variable should also be polynomials.

• Now, polynomials are easy to define, for instance $f(z) = z^2$ means

$$u = x^2 - y^2, \qquad v = 2xy$$

while $f(z) = z^3$ means

$$u = x^3 - 3xy^2, \qquad v = 3x^2y - y^3.$$

• Can the reader do $f(z) = z^n$ by heart? Euler could! In fact, Euler and Moivre could see the whole trigonometry (by using the polar form $z = re^{i\theta}$ and expanding the power z^n).

• Can you see something special in these pairs of functions, u and v? Cauchy and Riemann could! They saw the whole theory of *complex holomorphic functions*.

3.1.1. **The PDE code for complex variables.** What they saw is this hidden symmetry:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations are called Cauchy-Riemann (CR) equations for complex variables. They are one of the most important examples of a PDE system with extraordinary geometric and analytic consequences.

• We see that $\nabla u = (a, -b)$ is orthogonal to $\nabla v = (b, a)$. Consequence: the level lines $u = c_1$ and $v = c_2$ are orthogonal sets of curves.

• The linear algebra of infinitesimal calculus at every point is not 4-dimensional but two-dimensional. In fact, the system

$$du \sim adx - bdy, \quad dv \sim bdx + ady,$$

can be written together in the complex form $df \sim Jf(z)\,dz$, where Jf is the Jacobian matrix that we begin to call $f'(z) = a + ib$.

• *We will assume from such glorious moment on that this is the correct derivative of a 2-function of 2 variables that is a candidate to be a good complex differentiable function.*

3.1.2. **Complex Variables, analysis and geometry.** Hence we know some magic formulas:

$$f'(z) = a + bi = f_x, \quad f_y = -b + ai = if'(z).$$

Thus $df = f'dx$ along the x-axis and $df = if'dy$ along the y axis.

Comming back to the Jacobian,

$$Jf(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aE + bJ.$$

This is a similarity matrix with determinant

$$\det(Jf) = a^2 + b^2 = u_x^2 + u_y^2 = u_x^2 + v_x^2 = v_x^2 + v_y^2 = u_y^2 + v_y^2,$$

which can be written as

$$|Jf| = |f'(z)|^2 = \|f_x\|^2 = \|f_y\|^2 = \|\nabla u\|^2 = \|\nabla u\|^2.$$

The infinitesimal transformation preserves the angles (of tangent curves) and scales the size by $Jf = |f'(z)|^2$.

If the 2-2 function f is CR, then it defines a conformal transformation of the part of the plane where $f'(z) \neq 0$. Riemann's geometric theory of one complex variable is based on this idea!

3.1.3. **Complex Variables and PDEs. Solving the equations. Potential theory.** Once we enter that framework, the question is: how to find pairs of functions satisfying CR?

Of course, real and imaginary parts of algebraic complex functions satisfy CR. The Taylor Series, typical of the Cauchy–Weierstrass approach, also satisfies CR.

But PDE people want their way. Here is the wonderful trick:

$$\Delta u = u_{xx} + u_{yy} = v_{yx} + (-v_x)_y = 0.$$

Idem $\Delta v = 0$. Solutions of this equation are *harmonic functions*, and they count among the most beautiful C^∞ functions in analysis and among the most important in physics, where solving $\Delta u = -\rho$ means finding the *potential* of ρ, where ρ is a volume distribution of mass or of electric charge.

Note a novelty full of promise: harmonic functions live in all dimensions, not just two. But in $d = 2$ they produce complex holomorphic functions. Given u, we find v, its conjugate partner, by integration of the differential form

$$dv = Pdx + Qdy, \quad \text{with } P = v_x = -u_y, Q = v_y = u_x.$$

This is an exact differential thanks to CR.

3.2. **Complex Variables and ideal fluids in $d = 2$.** Recall that Riemann was a friend of Weber, the famous physicist.

The velocity of a 2-D fluid is a field $\mathbf{v} = (v_1(x, y), v_2(x, y)$. Irrotational means that $\nabla \times \mathbf{v} = 0$. Incompressible means $\nabla \cdot \mathbf{v} = 0$. For the PDE person this is easy:

$$v_{2,x} - v_{1,y} = 0, \quad v_{1,x} + v_{2,y} = 0.$$

Does this look like what we saw in the previous subsection? Yes, combining both we get $\Delta v_1 = \Delta v_2 = 0$.

Is v_2 harmonic conjugate to v_1? No, but $-v_2$ is.

Idea to eliminate sign problems. Go to the scalar potential of the vector field $\mathbf{v}$:

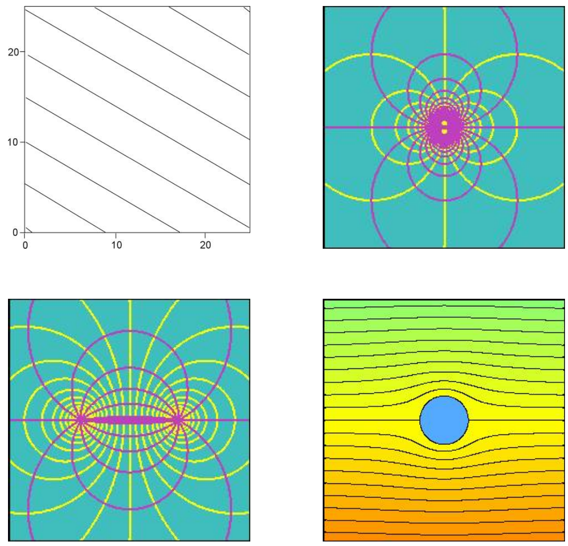
$$d\Phi = v_1 dx + v_2 dy$$

(it is exact by irrotationality). Take the harmonic conjugate Ψ and define the complex potential of the flow as $F = \Phi + i\Psi$, a complex holomorfic function. In that case

$$F'(z) = F_x = \Phi_x + i\Psi_x = \Phi_x - i\Phi_y = \overline{\mathbf{v}}.$$

Consequence: $\overline{\mathbf{v}} = v_1 - iv_2$ is a complex holomorphic function.

3.3. **Some Pictures of 2D glory.** The following pictures come in every book about two-dimensional perfect fluids and conformal transformations. We ask the reader to identify them as a linear flow, a dipole configuration, a source-sink combination, and lastly the famous streamlines for the planar flow around an obstacle. This complex variable theory stands at the core of the science of aerodynamics.



3.4. **Summary. The big picture in 2D.**

- There is an equivalence between holomorphic complex variable theory $\Leftrightarrow$ conformal geometry $\Leftrightarrow$ harmonic functions $\Leftrightarrow$ ideal fluids.
- Any two dimensional ideal fluid generates an analytic function, and conversely, and it is a conformal mapping, and viceversa.
- The complex derivative of the complex potential is just the conjugate of the velocity field.

- The stream function Ψ indicates the lines of current via the formula $\Psi = c$.
- What happens when $F'(z) = 0$, i.e., when $\mathbf{v} = \mathbf{0}$? These are singular points, called in physics the *stagnation points*. Many things can happen on a singularity, but essentially only one thing may happen on a regular point, where the implicit function theorem is valid. Riemann was an expert in singular points.

4. Riemmann and the equations of geometry

4.1. **From 2D to 3D.** Riemann was able to understand very well the Two-Dimensional Space with its functions, analysis, geometry and physics.

It is not as easy as it seems because complex holomorphic functions *try to follow their name and be globally defined,* actually they have analytic continuation. But they may have singularities blocking their way to global (global is called here *entire*).

Riemann's main contribution to 2D analysis+geometry is the concept of *Riemann surface* (RS) with the curious branching points. A simple Riemann surface may be a part of $\mathbb{R}^3$ but more complicated RS live in a very strange situation, *a different world.*

But we want now to forget 2D and remember that we live in 3D. Thinking about the geometry of 3D is an old pastime, masterfully encoded by Euclid of Alexandria (325 BC-265 BC).

The 3D world is much more complicated that 2D and no part of the equivalence between analysis, Taylor series, elementary PDEs, conformal geometry and ideal Physics survives.

4.1.1. **What is Geometry according to Riemann.** Let us follow the Encyclopædia Britannica article on B. Riemann.

In 1854 Riemann presented his ideas on geometry for the official postdoctoral qualification at Göttingen; the elderly Gauss was an examiner and was greatly impressed.

Riemann argued that the fundamental ingredients for geometry are a *space of points* (called today a manifold) and *a way of measuring distances* along curves in the space.

He argued that the space *need not be ordinary Euclidean space* and that it could have *any dimension* (he even contemplated spaces of infinite dimension). Nor is it necessary that the surface be drawn in its entirety in three-dimensional space. According to Riemann, *many spaces are possible*. This happened more than four decades before Relativity!

It seems that Riemann was led to these ideas partly by his dislike of the (Newton's) concept of *action at a distance* in contemporary physics and by his wish to endow space with the ability to transmit forces such as electromagnetism and gravitation.

A few years later this inspired the Italian mathematician Eugenio Beltrami to produce just such a description of non-Euclidean geometry, the first physically plausible alternative to Euclidean geometry. More italians influenced by B. Riemann: Ricci, Levi-Civita, Bianchi.

4.1.2. **Habilitationsvortrag, 1854. Riemannian Geometry.** This is one of the most famous and influential habilitation documents in the history of Mathematics.

Space around only has a definite sense locally around the place. The basic tool to do geometry is the *metric*, which is given by

$$ds^2 = \sum g_{ij} dx^i dx^j$$

It is local since it works on local entities, tangent vectors. Forget Pithagoras but remember $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ on the sphere.

The metric field changes from point to point, $g_{ij}(x)$, x is locally a set like $\mathbb{R}^d$. He even says that for a space of functions d can be infinite.

So there is no sense in principle of parallel vectors (at least if we do not work more). We can instead define the derivative of a tangent vector

$X = \sum_i a_i \mathbf{e}_i$ when we move along another vector $Y = \sum_j b_j \mathbf{e}_j$. This is the famous covariant derivative ∇:

$$\nabla_Y X = \sum_i Y(a_i)\mathbf{e}_i + \sum_{ijk} a_i b_j \Gamma_{ij}^k \mathbf{e}_k.$$

which depends on a set of functions Γ_{ij}^k, the so-called Christoffel symbols. For the correct covariant derivative, called Levi-Civita connection, the Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Objects with several indices are usually tensors. Note that although the Christoffel symbols have three indices on them, they are not tensors. Sorry, local coordinates are intuitive but messy to work with, this really cumbersome aspect of modern geometry is also part Riemann's inheritance!!

4.1.3. **Curvatures at the center of geometry.** The covariant derivative opens the way to a whole new Differential Calculus, where curvature tensors and Laplacians play a key role.

Curvature tensor. The Riemann curvature tensor is given by

$$R(X,Y,Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

In that case we have

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^s_{jk}\Gamma^i_{sl} - \Gamma^s_{jl}\Gamma^i_{sk}.$$

Contraction gives the low tensor $R_{ijkl} = g_{im} R^m_{jkl}$. Wikipedia gives

$$R_{iklm} = \frac{1}{2}\left(\partial^2_{kl} g_{im} + \partial^2_{im} g_{kl} - \partial^2_{km} g_{il} - \partial^2_{il} g_{km} + \right) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} + \Gamma^n_{km}\Gamma^p_{il}).$$

Ricci curvature. The Ricci curvature of g is a contraction of the general curvature tensor:

$$R_{ij} = \sum_j R^s_{isj} = \sum_{s,m} g^{sm} R_{isjm}$$

The Ricci tensor has the same type $(0, 2)$ (twice covariant) of the metric tensor. In coordinates we have (Nirenberg's sign, Wikipedia)

$$R_{ij} = \frac{\partial \Gamma^l_{ij}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^j} + \Gamma^l_{ij}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{jm}.$$

4.1.4. **The Laplacian operator in such geometries.** Here is the definition of the geometer's Laplacian (Laplace–Beltrami operator):

$$\Delta_g(u) = -g^{ij}(\partial_{ij}u - \Gamma^k_{ij}\partial_k u) = -\frac{1}{|g|^{1/2}}\partial_i(|g|^{1/2}g^{ij}\partial_j u).$$

This is minus the contraction of the second covariant derivative tensor

$$(\nabla^2 u)_{ij} = \partial_{ij}u - \Gamma^k_{ij}\partial_k u.$$

A coordinate chart (x^k) is called **harmonic chart** if and only if $\Delta_g x^k = 0$ for all i. Note that

$$\Delta_g(x^k) = -g^{ij}\Gamma^k_{ij}.$$

Therefore, (x^i) is harmonic iff $g^{ij}\Gamma^k_{ij} = 0$ for all k.

The Laplacian is convenient for doing analysis and PDEs on manifolds because the basic integration by parts formula

$$\int_M (\Delta_g u)\, v\, d\mu + \int_M \langle \nabla_g u, \nabla_g v\rangle d\mu = 0$$

makes sense if you use the correct definitions.

4.1.5. **Yamabe problem. Ricci flow.** Like in the rest of disciplines of Mathematics, the combination of differential geometry and PDEs has given rise to clasical problems that have focused the interest of generations. We will comment here on two problems that have attracted attention in the last decades.

• **Yamabe Problem.** Let g be a metric in the conformal class of a metric g_0. Let D denote the Levi-Civita connection of g. We denote by $R = R_g$ and R_0 the scalar curvatures of the metrics g, g_0, respectively. Write Δ_0 for the Laplacian operator of g_0. Then we can write

$$g = u^{4/(n-2)}g_0$$

locally on M for some positive smooth function u. Moreover, we have the formula

$$R = -u^{-N} Lu \quad \text{on } M,$$

with $N = (n+2)/(n-2)$ and

$$Lu = \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n+2}.$$

Note that $\Delta_0 - \frac{n-2}{4(n-1)} R_0$ is the conformal Laplacian relative to the background metric. Write equivalently, $R_g u^N = R_0 u - \kappa \Delta_0 u$.

The standard Yamabe Problem can be stated thus: *given* g_0, R_0 *and* R_g, *find* u. This is a nonlinear elliptic equation for u.

An evolution version of the Yamabe problem leads to the so-called Fast Diffusion Equation, $u_t = \Delta u^m$ with exponent $m = (n-2)/(n+2) < 1$. The problem is described for instance in [17].

• **Ricci flow.**The Ricci curvature features prominently in R. Hamilton's program, 1982, to classify three dimensional manifolds by continuous deformation of the original metric. This is a remarkable idea to try to solve by PDE methods the old *Poincaré conjecture* (one of the 7 problems of the Clay list). The proposed flow is

$$\partial_t g_{ij} = -R_{ij}. \tag{4.1}$$

In view of the expression of R_{ij} in terms of g_{ij} and its partial derivatives, this turns out to be a system of nonlinear partial differential equations for the evolution of the metric tensor g_{ij}. It is formally of parabolic type, so Hamilton established for it maximum principles and Harnack inequalities. But the program faced difficulties related to the blow-up of solutions in finite time

In 2002-03 G. Perelman posted three papers with a complete solution of that problem, and this seems to be one of the main mathematical events of the running century. In this way Riemann's legacy is more present than ever before for the mathematical research community.

More details on this topic can be found in J. Porti's contribution in this volume.

4.2. **General relativity. Einstein's equation.** Riemann's ideas went further and they turned out to provide the mathematical foundation for the four-dimensional geometry of space-time in Einstein's theory of general relativity.

The Einstein tensor $\mathbf{G}$ is a 2-tensor on pseudo-Riemannian manifolds which is defined in index-free notation as

$$\mathbf{G} = \mathbf{R} - \frac{1}{2}R\mathbf{g}, \tag{4.2}$$

where we use the following notations: $\mathbf{R}$ is the Ricci tensor, $\mathbf{g}$ is the metric tensor and R is the Ricci scalar (or scalar curvature). In components, the above equation reads

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij},$$

and Einstein's field equations (EFE's) are:

$$G_{ij} = \frac{8\pi G}{c^4}T_{ij},$$

a system of second order partial differential equations in 4 variables. The quantities T_{ij} are the components of the stress-energy tensor, so-called because it describes the flow of energy and momentum.

5. Riemmann and the PDEs of Physics

5.1. **Riemann's interest in Physics.** The influence of the famous experimental physicist W. Weber was important in Riemann's view of mathematics. Apart from his contributions to the mechanics of air waves, he was keenly interested in the contemporary developments, and in teaching. Let us mention here the famous book [15]: Riemann's lectures on the partial differential equations of mathematical physics and their application to heat conduction, elasticity, and hydrodynamics were published after his death by his former student, Hattendorff. Three editions appeared, the last in 1882; and few books have proved so useful to the student of theoretical physics. The object of Riemann's lectures was twofold: first, to formulate the differential equations which are based on the results of physical experiments or hypotheses; second, to integrate these equations and explain their limitations and their application to special cases.

5.2. **Paper "Über die Fortpflanzung...", 1860. The equations of gas dynamics.** Let us go back to creative science. Though Riemann's fame is usually associated among mathematicians with pure mathematics (Riemann's Hypotheses, Riemann surfaces, Riemannian geometry), his contribution to applied science is fundamental in the area of aerodynamics.

Let us present his contribution in a brief form. One-dimensional isentropic gas flow is a mathematical abstraction described by the system of differential equations

$$\begin{cases} u_t + u\,u_x + p_x/\rho = 0, \\ \rho_t + (\rho\,u)_x = 0 \end{cases} \tag{5.1}$$

plus the algebraic equation $p = p(\rho)$.

In applications x is interpreted as length along a tube, whose transversal dimensions are supposed to be irrelevant, u is interpreted as *fluid particle speed* and ρ as *density*. The equation $p = p(\rho)$ is called *equation of state* and for ideal gases it takes the form $p = C\rho^\gamma$ where $\gamma = 1,4$. Evidence on the determination of this γ really worried Riemann as he says at the beginning of his paper (he was not an absent-minded theoretician!).

In a more modern style, we may write the equations in a compact way

$$\mathbf{U}_t + A(\mathbf{U})\mathbf{U}_x = \mathbf{0}, \tag{5.2}$$

which encodes the system:

$$\begin{pmatrix} u_t \\ \rho_t \end{pmatrix} + \begin{pmatrix} u & p'(\rho)/\rho \\ \rho & u \end{pmatrix} \begin{pmatrix} u_x \\ \rho_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.3}$$

5.2.1. **Hyperbolic systems.** In order to continue we do linear algebra, calculating the eigenvectors and eigenvalues of the matrix A. We obtain

$$\lambda_1 = u + c, \qquad \lambda_2 = u - c$$

where $c^2 = p'(\rho)$ (c is called the speed of sound). Note that $\lambda = \lambda(u, \rho)$, hence it changes with (x, t) depending on the flow you are solving at this time.

If $\rho \neq 0$ then $c \neq 0$ and we have two different eigenvalues and we are entering with Riemann into the theory of Nonlinear Hyperbolic Differential Systems, still frightening today. Peter Lax, Courant Institute, Abel Prize winner, is a world leader in the topic. See his monograph [8].

We now get a map from (x, t) into (u, ρ), with two nice directions for the linearization of the evolution equation,

$$\mathbf{U}_t + A(\mathbf{U_0})\mathbf{U}_x = \mathbf{0}.$$

If you are Riemann, or you are able to follow his train of thought, this allows you to construct some magical local coordinates where the flow is not complicated. Correct coordinates are Riemann's specialty.

5.2.2. **Riemann invariants.** The eigenvectors of the system are

$$\mathbf{U}_1 = (c/\rho, 1), \quad \mathbf{U}_2 = (c/\rho, -1)$$

Now Riemann tells us to find the characteristic lines: if we think that the solution is known, then solve the ODE Systems

$$\frac{dx}{dt} = \lambda(x, t, \mathbf{U})$$

He tells you then to find functions F_1, F_2, called the *Riemann invariants*, which are independent and constant along the corresponding characteristics. In the gas example they are

$$F_{\pm} = u \pm \int \frac{c(\rho)}{\rho} d\rho.$$

Since these functions are constant on the characteristics, they allow to see what the characteristics do and this says what the flow does at any moment. Replace (u, ρ) by F_1, F_2 and try to see something. The reader can follow the detailed development in Chorin-Marsden [3] or Smoller [16].

5.2.3. **Shocks.** The theory Riemann develops allows to solve the system in a classical way if and only if the characteristics of the same type for different points do not cross. In that case the invariant takes two values, a shock appears.

Shocks appear in the examples even in $d = 1$, which is the Burger's equation

$$u_t + u\,u_x = 0.$$

Since it happens in the simplest nontrivial mathematics, Riemann concludes that you cannot avoid shock formation, and that a theory of solutions with discontinuities that propagate in some magical way is needed. This is today the theory of shocks and discontinuous solutions of conservation laws.

Very soon the physical community recognized this work as a fundamental new insight into the complexity inherent to compressible fluids.

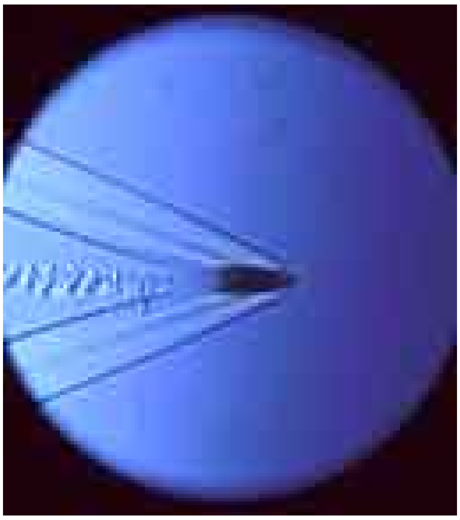
Rankine and Hugoniot completed the work of Riemann when the gas is not isentropic and the system is three dimensional. They even found that Riemann made an error in that general case! (cf. [6], [11]).

Aftermath. The story of how *discontinuous functions can be correct solutions of a partial differential equation of mathematical physics*, and even more, *how important is what happens at the point where classical analysis breaks down*, is one of the deepest and most beautiful aspects of PDEs in the 20th century. The catch word is *entropy solutions*, a theory that counts famous names in the last decades like P. Lax, O. Oleinik, S. Kruzhkov, J. Glimm, and continues for instance with the recent work by A. Bressan.

Before Riemann nobody really dared to face those problems, after him all of us must!

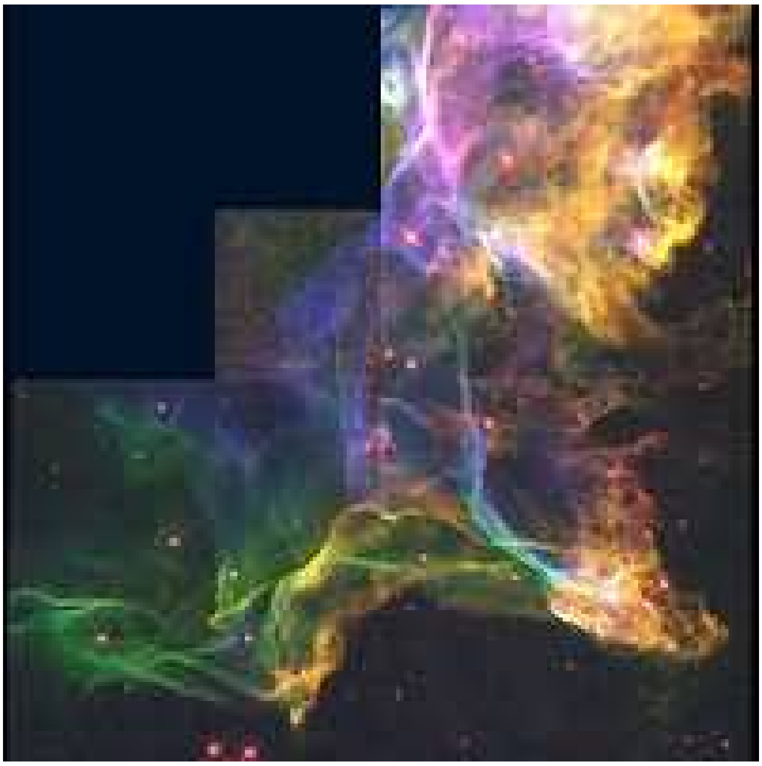
Follow the whole "shocking story" in references like [8], [3], [16], and more recently, [1], [2]. Connections of shocks and general relativity are described in [4].

6. Picture gallery. Some shock waves in Nature



Schlieren Image – Convection Currents and Shock Waves, *Steve Butcher, Alex Crouse, and Loren Winters – August, 2001.*

The projectiles were 0.222 calibre bullets fired with a muzzle velocity of 1000 m/s (Mach 3). The Schlieren lighting technique used for these images makes density gradients in fluids visible. Color filtration provides false color images in which the colors provide information about density changes.



This Hubble telescope image shows a small portion of a nebula called the "Cygnus Loop."

This nebula is an expanding blast wave from a stellar cataclysm, a supernova explosion, which occurred about 15,000 years ago. The supernova blast wave, which is moving from left to right across the picture, has recently hit a cloud of denser-than-average interstellar gas. This collision drives shock waves into the cloud that heats interstellar gas, causing it to glow.

Sandia Releases New Version of Shock Wave Physics Program; it can be found in:

http://composite.about.com/library/PR/2001/blsandia1.htm

As a conclusion of this work, as practitioners of PDEs interested in understanding how the real world works, and above all as mathematicians, we would like to say

Danke schön, Herr Riemann!

References

[1] A. Bressan, "Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem". Oxford University Press, 2000.

[2] A. Bressan, D. Serre, M. Williams, K. Zumbrun, "Hyperbolic Systems of Balance Laws": Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, July 14-21, 2003 (Lecture Notes in Mathematics / Fondazione C.I.M.E., Firenze). Springer, 2007.

[3] A. Chorin, J. Marsden, "A Mathematical Introduction to Fluid Mechanics". Springer, 1990.

[4] J. Groah, J. Smoller, B. Temple, "Shock Wave Interactions in General Relativity". Springer Science + Business Media, LLC, New York, NY USA, 2007.

[5] José Ferreirós, "Riemann, Bernhard. Riemanniana selecta" (Spanish). Edited and with an introductory study by José Ferreirós. Clásicos del Pensamiento. CSIC. Consejo Superior de Investigaciones Cientificas, Madrid, 2000.

[6] H. Hugoniot, *Propagation des Mouvements dans les Corps et spécialement dans les Gaz Parfaits*, Journal de l'École Polytechnique, **57**, (1887), p. 3; **58** (1889), p. 1.

[7] D. Laugwitz, "Bernhard Riemann (1826-1866): Turning Points in the Conception of Mathematics". Birkhäuser (1999).

[8] P. D. Lax, "Hyperbolic partial differential equations". With an appendix by Cathleen S. Morawetz. Courant Lecture Notes in Mathematics, 14. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2006.

[9] T. Levi-Civita, *Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura riemanniana,* Rend. Cir. Mat. Palermo, 42 (1917), 172–205.

[10] M. Monastyrsky, "Riemann, Topology, and Physics". Birkhäuser, 2nd ed., 1999.

[11] W. J. M. RANKINE, *On the thermodynamic theory of waves of finite longitudinal disturbances*, Phil. Trans. Roy. Soc. London, 160 (1870), p. 277.

[12] B. RIEMANN, "Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge" [Collected mathematical works, scientific Nachlass and addenda] Based on the edition by Heinrich Weber and Richard Dedekind. Edited and with a preface by Raghavan Narasimhan. Springer-Verlag, Berlin; BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1990.

[13] B. RIEMANN, *Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite [Propagation of plane air waves of finite oscillation amplitude],* Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 8 (1860), 43-65. Classic papers in shock compression science, 109–128, High-press. Shock Compression Condens. Matter, Springer, New York, 1998.

[14] B. RIEMANN, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen [On the Hypotheses which lie at the Bases of Geometry],* Habilitationsschrift (1854), Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 13 (1868).

[15] B. RIEMANN, "Die Partiellen Differentialgleichungen der Mathematischen Physik" [Partial Differential Equations Of Mathematical Physics], Herausgeg. von K. Hattendorf. Braunschweig: F. Vieweg. 1938. XII, 325.

[16] J. SMOLLER, "Shock Waves and Reaction-Diffusion Equations", Springer, 2nd ed., 1994.

[17] J. L. VÁZQUEZ, "Smoothing and decay estimates for nonlinear diffusion equations", vol. 33 of Oxford Lecture Notes in Maths. and its Applications. Oxford Univ. Press, 2006.