AN OVERVIEW OF RIEMANN'S LIFE AND WORK

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Abstract. Riemann made fundamental contributions to mathematics –number theory, differential geometry, real and complex analysis, Abelian functions, differential equations, and topology– and also carried out research in physics and natural philosophy.

The aim of this note is to show that his works can be interpreted as a unitary programme where mathematics, physics and natural philosophy are strictly connected with each other.

A brief biography

Bernhard Riemann was born in Breselenz –in the Kingdom of Hanover– in 1826. He was of humble origin; his father was a Lutheran minister. From 1840 he attended the Gymnasium in Hanover, where he lived with his grandmother; in 1842, when his grandmother died, he moved to the Gymnasium of Lüneburg, very close to Quickborn, where in the meantime his family had moved. He often went to school on foot and, at that time, had his first health problems –which eventually were to lead to his death from tuberculosis.

In 1846, in agreement with his father's wishes, he began to study the Faculty of Philology and Theology of the University of Göttingen; however, very soon he preferred to attend the Faculty of Philosophy, which also included mathematics. Among his teachers, I shall mention Carl Friedrich Gauss (1777-1855) and Johann Benedict Listing (1808-1882), who is well known for his contributions to topology.

In 1847 Riemann moved to Berlin, where the teaching of mathematics was more stimulating, thanks to the presence of Carl Gustav Jacob Jacobi (1804-1851), Johann Peter Gustav Lejeune Dirichlet (1805-1859), Jakob Steiner (1796-1863), and Gotthold Eisenstein (1823-1852). Therefore, in Berlin classical mechanics and the theory of elliptic and Abelian functions (with Jacobi), number theory (with Dirichlet and Eisenstein), geometry (with Steiner), real analysis and series of functions (with Dirichlet) were all well represented and taught; and in fact Riemann's stay in Berlin strongly influenced his future research.

In 1848 in Berlin –as well as in all Europe– there was a democratic uprising; Riemann was among the conservatory students who opposed it. This attitude was in contrast with the ideas of his teachers; Dirichlet, Eisenstein and Jacobi had indeed an active role in the fight against the King of Prussia.

In 1849 Riemann followed his father's wishes and came back to the University of Göttingen, where he began to attend the seminar on mathematical physics held by Gauss and Wilhelm Eduard Weber (1804-1891) who was appointed professor of physics in 1831. Gauss was also conservative; when Victoria became Queen of Britain in 1837 her uncle became ruler of Hanover and revoked the liberal constitution. Weber was one of seven professors at Göttingen to sign a protest and all were dismissed. Gauss did not support them, even though Weber –who was very close to him– and his son-in-law (the orientalist G.H.A. von Ewald) were among the seven professors who lost their positions at the University. Weber remained at Göttingen without a position until 1843, when he became professor of physics at Leipzig; but in 1848 he came back to his old position in Göttingen.

Riemann held his *Inauguraldissertation* (*Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse*) on complex analysis –supervised by Gauss– in 1851; his *Habilitationsschrift* (*Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*) on real analysis and his *Habilitationsvortrag* (*Über die Hypothesen, welche die Geometrie zu Grunde liegen*) on the foundations of geometry were held in 1853 and in 1854 respectively, and published posthumously in 1868.

In 1855 Gauss died and Dirichlet took his place. Two years later, in 1857, Riemann was appointed extraordinary professor at the University of Göttingen. In the same year he published his celebrated paper on the theory of Abelian functions ("Theorie der Abel'schen Functionen").

In 1859 he visited Paris, where he knew some important mathematicians –such as Joseph Bertrand (1822-1900), Jean-Baptiste Biot (1774-1862), Jean Claude Bouquet (1819-1885), Charles Hermite (1822-1901), Victor Puiseaux (1820-1883) and Joseph Serret (1819-1885). In the same year he was appointed ordinary professor at the University of Göttingen and published his paper on number theory ("Über die Anzahl der Primzahlen unter einer gegebener Grösse"), where he stated the famous Riemann hypothesis on the so-called zeta function.

In 1862 Riemann married Elise Koch, a friend of his sisters.

From 1863 to 1865 he was at the University of Pisa, invited by Enrico Betti (1823-1892). In fact, in this period Riemann's health problem worsened and he tried to recover by staying for a time in a mild climate.

In 1866 he went again to Italy because of his illness, but with no positive effect; he died on July 20 in Selasca.

Analysis, mathematical physics, topology and differential equations

In Riemann's work, research fields seem to be closely connected; as an explanatory example I show the case of Riemann's *Inauguraldissertation.* His starting point is given by the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

which were supposed to be satisfied by a function $w = u + iv$ of a variable $z = x + iy$; they lead to the conditions $\Delta u = 0$, $\Delta v = 0$ – that is to say, u and v satisfy the Laplace equations and, therefore, are called harmonic functions.

It is well known that Riemann's complex function theory is strongly connected with potential theory in two dimensions, which Riemann knew very well. As a student he had indeed participated in the physics seminar by Gauss and Weber. In potential theory, an important task concerned the so-called "Dirichlet problem", which many mathematicians of the 19th century dealt with. It can be stated as follows:

> To find a function u with continuous first partial derivatives on a given bounded domain, which satisfies the Laplace equation within the domain and has given values on the boundary.

Both Gauss and Dirichlet had also confronted such a problem. Gauss (1839) was led to it by studying the distribution of masses or electric charges on a closed surface S, assuming the potential constant on S; while Dirichlet published a treatise on potential theory (*Vorlesungen über die im umgekehrten Verhältnisse des Quadrats der Entfernung wirkenden Kräfte*) in 1876 where this problem has an important role.

The Dirichlet problem plays a fundamental role also in Riemann's theorem concerning the existence of a conformal mapping between two simply connected plane regions. Riemann proved this theorem by solving a special Dirichlet problem and remarked that the assumption "simply connected" can be removed and that, as a consequence, the theorem is valid for Riemann surfaces as well. A "Riemann surface" is introduced for the first time in his *Inauguraldissertation* in order to study multi-valued functions –such as algebraic functions and their integrals. The Riemann surface associated with a function is composed of as many sheets as there are branches of the function, connected in a particular way –so that continuity is preserved and a single-valued function on the surface is obtained. Therefore, he gave an abstract conception of the space of complex variables by using a geometrical formulation.

The concept of "transversal cut" (*Querschnitte*) probably come from mathematical physics too. This definition allowed him to make the surface simply connected with suitable transversal cuts and to study the behaviour of the function in the neighbourhood of the singularities. On the basis of some letters by Riemann himself, it seems that the idea of a transversal cut on a surface struck Riemann after a long discussion with Gauss on a mathematical-physical problem. The origin of the ideas of Riemann surfaces and "Querschnitte" can indeed be found in his note on a problem of electrostatic or thermic equilibrium on the surface of a cylinder with transversal cuts ("Gleichgewicht der Electricität auf Cylindern mit kreisförmigen Querschnitt und parallelen Axen", 1876). In this unpublished note, he was led to consider a Dirichlet problem on a simply connected and simple sheeted surface.

Finally, I note that the Laplace equation –which appears in Dirichlet's problem– is a special partial differential equation of the elliptic kind. Green (1828) showed that –at least theoretically– a Dirichlet problem can be solved by using the so-called Green function. Unfortunately the solution can be explicitly deduced only for special cases. Many mathematicians of the 19th century –such as Hermann von Helmholtz (1821-1894), Rudolph Lipschitz (1832-1903), Betti, Carl Neumann (1832-1925), Franz Neumann (1798-1895), and Riemann himself– deduced functions similar to Green's function in order to solve problems in acoustics, electrodynamics, magnetism, theory of heat, and elasticity.

In a paper ("Ueber die Fortpflanzung ebener Luftwelle von endlicher Schwingungsweite") published in 1860, Riemann applied the method of Green's function in order to integrate the differential equation of hyperbolic type describing the diffusion of acoustic waves. He introduced a function which plays the same role as Green's function did for the Laplace equation and is today called "Green's function for the hyperbolic problem".

Naturphilosophie

In a undated note, written after the completion of his *Inauguraldissertation*, Riemann wrote that his "main work" involved "a new interpretation of the known laws of nature –whereby the use of experimental data concerning the interaction between heat, light, magnetism and electricity would make possible an investigation of their interrelationships. I was led to this primarily through the study of the works of Newton, Euler and, on the other side, Herbart". The note remained unpublished and appeared only in 1876, posthumously.

Riemann agreed "almost completely" with Herbart's psychology, which inspired both Riemann's model of the ether (the elastic fluid which was supposed to fill all the universe) and his principles of *Naturphilosophie.* According to Herbart, the "psychic act" (or "representation") is an act of self-preservation with which the "ego" opposes the perturbations coming from the external world. A continuous flow of representations go from the ego to the conscious and back. Herbart studied the connections between different representations in mechanical terms as compositions of forces.

Riemann followed Herbart's psychology and tried to apply Herbart's theory to his conception of the universe in a paper drafted in March 1853 ("Neue mathematische Principien der Naturphilosophie") and in other notes on *Naturphilosophie* which he intended to publish, as he wrote to the brother Wilhelm in December 1853. However, he never published his writings on this subject and all of them only appeared in the edition of his collected works (1876).

In his "Neue mathematische Principien der Naturphilosophie" Riemann followed Herbart's ideas and supposed that the universe is filled with a substance (*Stoff*) flowing continually through atoms and there disappearing from the material world (*Körperwelt*). From this obscure assumption, Riemann tried to build a mathematical model of the space surrounding two interacting particles of substance: if a single particle of substance is concentrated at the point $O(x_1, x_2, x_3)$ at time t and at the point $O'(x'_1, x'_2, x'_3)$ at time t', then he considered the two homogeneous forms:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2, \quad ds'^2 = dx'^2_1 + dx'^2_2 + dx'^2_3. \tag{2}$$

Riemann considered an appropriate new basis and compared the two forms associated to the "particle of substance" at times t and t'. The difference between the two forms is given by:

$$\delta(ds) = ds'^2 - ds^2 = (G_1^2 - 1)ds_1^2 + (G_2^2 - 1)ds_2^2 + (G_3^2 - 1)ds_3^2. \tag{3}$$

If $\delta(ds) = 0$, then the particle will not change its form from the time t to the time t'; in this case, the particle does not propagate any force since space is not submitted to deformation by a force. On the contrary, if $\delta(ds)$ is different from zero, a physical phenomenon is propagated through space.

Riemann tried to connect this formula with the different forces, such as gravity, and heat and light propagation. Riemann could not explicitly show these connections. He limited himself to state that gravity, and light and heat can be explained by assuming that every particle of the homogeneous substance filling space has a direct effect only on its neighbourhood and the mathematical law according to which this happens is due to:

> 1) The resistance with which a particle opposes a change of its volume, and

2) The resistance with which a physical line element opposes a change of length.

Gravity and electric attraction and repulsion are founded on the first part, light, heat propagation, electrodynamic and magnetic attraction and repulsion on the second part.

Riemann's *Naturphilosophie* is connected both to some of his physical concepts on electricity and electromagnetism and to his ideas on differential geometry. In fact, as regards physics, I mention a paper on Kohlrausch's experiment written by Riemann in September 1854. According to Kohlrausch's experiment, in a Leyden jar - the first "capacitor" in the history of physics - which had been charged, then discharged and left for some time, a residual charge appeared. Riemann tried to explain "the residual charge" by using the model of ether developed in his *Naturphilosophie* and developed a physical explanation of the electromotive force and of electric propagation. Riemann addressed this paper to the famous journal *Annalen der Physik und Chemie*, but Kohlrausch –the editor of the journal– asked for so many changes that Riemann retracted the paper.

In this paper Riemann proposed a new theory of electricity by assuming that the electric current was caused by a reaction of the body opposed to the change of its own electric state. This reaction is proportional to the charge density, and it decreases or increases the electric density depending on whether the body contains positive or negative electricity. Therefore the transmission of electricity could not be instantaneous but electricity moves "against ponderable bodies" with a certain speed.

In the paper, "Ein Beitrag zur Electrodynamik" (1858), he developed a new theory of electromagnetism, by assuming that electric phenomena travel with the velocity of light and that the differential equations for the electric force are the same as those valid for light and heat propagation. Riemann's research on electromagnetic forces was also influenced by Gauss and Weber. In an unpublished note –dated July 1835– Gauss had already suggested a new theory of electrodynamics. According to Gauss, two elements of electricity attract and repulse each other with a force depending on their moving state. In a letter to Weber written in 1845, Gauss supposed that electricity propagated from one point to another not instantaneously, but in time, as in the case of light.

He never published his ideas on electrodynamics during his lifetime; therefore, Weber's theory of electrodynamics, published in *Electrodynamische Maasbestimmungen* (1846), was the first result of this kind –where electricity propagates in time– known to the scientific world.

In his lectures on partial differential equations (*Partielle Differentialgleichungen und deren Anwendungen auf physikalische Fragen*, ed. by Hattendorff), published in 1876, Riemann tried to describe the ether surrounding two interacting electric particles. For this purpose, he assumed that the ether possesses physical properties which guarantee electric propagation. He actually deduced a differential equation expressing the flux of ether in space, by using classical Lagrangian mechanics.

His model of the ether and, more generally, his ideas contained in the notes on *Naturphilosophie* are not only connected to physics, but also to his geometrical concepts expressed in the *Habilitationsvortrag.* Riemann indeed extended the "local" investigation of particles of ether –developed in his writings on natural philosophy– to the "global" analysis of n-dimensional spaces. In fact, if one considers an ether filling all space, then a deformation of space will be closely linked to the force which has to be propagated. Force and curvature of space are then deeply connected; one could say that it is space which propagates forces by changing its curvature.

A deep analysis on the connection between n-dimensional manifolds and their curvatures is widely developed in a paper on mathematical physics ("Commentatio Mathematica…"), which Riemann wrote in 1861, when trying to answer a question proposed by the Paris Academy of Sciences on heat conduction in homogeneous solid bodies.

Riemann's ideas on space and curvature, which seem to anticipate the theory of relativity, were shared by other mathematicians of the 19th century –such as Eugenio Beltrami (1835-1900), Nicolai Lobachevskij (1792-1856), and William Kingdon Clifford (1845-1879). In this connection, in *The common sense of exact sciences* (1885) Clifford asked the question "whether physicists might not find it simpler to assume that space is capable of a varying of curvature, and of a resistance to that variation…", and that this resistance was the responsible of the propagation of phenomena.

The idea of being in a curved space which, thanks to his changes of curvature, transmitted physical force was at the basis of many mathematical and physical reflections during the 19th century and, in any case, long before Einstein's theory of relativity.

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