RIEMANN & PHYSICS

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Abstract. Riemann's interest in physics is clear from his legacy, which is discussed in detail in all the contributions to this volume. Here, after providing a very concise review of the main publications of Bernhard Riemann on physical problems, we will turn to consider some rather less known (but not less interesting) connections between Riemann's papers and contemporary Physics. More specifically, we will address, among other aspects: (a) the influence of Riemann's work on the zeta function, its functional formula, and related extensions of those concepts, to the regularization of quantum field theories in curved space-time (in particular, that of the quantum vacuum fluctuations); and (b) the uses of the Riemann tensor in general relativity and in very recent generalizations of this celebrated theory, which aim at understanding the presently observed acceleration of the universe expansion (the dark energy issue). We shall argue that the importance of the influence in Physics of Riemann's *purely mathematical* works exceeds by far that of his papers which were directly devoted to *physical* issues.

1. Introduction

The presentation at the meeting, on which this paper on Riemann's work is based, took place at the very end of a long day of dense and interesting discussions. It is in this framework that the contents which follow have to be pondered. The author is somehow afraid he tried to present in the talk rather deep concepts in a light, almost casual way. This was however maybe not too bad, since it gave rise to a number of clever questions from the audience. They can be found, hopefully, in the recordings of the lecture, to which the reader is addressed for additional information, when needed. We will try here to avoid repeating concepts and arguments already contained in the other contributions to this volume. For all these reasons—and also for lack of space—the present article will not be self-contained, but the reader will be addressed to the relevant references at the appropriate places. Even then, some repetition will be unavoidable, but hopefully the viewpoints will be different and maybe enriching.

It is quite clear that Riemann was definitely interested in physics. This may sound to more than one a weird statement nowadays, when he is considered to be a pure mathematician, who gave name to so many concepts in different fields of mathematics, as the Riemann integral, the Riemann surface, the Cauchy-Riemann equations, the Riemann-Roch theorem, the Riemannian manifolds, the Riemann curvature tensor and, most notably the Riemann zeta function, with its associated conjecture—the only one of Hilbert's problems that after the turn of the XXth Century has entered the new list of *Million Dollar Problems* of the XXIst (awarded by the Clay Foundation). However, historians of science assure that during his life and till as late as 60 to 80 years after his death, Riemann was counted among the list of important physicists, whose ideas on the unification of all known forms of energy preceded the ground-breaking work of Hilbert and Einstein (see later, and also the other contributions in this volume). Even more surprising is to learn that Riemann was not only a theoretical physicist, but also an experimental one, and that he made use of physical proofs with charged surfaces in order to establish supplementary checks of the validity of some mathematical theorems (as boundary problems involving partial differential equations).

Let me here just recall that, as a student at Göttingen university, Riemann worked with Weber on electromagnetism, which happened starting around 1849. Like Riemann, Weber was also a student of Gauss, but at that time Weber had already a faculty position. He had proposed a theory of electromagnetism which gained him a name in history, as every physicist knows, although not through his theory in fact, that was eventually superseded by Maxwell's one, the real landmark in classical electromagnetism. Gauss himself is also famous for his important work on this subject.

Riemann publications include some fifteen papers, four of which where published after his death. Needless to say, this does not include a number of important notes, letters, books and other writings that also form part of his written scientific production. In the first section of the present paper a brief summary will be provided (the reader is again addressed to the other contributions in this issue for more detailed discussions) of the six papers (among the mentioned fifteen) which are devoted to physical problems. Then, sort of a panoramic view will be presented of the enormous influence of Riemann's work on pure mathematics to past and present Physics. In the last part of the paper I will concentrate more specifically on a couple of issues of my own speciality, namely, on the one hand, the use of zeta functions as a very elegant regularization tool in quantum field theory, including a brief description of its uses for the calculation of quantum vacuum fluctuations, the Casimir effect, and the related cosmological constant problem. The other issue to be addressed is the very well known applications of the Riemann curvature tensor and all his geometrical formalism in general relativity and the, much less known but very important nowadays, proposed modifications of the Einstein-Hilbert Lagrangian with additional terms—a function of the curvature scalar, the so-called $f(R)$ theories. Only ca. hundred years after the formulation of general relativity, on response to the demand of the observed acceleration of the universe expansion (the crucial dark energy issue), have some attempts at a modification of Einstein's equations started to appear. But again, notably, in terms of its basic Riemannian building blocks, as we shall later see.

2. Six Riemann pieces on physical problems

The starting reference list of works by Riemann, previously mentioned, consists of *The Mathematical Papers of Georg Friedrich Bernhard Riemann (1826-1866)*, a collection which contains scientific papers of Bernhard Riemann as transcribed and edited by David R. Wilkins [1]. These texts are based on the second edition of the *Gesammelte Mathematische Werke* and, in the case of some of the papers, the original printed text in the *Journal für die reine und angewandte Mathematik, Annalen der Physik und Chemie* and *Annali di Matematica.* Included in Ref. [1] are all papers published in Riemann's lifetime, papers and correspondence published after Riemann's death by Dirichlet and others prior to the publication of the first edition of the *Gesammelte Mathematische Werke* (with the exception of the fragment *Mechanik des Ohres*, which is non-mathematical in character), and one of the papers from his *Nachlass*, first published in the *Gesammelte Mathematische Werke*. There is also a translation by W. K. Clifford of Riemann's inaugural lecture on the foundations of geometry, and a biographical sketch by Richard Dedekind that was included in the *Gesammelte Mathematische Werke*.

However, I will not go here through *all* these works. I will restrict my attention to a subset which, although not complete as viewed by a historian of science, I think it is fair enough in order to establish my point that Riemann's physical production was actually a good part of his complete scientific work. I will reduced the whole sample in Ref. [1] to that of the published papers—during Riemann's lifetime and posthumously—and limit my study to the papers on physical issues among them. The list of these published articles is as follows.

2.1 Papers published in Riemann's lifetime

(1) *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, Inauguraldissertation, Göttingen (1851).

(2) *Ueber die Gesetze der Vertheilung von Spannungselectricität in ponderabeln Körpern, wenn diese nicht als vollkommene Leiter oder Nichtleiter, sondern als dem Enthalten von Spannungselectricität mit endlicher Kraft widerstrebend betrachtet werden*, Amtlicher Bericht über die 31. Versammlung deutscher Naturforscher und Aerzte zu Göttingen (im September 1854).

(3) *Zur Theorie der Nobili'schen Farbenringe*, Annalen der Physik und Chemie, **95** (1855) 130-139.

(4) *Beiträge zur Theorie der durch die Gauss'sche Reihe* $F(\alpha, \beta, \gamma, x)$ *darstellbaren Functionen*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **7** (1857) 3-32.

(5) *Selbstanzeige: Beiträge zur Theorie der durch die Gauss'sche Reihe darstellbaren Functionen*, Göttinger Nachrichten (1857) 6-8.

(6) *Theorie der Abel'schen Functionen*, Journal für die reine und angewandte Mathematik, **54** (1857) 101-155.
(7) *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie (November, 1859) 671-680.
(8) *Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **8** (1860) 43-65.
(9) *Selbstanzeige: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Göttinger Nachrichten (1859) 192-197.
(10) *Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **9** (1860) 3-36.
(11) *Ueber das Verschwinden der Theta-Functionen*, Journal für die reine und angewandte Mathematik, **65** (1866) 161-172.

2.2 Posthumously published papers of Riemann

(12) *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, Habilitationsschrift, 1854, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **13** (1868).
(13) *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, Habilitationsschrift (1854), Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **13** (1868).
(14) *Ein Beitrag zur Elektrodynamik* (1858), Annalen der Physik und Chemie, **131** (1867) 237-243.
(15) *Ueber die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **13** (1868).

Six among these fifteen papers (namely, those with numbers 2, 3, 8, 9, 10, 14) are the ones that I have selected because they directly address issues of theoretical and experimental physics. I now provide a free translation of their titles, together with a short summary of each of them (indeed very brief, since they are also described in some detail in the other contributions to this volume).

2.3 Riemann papers on Physics

2. *About the distribution laws of electric tension in ponderable bodies, when these cannot be considered as absolutely conductors or non-conductors, but as opposing with a finite force to the electric tension they contain*, Official Report at the 31st Meeting of German Scientists and Physicians at Göttingen (September, 1854).

   Riemann considers in this paper Leyden jars, where an electric charge is kept, and studies in particular how, once the bottle has been emptied, a certain amount of charge still remains, which gradually disappears with time. He studies in detail the corresponding law describing this phenomenon. Riemann deals, in particular, as the title clearly indicates, with bodies that are neither perfect conductors nor perfect isolators. He elaborates on previous work by Ohm, Weber, Kirchhoff and Kohlrausch. An important point in the whole development is the contact with the corresponding experimental results. The mathematical basis of the paper are partial differential equations, as is also the case in the ones to follow.
3. *On the theory of noble color rings*, Annals of Physics and Chemistry, **95** (1855) 130-139.

   Here an experimental study of the propagation and of the distribution of an electrical current in a conductor is presented. The rings mentioned in the title are generated when one covers a plate of a noble metal, as platinum, gold plated silver, or similar, with a solution of lead oxide. Then, an electric current, produced by a battery, is connected to the plate. In this way, the so-called Newton color rings are produced. Riemann elaborates here on previous results by Becquerel, Du-Bois-Reymond and Beetz, improving their calculation and discussing about the hypothesis previously considered by these authors.
8. *About the propagation of plane airwaves of finite oscillation amplitude*, Sessions of the Royal Science Society at Göttingen, **8** (1860) 43-65.

Riemann integrates in this paper the differential equations corresponding to the movement of gases, under different conditions of pressure and temperature. He remarks that he can bring his calculations further away in the order of approximation, with respect to those previously carried out by Helmholtz, for instance, who only got to the second order in the perturbative expansion. He refers to previous results by Helmholtz, Regnault, Joule and Thomson, improving their calculations, discussing the set up and improving the hypothesis in the works by these authors. With 22 pages, this is quite a long paper as compared with other papers of Riemann.

9. *Self-announcement: About the propagation of plane airwaves of finite oscillation amplitude*, Göttingen Notices (1859) 192-197.

   This is a very short compendium of the main mathematical formalism that is used in the former paper, of the same title, in order to obtain the results. In spite of its title, this one could be considered as a mathematical article. Indeed, it deals with the theory of propagation of a gas, but the only physical input in the whole paper is the mathematical equation giving the behavior of gas pressure as a function of the density (that is, its equation of state), in the absence of any heat exchange. He develops the mathematical formalism in detail and compares with previous results by other mathematical physicists as Challis, Airy, Stokes, Petzval, Doppler and von Ettinghausen (most of them have given names to quite famous equations).

10. *A contribution to the investigation of the movement of a uniform fluid ellipsoid*, Sessions of the Royal Science Society at Göttingen, **9** (1860) 3-36.

   Again as clearly indicated in the title, Riemann deals here with the movement of a uniform fluid ellipsoid, which is considered to be constituted by isolated points that attract themselves under the influence of gravity. This is considered one of the finest papers by Riemann within the class of those considered here, i.e. the ones dealing with actual physical problems. In the paper, the equilibrium configurations of the ellipsoid are identified, what has many and important applications, e.g., to the study of the possible forms of celestial bodies as galaxies or clusters. Riemann studies in particular the evolution of the principal axis of the ellipsoid and the relative movement of its components. As the one before, this is also a rather mathematical paper, since the only physics it contains is practically reduced to the initial conditions and Newton's law. Previous results of Dirichlet and Dedekind on this problem are extended.

14. *A contribution to Electrodynamics*, Annals of Physics and Chemistry, **131** (1867) 237-243.

    This paper is generally considered to incorporate the main results of Riemann's physical (and also philosophical) ideas on the 'unification' of gravity, electricity, magnetism, and heat. It contains indeed his observation on how a theory of electricity and magnetism is closely related with those for the propagation of light and heat radiation. He presents in the paper a complete mathematical theory, with *"an action that does not differentiate"* the already mentioned four cases of *"gravity, electricity, magnetism, and temperature".* The finite velocity of propagation of the interaction (as opposed to the predominant concept, at the epoch, of action at a distance) is clearly presented, identifying such velocity with that of light, which has been considered by many to be a really remarkable achievement of Riemann's genius. The paper, which with only six pages is in fact quite short, relies on experimental results by Weber and Kohlrausch, Busch, and by Bradley and Fizeau.

2.4 Some additional considerations

(1) Once more, those above are not all the works on physical issues Riemann wrote, but just the ones extracted from a uniform sample, namely his published articles.

(2) A good example of a work not in the list is the well-known book by H. Weber and B. Riemann, *Die partiellen Differential-Gleichungen der mathematischen Physik nach Riemanns Vorlesungen*, 6. unveranderte Aufl., 2 vols (Vieweg, Braunschweig, 1919), that was used for many years as a textbook in different universities, together with several other papers by Riemann.

(3) An interesting biography of Riemann is the book by Monastyrsky [2]. A lot of emphasis is made there on the importance of the

contribution to physics of paper 14 of the list above. In particular, it is underlined how Riemann was searching for *"... a completely self-contained mathematical theory ..., which leads from the elementary laws up to the actions in an actually given filled space, without making a difference between gravity, electricity, magnetism or the equilibrium of temperature."*

(4) In the celebrated biography of David Hilbert by Constance Reid [3] we can read that Hilbert sustained the opinion (referring to what is nowadays known as the Einstein-Hilbert action) that *"... the invariance of the action integral unifies electromagnetism with gravity ..."*, yielding in this way a solution to a problem that, as he recognizes, *"... was already posed by Riemann: the connection between gravitation and light."* Hilbert goes on to observe that, since then, many investigators had tried to arrive at a deeper understanding of this connection by merging the gravitational and electromagnetic potentials into a unity. The one example Hilbert mentions explicitly is Weyl's unification of the two fields in a *"unified world metric,"* as he calls it, by means of Weyl's notion of gauge invariance.

(5) Remarkably enough, in what is probably one the most exhaustive biographies of Riemann ever written, Laugwitz [4] forgets almost completely about Riemann's work on physical issues. This is, in my view, to push to an extreme the opinion that I maintain here, which is much more moderate and doublefaced.

3. Influence in Physics of Riemann's purely mathematical papers

It is the opinion of the author, shared also by others (see, e.g., Ref. [4]), that the influence in Physics of Riemann's purely mathematical papers exceeds by far, in its manifest importance, the above mentioned contributions on actual physical problems; even if the interest of the last attains already, as we have pointed out in the preceding section, a fair high level.

I would need more space and time than I have at disposal in order to describe all such intertwining influences. In the following, to start, a rather short list of items will provide some basic ideas about those influences. Then we will elaborate on some of them in more detail, not only because of their importance, but also because of the fact that they have to do with my own scientific expertise and published record — mostly join works with a number of different colleagues— in the last few years.

3.1. **On the concept of space.** One reason why the discovery of non-Euclidean geometry took so long might have been the fact that there was universal belief that Euclidean geometry was special because it described the space we live in. Stemming from this uncritical acceptance of the view that the geometry of space is Euclidean was the conviction that there was *no other* geometry. Philosophers like Emmanuel Kant argued that the Euclidean nature of space was a fact of nature, and the weight of their authority was very powerful. From our perspective, we know of course that the question of the geometry of space is in fact entirely different from the question of the existence of geometries which are non-Euclidean. Gauss was the first who clearly understood the difference between these two issues. In Gauss' *Nachlass* one can find his computations of the sums of angles of each of the triangles that occurred in his triangulation of the Hanover region. His conclusion was that the sum was always two right angles, within the limits of observational errors.

Nevertheless, quite early in his scientific career Gauss became convinced of the possibility of constructing non-Euclidean geometries, and in fact came up with the *'theory of parallels,'* but because of the fact that the general belief in Euclidean geometry was deeply ingrained, Gauss decided not to publish his researches in the 'theory of parallels' and the construction of non-Euclidean geometries for fear that there would arise criticisms of such investigations by people who did not understand those things *('the outcry of the Boeotians').*

Riemann took this entire circle of ideas to a higher, completely different level. In his famous inaugural lecture of 1854, written under the advice (or, better, compulsory choice) of Gauss himself, he touched upon all of the aspects that his thesis advisor had considered. He pointed out, to start, the very crucial idea that a space does not have any structure except that it is a continuum in which points are specified by the values of n coordinates, n being the *dimension* of the space. On such a space one can then impose many geometrical structures. His great insight was that a geometry should be built from these infinitesimal parts. He treated in depth geometries where the distance between pairs of infinitely near points is pythagorean, formulated also central questions about such geometries, and discovered the set of functions—the sectional curvatures—whose vanishing characterized the geometries which are Euclidean, namely those whose distance function is pythagorean not only for infinitely near points, but even for points which are a finite but small distance apart.

If the space is the one we live in, he formulated the principle that its geometrical structure could only be determined empirically. In fact he stated explicitly that the question of the geometry of physical space does not make sense independently of physical phenomena, i.e., that space has no geometrical structure until we take into account the physical properties of matter in it, and that this structure can be determined by measurement only. Indeed, he went so far as to say that *"the physical matter determines the geometrical structure of space".* This groundbreaking idea took definite form some half a century later with Einstein equations.

Indeed, it is also important to remark that Riemann's ideas constituted a profound departure from the perceptions that had prevailed until his time. No less an authority than Newton had asserted that space by itself is an *"absolute entity endowed with Euclidean geometric structure"*, and had built his entire theory of motion and celestial gravitation on that premise. Riemann went completely *away* from this point of view. Thus, for Riemann, space *derived its properties from the matter that occupied it*, and he asserted that the only question that could be studied was *whether* the physics of the world made its geometry Euclidean. It followed from this idea that only a mixture of geometry and physics could be tested against experience. For instance, measurements of the distance between remote points clearly depended on the assumption that a light ray would travel along shortest paths. This merging of geometry and physics, which is a central and dominating theme of modern physics, since Einstein's work, may be thus traced back to Riemann's inaugural lecture [5].

3.2. **Linear algebra, the concept of $n-$dimensional space (linear, or trivial 'variety').** It has been often reported (and this seems indeed to be the case) that linear algebra was a 'trivial matter' for Riemann. However, in Laugwitz's book [4] (p. 242) we can read that the early developments of Riemannian geometry were 'prolix and opaque' because 'the development of linear algebra failed for a long time to keep pace with the progress of analysis.' This may be true, in fact: although nowadays $n-$dimensional linear spaces and their algebraic properties are considered to be one of the simplest theories in Mathematics, and its uses in classical and quantum physics are so basic and widespread (including infinite dimensional spaces, topological spaces, Banach and Hilbert spaces, etc.), that even the most basic issues of modern physics would not be possible without such concepts. One cannot simply translate this view to Riemann's time. But it was already clear that these abstract linear spaces had nothing to do with the space we live in, and were not even called 'spaces' by Riemann or Gauss, but rather 'varieties' or 'manifolds' again.

3.3. **Riemann's integral.** Riemann may have arrived at his notion of an 'integral' in answer to the question of whether the Fourier coefficients, c_n, of a given function tend to 0 (as n goes to infinity). Yet Laugwitz [4] characterizes Riemann's introduction of his integral as *ad hoc* and remarks that "History would have been different if he had asked himself the question: what kind of integral implies the equality

$$\lim \int_a^b f_n = \int_a^b f,$$

where f_n is a monotonically increasing sequence of integrable functions that converge pointwise to the limit f?"

It is of course true that Lebesgue's integral is the ultimate extension of Riemann's one, a most fundamental tool in measure theory. But it is not less certain that, concerning the subject at discussion here, Riemann's integral is much more intuitive for a physicist. This I know well since I have been teaching both kind of integrals to physicists during many years. The Riemann integral is such a physically fashionable object, in particular the incremental version before the limit is taken, before an 'increment' is transformed into a 'differential', which is a far more elusive concept indeed! (I know well from my students). The corresponding upper and lower finite sums[1], ... no realization of an integral could be more suited to the physicist's mind.

3.4. **Complex Variables, Cauchy-Riemann equations.** Riemann's Thesis studied the theory of complex variables and, in particular, what we now call Riemann surfaces. It therefore introduced topological methods into complex function theory. The work elaborates on Cauchy's foundations of the theory of complex variables built up over many years and also on Puiseux's ideas of branch points. However, Riemann's Thesis is a strikingly original piece of work which examined geometric properties of analytic functions, conformal mappings and the connectivity of surfaces. Riemann's work was always based on intuitive reasoning which would fall at instances a little below the rigor required to make the conclusions watertight. However, the brilliant ideas which his works contain are so much clearer because his papers are not overly filled with lengthy computations [6], and this is why they were so frequently used in lecture courses (in special in Italy) afterwards. Again, one recognizes the physicist's approach in many of his discussions, but more important than this is the enormous use that both classical and quantum physics has made of the complex calculus that Riemann (among others) contributed to expand and popularize in a very efficient way.

3.5. **Riemann surface, sphere, manifold.** In principle, those are very abstract concepts, but which have been applied, e.g., by engineers to the study of aerodynamics and hydrodynamics. At a different level, theoretical physicists have more recently drawn upon them very heavily in their formulations of string theory.

String theory is the modern version of a Theory of Everything (TOE). It suggests replacing pointlike particles with infinitesimal vibrating strings as the basic units of the physical world. Some ten to fifteen years ago, when string theory was overwhelmingly dominating the landscape in theoretical physics[2], there have even been jokes about the typical

[1] With references to the determination of areas of real fields in ancient Egypt and Mesopotamia.

[2] There is a fashionable string theory *landscape* right now which contains an enormous amount (maybe 10^{500}) of possible vacuum solutions of the corresponding theory. Choosing one among them seems hopeless, for the moment, and it is one of the main problems of M theories (M stands for 'Mother', or 'Mysterious').

theoretical physicist always carrying Farkas and Kra's book (*Riemann Surfaces*) under his arm, everywhere from place to place [7]. Edward Witten, from the Institute for Advanced Study at Princeton, has been, and continues to be, one of the main architects of string theory. He has given talks from time to time on Riemann's work, when discussing some of the relations between physics and mathematics in the 20th and 21st centuries, to which the reader is addressed for material that complements a lot, from a different, much more ambitious perspective, what I will discuss below.

3.6. **Analytic continuation, complex power series.** Most of Riemann's predecessors concentrated on a power series expansion rather than on the function that it represents. By shifting emphasis to the latter, Riemann could eliminate superfluous information, determining a complex function from its singularities. Riemann's work used simple concepts in place of the lengthy and sometimes obscure computations typical of his predecessors and contemporaries. The steady decrease in the amount of attention Riemann seems to have paid to power series between 1856 and 1861 indicates how Riemann's thought matured, shifting further away from computation. Even when using his great computational abilities, Riemann still focused upon concepts rather than the computation itself. Since relations obtained from series expansions of functions retain their validity outside their regions of convergence, he asked himself *what actually continues functions from region to region?* For example, Riemann constructed a function that has simple zeros at $z = 0, 1, 2, \dots$ and is finite for all finite z (see Laugwitz [4]). The road to his function $g(z)$ was heuristic, but this was of no consequence to Riemann. All he wanted was to find some function with the prescribed zeros. By contrast, Weierstrass always aimed to obtain formula representations of given functions. The Riemann approach to this issue is one of the main starting points in a big part of the present author's work, as it will be commented later in more detail.

3.7. **Curvature tensor. Differential Geometry.** In his general theory of relativity, Einstein used Riemann's concept of curved space as the basis for his elegant explanation of gravitation. Massive objects put a dimple in space in their vicinity. So when other physical objects, including photons, which do not have any mass, wander into the object's vicinity, they encounter this curved space. Such curvature determines the path the objects follow, in a way that was formerly attributed to the force of attraction that we call gravity.

In much the same way that Riemann conceived of curving and twisting space in innovative ways, he also described a set of abstract surfaces that were created by cutting and pasting together normal surfaces in ways that cannot be employed with real surfaces, but can be thought abstractly. You can do a lot of mathematics on those abstract surfaces. So this has been an amazingly important idea for many parts of mathematics, and now for physics.

General relativity, quantum field theory in curved spaces, string theories, gravitation, modern cosmology, would had been impossible without those basic concepts introduced by Riemann.

3.8. **The Riemann zeta function.** This is known to be of extreme importance in analytic number theory. But also, through its analytical continuation (the so called functional equation or reflection formula of the zeta function), and extrapolating the concept of the zeta function to the domain of pseudodifferential operators (the spectral values of the operator replace the natural numbers in the zeta function definition), as a regularization tool in quantum field theory (notably in curved space-time), for dynamical systems (classical and quantum), the concept of chaos (also present nowadays in the issue of the distribution of non-trivial zeros, or Riemann conjecture), etc. The interconnections between pure mathematics and physical uses here is becoming more on more profound as decades advance.

4. Selected hot subjects: concept of space, zeta regularization, modified gravity theories

4.1. The concept of space.

4.1.1. *Historical evolution of the concept of space.* A summary of the evolution of the concept of space, from the very remote times of its inception, could be as follows.

(1) The introduction of the concept of space seems to go back to the pre-Socratic philosophers, who already had coined this notion, together with some other very important ones as those of substance, number, power, infinity, movement, being, atom, and of course time, among others.
(2) The Pythagorean school should be mentioned as another important step, in its attempt at bringing all these concepts, in particular the one of space to the domain of numbers *("all things are numbers")*. Just recall the importance of Pythagoras theorem, that has so much to do with space and with Gauss' search to check if the space we live in was or not Euclidean.
(3) Euclid's *"Elements"*, this goes without saying as one of he most important pieces of work in the History of Mankind. It was so influential, for generations, that departing from the concept of Euclidean space was absolutely impossible for many centuries to come.
(4) Indeed, still for Isaac Newton *"space is, by itself, an absolute entity embedded with a Euclidean geometrical structure"*.
(5) On the side of the philosophers, for Immanuel Kant *"that space is Euclidean is a property of nature itself"*.
(6) Now Bernhard Riemann came to clearly say, as we have advanced before, the following: *"many spaces are possible; it is the physical matter that determines the geometrical structure of space"*.
(7) And Albert Einstein gave a precise mathematical formulation of this concept, with the important help of Grossmann and making use of Riemann's manifolds and tensors: *space-time is curved by matter*, as prescribed by Einstein's equations (in terms of the Riemann curvature tensor).
(8) Finally, an embracing reflection by Eugene Wigner, which can be extended to the whole development of the concept of space, is that of *"the unreasonable effectiveness of Mathematics in the Natural Sciences"*.

4.1.2. *On the topology and curvature of our universe.* Let us now connect, briefly, these philosophical ideas about space with recent precise determinations of the topology and geometrical curvature of the universe we live in—what can be considered as the modern version of the pioneering attempts by Gauss, already mentioned, to determine its possible curvature.

The Friedmann-Robertson-Walker (FRW) model, which can be obtained as the *only* family of solutions to the Einstein's equations compatible with the assumptions of *homogeneity* and *isotropy* of space, is the generally accepted model of the cosmos nowadays. But, as the reader surely knows, the FRW is a family with a free parameter, k, the curvature, that can be either positive, negative or zero (the flat or Euclidean case). This curvature, or equivalently the curvature radius, R, is not fixed by the theory and should be matched with cosmological observations. Moreover, the FRW model, and Einstein's equations themselves, can only provide local properties, not global ones, so they cannot tell about the overall topology of our world: is it closed or open? is it finite or infinite? Those questions are very appealing to any human being. All this discussion will only concern three dimensional space curvature and topology, time not being for the moment involved.

Serious attempts to measure the possible curvature of the space we live in go back to Gauss, who measured the sum of the three angles of a big triangle with vertices on the picks of three far away mountains (Brocken, Inselberg, and Hohenhagen). He was looking for evidence that the geometry of space is non-Euclidean. The idea was brilliant, but condemned to failure: one needs a much bigger triangle to try to find the possible non-zero curvature of space. Now cosmologist have recently measured the curvature radius R by using the largest triangle available, namely one with us at one vertex and with the other two on the hot opaque surface of the ionized hydrogen that delimits our visible universe and emits the cosmic microwave background radiation (CMB, some 3 to 4 $\times 10^5$ years after the Big Bang) [8]. The CMB maps exhibit hot and cold spots. It can be shown that the characteristic spot angular size corresponds to the first peak of the temperature power spectrum, which is reached for an angular size of $.5^o$ (approximately the one subtended by the Moon) if space is flat. If it has a positive curvature, spots should be larger (with a corresponding displacement of the position of the peak), and correspondingly smaller for negative curvature.

The joint analysis of the considerable amount of data obtained during the last years by balloon experiments (BOOMERanG, MAXIMA, DASI), combined also with galaxy clustering data, have produced a lower bound for $|R| > 20h^{-1}$Gpc, that is, twice as large as the radius of the observable universe, of about $R_U \simeq 9h^{-1}$Gpc.

General Relativity does not prescribe the topology of the universe, or its being finite or not. The universe could perfectly be flat and finite. The simplest non-trivial model from the theoretical viewpoint is the toroidal topology (that of a tyre or a donut, but in one dimension more). Traces for the toroidal topology (and more elaborated ones, as negatively curved but compact spaces) have been profusely investigated, and some circles in the sky with near identical temperature patterns were identified [9]. And yet more papers appear, from time to time, proposing a new topology [10]. However, to summarize all these efforts and the observational situation, and once the numerical data are interpreted without bias (what sometimes was not the case, and led to erroneous conclusions), it seems at present that available data still point towards a very large (we may call it *infinite*) flat space.

4.2. **On zeta-function regularization and its uses in quantum field theory.** The fact that the infinite series

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

has the sum $s = 1$ is nowadays clear to any school student. It was not so, even to well learned persons, for many centuries, as we can recall from *Zeno of Elea's paradox* (or Zeno's paradox of the tortoise and Achilles), transmitted by Aristotle and based on the pretended impossibility to do an *infinite* number of summations (or recurrent 'jumps' or steps of any kind, in a *finite* amount of time). In fact there are still modern versions of the Zeno paradox (e.g. the *quantum Zeno paradox*) which pop up now and then [11, 12].

In a modern version of this paradox, extrapolated to its more far reaching consequences, Krauss and Dent affirm, in a recent paper appeared in *The New Scientist* that, incredible as it may seem, our detection of dark energy could have reduced the life-expectancy of the universe (!). This idea is a worrying new variant of the *quantum Zeno paradox*, as these cosmologists claim that astronomers may have accidentally nudged the universe closer to its death by observing dark energy, the anti-gravity force which is thought to be accelerating the expansion of the cosmos. These allegations suggest that by making this observation in 1998 we may have caused the cosmos to revert to an earlier state when it was more likely to end. Krauss and Dent came to such astonishing conclusion by calculating how the energy state of our universe—a summation of all its particles and all their energies—has evolved since the big bang of creation some 13.7 billion years ago.

The *quantum Zeno effect* is a well known phenomenon in quantum physics, with sufficiently precise experimental proofs. It says that, whenever we observe or measure a quantum system repeatedly, we make its evolution slower and slower, until it could stop decaying. That is, if an observer makes repeated, quick observations of a microscopic object undergoing change, the object can stop changing (just as, according to common lore, a watched kettle never boils).

A couple of months ago, under the request of some journalists, I was asked to report on this issue in the scientific sections of a couple of Spanish newspapers. What I said, in short, is that even being the quantum Zeno effect a widely accepted phenomenon, the extrapolation made by Krauss and Dent is far from clear. Actually, I was able to find some loopholes in the mathematical derivation, what points out to the conclusion that, even in the best of cases, the computed result wold be many orders of magnitude smaller than the one reported and, therefore, negligible.

I shall not discuss on this philosophical point here any further, but rather concentrate on the beautiful mathematics behind the Zeno paradox. Let us continue with the very simple example above. It is quite clear that, by taking the first term, 1/2, to the left what remains on the r.h.s. is just one half of the original series (extracting 1/2 as a common factor), so that

$$s - \frac{1}{2} = \frac{s}{2} \quad \Longrightarrow \quad s = 1.$$

Thus the conclusion follows that when to one half of an apple pie we add a quarter of it and then an eighth, and so on, what we get in the end is the whole pie.

Now, something more difficult: what is the sum of the following series?

$$s = 1 + 1 + 1 + \cdots + 1 + \cdots$$

Again, any of us will answer immediately: $s = \infty$. In fact, whatever ∞ is, everybody recognizes in this last expression the definition itself of the concept of infinity, e.g. the piling of one and the same object, once and again, without an end. Of course, this idea is absolutely true, but it is at the same time of little use to modern Physics. To be more precise, since the advent of Quantum Field Theory (QFT). In fact, calculations there are plagued with divergent series, and it is of no use to say that: look, this series here is divergent, and this other one is also divergent, and the other there too, and so on. One gets non-false but also non-useful information in this way, and actually we *do not observe* these many infinities in Nature. Thus it was discovered in the 30's and 40's that something very important was missing from the formulation or mathematical modelization of quantum physical processes.

Within the mathematical community, for years there was the suspicion that one could indeed give sense to divergent series. This has now been proven experimentally (with 10^{-14} accuracy in some cases) to be true in physics, but many years earlier mathematicians were the first to realise that it was possible. In fact, Leonard Euler (1707-1783) was convinced that *"To every series one could assign a number"* [13] (that is, in a reasonable, consistent, and possibly useful way, of course). Euler was unable to prove this statement in full, but he devised a technique (Euler's summation criterion) in order to 'sum' a large family of divergent series. His statement was however controverted by some other great mathematicians, as Abel, who said that *"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever"*. [14] There is a classical treatise due to G.H. Hardy and entitled simply *Divergent Series* [15] that can be highly recommended to the reader.

Actually, regularization and renormalization procedures are essential in present day Physics. Among the different techniques at hand in order to implement these processes, zeta function regularization is one of the most beautiful. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system, which could, e.g., contribute to the cosmic force leading to the present acceleration of the expansion of our universe. The zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically (from a mathematical point of view) to the computation of determinants of elliptic pseudodifferential operators (ΨDOs) [16]. It is thus no surprise that the preferred definition of determinant for such operators is obtained through the corresponding zeta function (see, e.g., [17, 18]).

4.2.1. *The zeta function as a summation method.* The method of zeta regularization evolved from the consideration of the Riemann zeta function as a 'series summation method'. The zeta function, on its turn, was actually introduced by Euler, from considerations of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots ,$$

which is logarithmically divergent, and of the fact that, putting a real exponent s over each term,

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} + \cdots ,$$

then for $s > 1$ the series is convergent, while for $s \leq 1$ it is divergent. Euler called this expression, as a function of s, the $\zeta-$function, $\zeta(s)$, and found the following important relation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\,\mathrm{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} ,$$

which is crucial for the applications of this function in Number Theory. By allowing the variable s to be complex, Riemann saw the relevance of this function (that now bears his name) for the proof of the prime number theorem[3], and formulated thereby the *Riemann hypothesis*, which is one of the most important problems (if not *the* most important one) in the history of Mathematics. More of that in the excellent review by Gelbart and Miller [19].

For the Riemann $\zeta(s)$, the corresponding complex series converges absolutely on the open half of the complex plane to the right of the

[3]Which states that the number $\Pi(x)$ of primes which are less than or equal to a given natural number x behaves as $x/\log x$, as $x \to \infty$. It was finally proven, using Riemann's work, by Hadamard and de la Vallé-Poussin.

abscissa of convergence Re $s = 1$, while it diverges on the other side, but it turns out that it can be analytically continued to that part of the plane, being then everywhere analytic and finite except for the only, simple pole at $s = 1$ (Fig. 1).[4] In more general cases, namely

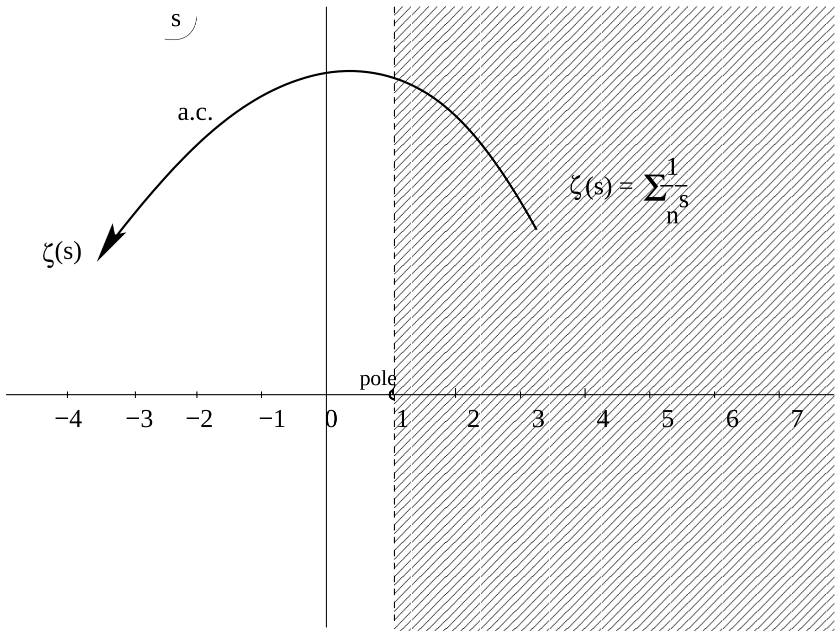

FIGURE 1. The zeta function $\zeta(s)$ is defined in the following way, on the whole complex plane, $s \in \mathbf{C}$. To start, on the open half of the complex plane which is on the r.h.s of the abscissa of convergence $\mathrm{Re}\, s = 1$, ζ is defined as the absolutely convergent series: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. In the rest of the $s-$complex plane, $\zeta(s)$ is defined as the (unique) analytic continuation of the preceding function, which turns out to be meromorphic. Specifically, it is analytic everywhere on the complex plane except for one simple pole with residue equal to 1, which is at the point $s = 1$ (notice that it corresponds to the logarithmically divergent harmonic series, as already discussed).

corresponding to the Hamiltonians which are relevant in physical applications, [20, 21, 22] the situation is in essence quite similar, albeit in

[4]Where it yields the harmonic series: there is no way out for this divergence.

practice it can be rather more involved. A mathematical theorem exists, which assures that under very general conditions the zeta function corresponding to a Hamiltonian operator will be also meromorphic, with just a discrete number of possible poles, which are usually simple and extend to the negative side of the real axis.[5]

The above picture already hints towards the use of the zeta function as a summation method. Let us consider two examples.

(1) We interpret our starting series

$$s_1 = 1 + 1 + 1 + \cdots + 1 + \cdots$$

as a particular case of the Riemann zeta function, e.g. for the value $s = 0$. This value is on the left hand side of the abscissa of convergence (Fig. 1), where the series as such diverges but where the analytic continuation of the zeta function provides a perfectly finite value:

$$s_1 = \zeta(0) = -\frac{1}{2}.$$

So this is the value to be attributed to the series $1+1+1+1+\cdots$.

(2) The series

$$s_2 = 1 + 2 + 3 + 4 + \cdots + n + \cdots$$

corresponds to the exponent $s = -1$, so that

$$s_2 = \zeta(-1) = -\frac{1}{12}.$$

A couple of comments are in order.

(1) In two following years, some time ago, two distinguished physicists, A. Slavnov from Moscow and F. Yndurain from Mdrid, gave seminars in Barcelona, about different subjects. It was quite remarkable that, in both presentations, at some point the speaker addressed the audience with these (or equivalent) words: *"As everybody knows,* $1 + 1 + 1 + \cdots = -1/2$*"*.[6]

[5]Although there are some exceptions to this general behavior, they correspond to rather twisted situations, and are outside the scope of this brief presentation.

[6]Implying maybe: *If you do not know this it will be no use for you to continue listening.* Remember by the way the *lemma* of the Pythagorean school: *Do not cross this door if you do not know Geometry.*

(2) That positive series, as the ones above, can yield a *negative* result may seem utterly nonsensical. However, it turns out that the most precise experiments ever carried out in Physics do confirm such results. More precisely: models of regularization in QED built upon these techniques lead to final numbers which are in agreement with the experimental values up to the 14th figure [23]. In recent experimental proofs of the Casimir effect [24] the agreement is also quite remarkable (given the difficulties of the experimental setup) [25].

(3) The method of zeta regularization is based on the analytic continuation of the zeta function in the complex plane. Now, how easy is to perform that continuation? Will we need to undertake a fashionable complex-plane computation every time? It turns out that this is not so. The result is immediate to obtain, in principle, once you know the appropriate functional equation (or reflection formula) that your zeta function obeys: in the case of the Riemann zeta $\xi(s) = \xi(1-s), \quad \xi(s) \equiv \pi^{-s/2}\Gamma(s/2)\zeta(s)$. In practice these formulas are however not optimal for actual calculations, since they are ordinarily given in terms of power series expansions (as the Riemann zeta itself), which are very slowly convergent near the corresponding abscissa. Fortunately, sometimes there are more clever expressions, that can be found, which converge exponentially fast, as the celebrated Chowla-Selberg [26] formula and some others [27, 28]. Those formulas are an speciality of the author, and give enormous power to the method of zeta function regularization.

4.2.2. *Zeta regularization in physics.* As advanced already, the regularization and renormalization procedures are essential issues of contemporary physics —without which it would simply not exist, at least in the form we now know it. [29] Among the different methods, zeta function regularization—which is obtained by analytic continuation in the complex plane of the zeta function of the relevant physical operator in each case—is maybe the most beautiful of all. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of any kind, in principle). Assume the corresponding Hamiltonian operator, H, has a spectral decomposition of the form (think, as simplest case, in a quantum harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i\in I}$, where I is a set of indices (which can be discrete, continuous, mixed, multiple, ...). Then, the quantum vacuum energy is obtained as follows [20]:

$$E/\mu = \sum_{i\in I} \langle \varphi_i, (H/\mu)\varphi_i \rangle = \mathrm{Tr}_\zeta H/\mu = \sum_{i\in I} \lambda_i/\mu$$
$$= \sum_{i\in I} (\lambda_i/\mu)^{-s}\bigg|_{s=-1} = \zeta_{H/\mu}(-1),$$

where ζ_A is the zeta function corresponding to the operator A, and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class).[7] Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation, inherent with the definition of the zeta function itself. Also, the unavoidable regularization parameter with dimensions of mass, μ, appears in the process, in order to render the eigenvalues of the resulting operator dimensionless, so that the corresponding zeta function can indeed be defined. We shall not discuss these important details here, which are just at the starting point of the whole renormalization procedure. The mathematically simple-looking relations above involve very deep physical concepts (no wonder that understanding them took several decades in the recent history of quantum field theory).

4.2.3. *The Casimir energy.* In fact things do not turn out to be so simple. One cannot assign a meaning to the *absolute* value of the zero-point energy, and any physical effect is an energy difference between two situations, such as a quantum field in curved space as compared with the same field in flat space, or one satisfying BCs on some surface as compared with the same in its absence, etc. This difference is the Casimir energy: $E_C = E_0^{BC} - E_0 = \frac{1}{2}\left(\mathrm{tr}\ H^{BC} - \mathrm{tr}\ H\right)$.

But here a problem appears. Imposing mathematical boundary conditions (BCs) on physical quantum fields turns out to be a highly non-trivial act. This was discussed in much detail in a paper by Deutsch

[7] The reader should be warned that this $\zeta-$trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained by the author [30]. Some colleagues are unaware of this fact, which has lead to important mistakes and erroneous conclusions too often.

and Candelas a quarter of a century ago [31]. These authors quantized em and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress-energy tensor, to find out that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite *physical* energy was obtained if the mathematical BCs were to be fulfilled. However, the authors argued that surfaces have non-zero depth, and its value could be taken as a handy (dimensional) cutoff in order to regularize the infinities. Just two years after Deutsch and Candelas' work, Kurt Symanzik carried out a rigorous analysis of QFT in the presence of boundaries [32]. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure and point splitting to be applied. In any case, the issue was proven to be perfectly meaningful within the domains of renormalized QFT. In this case the BCs and the hypersurfaces themselves were treated at a pure mathematical level (zero depth) by using (Dirac) delta functions.

Recently, a new approach to the problem has been postulated [33]. BCs on a field, ϕ, are enforced on a surface, S, by introducing a scalar potential, σ, of Gaussian shape living on and near the surface. When the Gaussian becomes a delta function, the BCs (Dirichlet here) are enforced: the delta-shaped potential kills *all* the modes of ϕ at the surface. For the rest, the quantum system undergoes a full-fledged QFT renormalization, as in the case of Symanzik's approach. The results obtained confirm those of [31] in the several models studied albeit they do not seem to agree with those of [32]. They are also in clear contradiction with the ones quoted in the usual textbooks and review articles dealing with the Casimir effect [34], where no infinite energy density when approaching the Casimir plates has been reported. This issue is also of importance at the cosmological level, in braneworld models.

4.3. **Present day cosmology from modified theories of gravity.**

4.3.1. *Uses of the Riemann tensor in cosmology.* As was mentioned before, Riemann's revolutionary ideas about the concept of physical space where given a definite form by Albert Einstein when he formulated the *Theory of General Relativity*, with the help of his, more mathematically minded, classmate and friend Marcel Grossmann [8]. The community of relativists celebrates Grossmann's contributions to physics by organizing the very important Marcel Grossman meetings, every three years (MG12 will take place in Paris, in 2009). Let us summarize the main points of the so called *"curved-space-time physics"* (excellent references are the books by Robert Wald [35]):

(1) Space-time, the set of all events, is a four-dimensional manifold endowed with a metric (M, g).
(2) The metric is physically measurable by rods and clocks.
(3) The metric of space-time can be put in the Lorentz form momentarily at any particular event by an appropriate choice of coordinates.
(4) Freely-falling particles, unaffected by other forces, move on time-like geodesics of the space-time.
(5) Any physical law that can be expressed in tensor notation in special relativity has exactly the same form in a locally-inertial frame of a curved space-time.

We cannot go into much detail in the standard theory of General Relativity, since we here aim at putting our emphasis on the very recent developments concerning its application to modern cosmology. Let us just recall Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu},$$

where on the lhs we have the curvature, the geometry of space-time, under the form of contractions of the Riemann curvature tensor:

$$R_{\mu\nu\rho}{}^{\sigma} = \Gamma^{\sigma}_{\mu\rho,\nu} - \Gamma^{\sigma}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\sigma}_{\alpha\mu},$$

[8]Who later became a Professor of Mathematics at the Federal Polytechnic Institute in Zurich, today the ETH Zurich, specializing in descriptive geometry.

the Γ's being, as usual, Christoffel symbols of the Riemannian connection, and

$$R_{\mu\rho} = R_{\mu\sigma\rho}{}^{\sigma}, \qquad R = R^{\mu}_{\mu}.$$

Einstein observed that the solution of these equations, subject to the constraints of the cosmological principle, led to a universe that was not static. He was disappointed because at that time (1915-20) the expansion of the Universe had not yet been discovered (Hubble, 1925-30) and the universe was considered by everybody to be in a stationary state. This led Einstein to introduce (almost against his actual will) a constant term in his equations (known now as the cosmological constant, Λ), that was perfectly compatible with all of the principles of his gravity theory (but otherwise unnecessary):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}.$$

When a few years later Hubble discovered that the universe was in fact expanding, Einstein said the introduction of the cosmological constant had been the greatest blunder of his life. He was right to be upset since, to get a static Universe, he had added an artificial term to his field equations that stabilized the Universe against expansion or contraction. Had he possessed sufficient confidence in his original equations, he could have predicted that either his theory was wrong or the Universe was expanding or contracting, well before there was any experimental evidence of the expansion![9]

An important historical issue (also for what will follow) was the derivation of Einstein's equations from a variational principle, starting from what is now called the Einstein-Hilbert action[10]:

$$S = \int d^4x \, \sqrt{-g} \, \left(L_G + L_m - \lambda \right),$$

where $\lambda = \Lambda/8\pi G$. Here the first two terms within the brackets are the Lagrangians corresponding to gravity and matter, and the last one is the cosmological constant term. By variation in the Euler-Lagrange

[9]What would have been an enormous accomplishment. This explains why Einstein got so angry.

[10]In fact Hilbert preceded Einstein by one day in the submission of his results for publication, in 1915.

sense, one obtains

$$\delta S_m = \int d^4x \ \sqrt{-g} \ \left(\frac{\partial L_m}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} L_m \right) \delta g^{\mu\nu},$$

$$T_{\mu\nu} = -2 \frac{\partial L_m}{\partial g^{\mu\nu}} + g_{\mu\nu} L_m,$$

wherefrom Einstein's equations follow.

4.3.2. *Cosmological constant and the quantum vacuum energy.* However, this was not the end of the story. Any attempt at a unification of all fundamental interactions—already envisaged by Riemann and to which Einstein devoted an important part of his entire life and scientific effort—that is, a physical theory describing the gravitational interactions of matter and energy in which matter and energy are described by quantum theory, has failed. In most theories that aim at doing this, gravity itself is quantized. Since the contemporary theory of gravity, general relativity, describes gravitation as the curvature of space-time by matter and energy, a quantization of gravity implies some sort of quantization of space-time itself. As all existing physical theories rely on a classical space-time background, this presents profound methodological and ontological challenges, in fact it is considered to be maybe the most difficult problem in physics. However, new theories must always contain the successful previous ones, that have proven already to be perfectly valid in their corresponding domains of applicability.

Thus, special relativity reduces to classical Newtonian mechanics when the velocities v involved are $v \ll c$, and corrections to the classical formulas start with terms of the form v/c and higher powers (post-Newtonian, post-post-Newtonian approaches, etc.). In this sense, some successful semi-classical approaches to quantum gravity have been constructed. Summing up, even if we do not have a quantum theory of gravity, it is by now clear that the quantum correction to the Einstein equations corresponding to the fluctuations of the quantum vacuum will show out as an additional term in the energy-momentum tensor $T_{\mu\nu}$, side by side with Einstein's cosmological constant contribution, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T_{\mu\nu} - E g_{\mu\nu} - \lambda g_{\mu\nu}),$$

where E denotes this vacuum energy density (and remember $\lambda = \Lambda/8\pi G$). More precisely, the combination of this two terms (including all fundamental physical constants) reads

$$\frac{\Lambda\, c^2}{8\pi G} \;+\; \frac{1}{\mathrm{Vol}} \frac{\hbar\, c}{2} \sum_i \omega_i,$$

ω_i being the energy modes (spectrum) of the Hamiltonian operator of the quantum theory. This fact will remain true in any quantum theory of gravitation, as far as vacuum fluctuations behave as an ordinary form of energy (e.g., they satisfy the equivalence principle), what seems indeed to be the case [36].

The dramatic consequence of this issue (already pointed out by Zel'dovich in the sixties) is that we cannot get rid any more of the cosmological constant as Einstein finally did. It will pop up, under this new form, as fluctuations of the quantum vacuum, that are allowed by the fundamental Heisenberg's uncertainty principle (unless, of course, all quantum vacuum fluctuations add up to zero, which is very difficult to realize; this is known as *the cosmological constant problem*).

4.3.3. *Cosmic acceleration.* Astrophysical observations clearly indicate that huge amounts of 'dark matter' and 'dark energy' are needed to explain the observed large scale structures and cosmic accelerating expansion of our universe. Up to now, no experimental evidence has been found, at the fundamental level, to convincingly explain such weird components. In particular, concerning the problem of the accelerating expansion, the only possibility to solve it within the domains of Einsteinian gravity is, again, through the cosmological constant term, that with the convenient sign provides the contribution needed to produce the observed acceleration (very similar to the way how Einstein tried to stabilize the universe against gravitational collapse, when he though it should be static). However, this is not easy to do. First, when computed with care, the contribution of the vacuum energy density is many orders of magnitude *larger* than the value needed to explain the small acceleration rate of the universe expansion[11] (what is called the 'new' problem of the cosmological constant, which is even worse

[11] It is of the order of 10^{123}, one of the largest discrepancies between theory and observation in the history of Physics.

than the older one). Second, it is not even clear (very specific models must be involved) whether the *sign* of the contribution of the vacuum fluctuations is the correct one in order to obtain expansion (and not contraction!). Making the story short, there are models where these two problems could be understood, but always with the help of some tailored hypothesis, and the general consensus is that the problem is far from having been solved yet.

This has led to consider completely different approaches (see, for instance, [37]). One of the most successful is the so-called $f(R)$ gravity, which is a deviation from Einstein's General Relativity in the way we are going to see (note that the R stands here again for Riemann: the Riemann tensor contraction). This is an alternative theory of gravity in which dark energy and dark matter could be effects—illusions, in a sense—created by the curvature of space-time (the same bending of space and time as in General Relativity, caused by extremely massive objects, like galaxies, but now a bit modified). This theory does not require the existence of dark energy and dark matter. The problem then could be completely reversed considering dark matter and dark energy as 'shortcomings' of General Relativity and claiming for a more 'correct' theory of gravity as derived phenomenologically by matching the largest number of observational data available. As a result, accelerating behavior of cosmic fluid and rotation curves of spiral galaxies have been reported to be reproduceable by means of 'curvature effects' [38].

4.3.4. $f(R)$ *gravity.* Modified gravity models constitute an interesting dynamical alternative to the ΛCDM cosmology—which is the standard approach nowadays—in that they are able to describe with success, and in a rather natural way, the current acceleration in the expansion of our Universe, the so called dark energy epoch (and even perhaps the initial de Sitter phase and inflation). As the name itself indicates, the modification in the action of the modified gravitational models consists of changing the R contribution by adding to it a term which is a (in principle arbitrary) function of R only. It thus reads

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g}\,[R + f(R)] + S_{(m)}.$$

Usually, one calls $F(R) = R + f(R)$ and sometimes the theory itself is named $F(R)$ theory (those are very recent concepts, and nomenclature is not yet completely fixed). The general equation of motion in $F(R) \equiv R + f(R)$ gravity with matter is obtained as

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{(m)\mu\nu},$$

where $T_{(m)\mu\nu}$ is the matter energy-momentum tensor.

Modified $f(R)$ gravity has undergone already a number of studies which conclude that this gravitational alternative to dark energy is able to pass the solar system tests, that is, the very severe constraints imposed by the observational proofs that Einstein's gravity (with R only) is able to describe to extremely high precision the evolution of our solar system. Recently the importance of those modified gravity models has been reassessed, namely with the appearance of the so-called 'viable' $f(R)$ models [39, 40]. Those are models which satisfy the stringent cosmological as well as the local gravity constraints, which had caused a number of serious problems to some of the first-generation theories of that kind. The final aim of all these phenomenological models is to describe a segment as large as possible of the whole history of our universe, as well as to recover all local predictions of Einsteinian gravity, which have been verified experimentally to very good accuracy, at the solar system scale.

In this last couple of years, we have investigated for two classes of 'viable' modified gravitational models what it means, roughly speaking, that they incorporate the vanishing (or fast decrease) of the cosmological constant in the flat ($R \to 0$) limit, and that they exhibit a suitable constant asymptotic behavior for large values of R. A huge family of these models, which we term first class—and to which most of the models proposed in the literature belong—can be viewed as containing all possible smooth versions of the following sharp step-function model. To discuss this toy model, at the distribution level, proves to be very useful in order to grasp the essential features that *all* models in this large family are bound to satisfy. In other words, to extract the general properties of the whole family in a rather simple fashion (which involves, of course, standard distribution calculus). This simple model (representative of the whole class) reads

$$f(R) = -2\Lambda_{\text{eff}}\,\theta(R - R_0),$$

where $\theta(R - R_0)$ is Heaviside's step distribution. Models in this class are characterized by the existence of one or more transition scalar curvatures, an example being R_0 in the above toy model.

The other class of modified gravitational models that has been considered contains a sort of 'switching on' of the cosmological constant as a function of the scalar curvature R. A simplest version of this kind reads

$$f(R) = 2\Lambda_{\text{eff}}(e^{-bR} - 1).$$

Here the transition is smooth. The two models above may be combined in a natural way, if one is also interested in the phenomenological description of the inflationary epoch. For example, a two-step model may be the smooth version of

$$f(R) = -2\Lambda_0\,\theta(R - R_0)\; - 2\Lambda_I\,\theta(R - R_I)\,,$$

with $R_0 \ll R_I$, the latter being the inflation scale curvature.

In a recent paper [41], we have developed a general approach to viable modified gravity in both the Jordan and the Einstein frames. We have focussed on the so-called step-class models mainly, since they seem to be most promising from the phenomenological viewpoint and, at the same time, they provide a natural possibility to classify all viable modified gravities. We have explicitly presented the cases of one- and two-step models, but a similar analysis can be extended to the case of an N-step model, with N being finite or countably infinite. No additional problems are expected to appear and the models can be adjusted, provided one can always find smooth solutions interpolating between the de Sitter solutions (what seems at this point a reasonable possibility), to repeat at each stage the same kind of de Sitter transition. We can thus obtain multi-step models which may lead to multiple inflation and multiple acceleration, in a way clearly reminiscent of braneworld inflation.

This looks quite promising, with the added bonus that the model's construction is rather simple. Use has been made of the simple but efficient tools provided by the corresponding toy model constructed with sharp distributions, a new technique that we have pioneered. It is to be remarked that, for the infinite-step models, one can naturally expect to construct the classical gravity analog of the string-theory landscape realizations, as in the classical ideal fluid model.

The existence of viable (or "chameleon") $f(R)$ theories with a phase of early-time inflation is not known to us from the literature. The fact that we are able to provide several classes of models of this kind that are consistent also with the late-time accelerated expansion is thus a novelty, worth to be remarked.

Both inflation in the early universe and the recent accelerated expansion could be thus understood in these theories in a unified way. If we start with large curvature, $f(R)$ becomes almost constant and plays the role of the effective cosmological constant, which would generate inflation. For a successful exit from the inflationary epoch we may need, in the end, more (say small non-local or small R^n) terms. When curvature becomes smaller, matter could dominate, what would indeed lower the curvature values. Then, when the curvature R becomes small enough and $R_0 \ll R \ll R_I$, $f(R)$ becomes again an almost constant function, and plays the role of the small cosmological constant which generates the accelerated expansion of the universe, that started in the recent past. Moreover, the model naturally passes all local tests and can be considered as a true viable alternative to General Relativity. Some remark is however in order. On general grounds, one is dealing here with a highly non-linear system and one should investigate all possible critical points thereof (including other time-dependent cosmologies), within the dynamical approach method. Of course, the existence of other critical points is possible; anyhow, for viable $f(R)$ models, to find them is not a simple task, and in Ref. [41] we have restricted our effort to the investigation of the de Sitter critical points. With regard to the stability of these points, the one associated with inflation should be unstable. In this way, the exit from inflation (what is always a very non-trivial issue) could be achieved in a quite natural way. In particular, for instance, this is in fact the case for the two-step model with an R^3 term.

In conclusion, we are on the way to construct realistic modified gravities. Some of these models ultimately lead to the unification of the inflationary epoch with the late-time accelerating epoch, under quite simple and rather natural conditions. What remains to be done is to study those models in further quantitative detail, by comparing their predictions with the accurate astrophysical data coming from ongoing and proposed sky observations. It is expected that this can be done rather soon, having in mind the possibility to slightly modify the early universe features of the theories here discussed, while still preserving all of their nice universal properties.

4.4. **Epilogue.** Let us finish this short overview of Riemann's work and its uses in modern Physics—a clear example of the very fruitful relation between the worlds of Physics and Mathematics—with an extremely touching sentence that appears in a letter written by Albert Einstein and addressed to Arnold Sommerfeld, in the year 1912—this means, some 60 years after the celebrated *Habilitationschrifft* of Bernhard Riemann—where Einstein comments on the efforts he is doing in trying to understand *Riemannian Geometry*:

"Aber eines ist sicher, dass ich mich im Leben noch nicht annähend so geplagt habe und dass ich große Hochachtung vor der Mathematik eingeflößt bekommen habe, die ich bis jetzt in ihren subtileren Teilen in meiner Einfalt für puren Luxus gehalten habe!"

What means, in a free English translation: *"But one thing is sure, that never before in my life had I invested such an effort, and that I never had had such a high opinion of Mathematics, which I considered till very recently, in my boldness, and for what respects its most subtle parts, as a mere luxury!"*

I, ja per acabar, en català: *"Però una cosa és segura, que mai en la meva vida no m'havia afanyat ni de bon troç com ara, i que mai no havia dispensat tan alta consideració a la Matemàtica, la qual tenia fins fa poc, en la meva ingenuitat, pel que fa a les seves parts més subtils, per un simple luxe!"*

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References

[1] D.R. Wilkins, http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Papers.html

[2] M. Monastyrsky, *Riemann, Topology, and Physics*, 2nd ed. (Springer, NY, 1999).

[3] C. Reid, *Hilbert* (Springer, Berlin, 1970) [Copernicus, NY, 1996].

[4] D. Laugwitz, *Bernhard Riemann, 1826-1866: Turning Points in the Conception of Mathematics* (Birkhäuser, Boston, 1999).

[5] http://arxiv.org/PS_cache/physics/pdf/0405/0405110.pdf, http://www.math.ucla.edu/~vsv/papers/ch2.pdf, http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/WKCGeom.html.

[6] http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html

[7] H. Farkas and I. Kra, *Riemann Surfaces* (Springer, Berlin-Heidelberg-New York, 1980); 2nd Ed. (1992).

[8] M. Tegmark, *Measuring spacetime: from Big Bang to Black Holes*; Science, **296**, 1427 (2002).

[9] N.J. Cornish et al., Phys. Rev. Lett. **92**, 201302 (2004).

[10] J.-P. Luminet et al., Nature, **425**, 593 (2003).

[11] B. Misra and E.C.G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977); A. P. Balachandran and S. M. Roy, *Phys. Rev. Lett.* **84**, 4019 (2000).

[12] L. Krauss and J. Dent, *The late time behavior of false vacuum decay: possible implications for cosmology and metastable inflating states*, arXiv:0711.1821 [hep-ph]; see also New Scientist, 21 November 2007.

[13] H. W. Turnbull, *The Great Mathematicians* (New York University Press, NY, 1961); E. J. Alton, *Encyclopedia of World Biography* (McGraw-Hill, NY, 1973) vol. 4, pp. 30-31.

[14] E. Maor, *To Infinity and Beyond: a Cultural History of the Infinite* (Princeton University Press, 1991).

[15] G. H. Hardy, *Divergent Series* (Oxford University Press,1949).

[16] P. Ramond, *Field Theory: a Modern Primer* (Addison-Wesley, Redwood City, 1989).

[17] E. Elizalde, J. Phys. **A22**, 931 (1989); E. Elizalde and A. Romeo, Phys. Rev. **D40**, 436 (1989); E. Elizalde, J. Math. Phys. **31**, 170 (1990).

[18] E. Elizalde, J. Phys. **A18**, 1637 (1985); E. Elizalde, Math. of Computation **47**, 175 (1986); E. Elizalde and A. Romeo, Int. J. Math. and Math. Sci. **13**, 453 (1990); D. Fried, Invent. Math. **84**, 523 (1984).

[19] S. S. Gelbart and S. D. Miller, *Bull. Amer. Math. Soc. (NS)* **41**, 59 (2004).

[20] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994); K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman & Hall, London, 2001); M. Bordag, E. Elizalde and K. Kirsten, J. Math. Phys. **37**, 895 (1996); J. Phys. **A31**, 1743 (1998); M. Bordag, E. Elizalde, K. Kirsten

and S. Leseduarte, Phys. Rev. **D56**, 4896 (1997); E. Elizalde, L. Vanzo and S. Zerbini,Commun. Math. Phys. **194**, 613 (1998); E. Elizalde, J. Comput. Appl. Math. **118**, 125 (2000); J. Phys. **A34**, 3025 (2001); E. Elizalde, M. Bordag and K. Kirsten, J. Phys. A **31**, 1743 (1998); E. Elizalde and K. Kirsten, J. Math. Phys. **35**, 1260 (1994); E. Elizalde and A. Romeo, J. Math. Phys. **30**, 1133 (1989).

[21] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions* (Springer-Verlag, Berlin, 1995).

[22] A.A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, *Analytic Aspects of Quantum Fields* (World Scientific, Singapore, 2004).

[23] R. S. Van Dyck, Jr. *Anomalous Magnetic Moment of Single Electrons and Positrons: Experiment*, in Quantum Electrodynamics (Ed. T. Kinoshita) (World Scientific, Singapore, 1990) pp. 322-388; T. Kinoshita, *Rept. Prof. Phys.* **59**, 1459 (1996).

[24] H.B.G. Casimir, Proc. K. Ned. Acad. Wet. **51**, 635 (1948).

[25] S. K. Lamoreaux, *Phys. Rev. Lett.* **78**, 5 (1997); U. Mohideen and A. Roy, *Phys. Rev. Lett.* **81**, 4549 (1998); B. W. Harris, F. Chen, and U. Mohideen, *Phys. Rev.* **A62**, 052109 (2000).

[26] S. Chowla and A. Selberg, Proc. Nat. Acad. Sci. U.S.A. **35**, 317 (1949).

[27] E. Elizalde, J. Phys. **A27**, 3775 (1994).

[28] E. Elizalde, Commun. Math. Phys. **198**, 83 (1998).

[29] J. Collins, *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion* (Cambridge University Press, 1984); W. D. McComb, *Renormalization Methods: A Guide for Beginners* (Oxford University Press, 2004).

[30] E. Elizalde, J. High Energy Phys. **9907**, 015 (1999).

[31] D. Deutsch and P. Candelas, Phys. Rev. **D20**, 3063 (1979).

[32] K. Symanzik, Nucl. Phys. **B190**, 1 (1981).

[33] R. L. Jaffe, *Unnatural acts: unphysical consequences of imposing boundary conditions on quantum fields*, hep-th/0307014; N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, *Casimir energies in light of quantum field theory*, hep-th/0207205; N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, Nucl. Phys. **B645**, 49 (2002); N. Graham, R. L. Jaffe and H. Weigel, Int. J. Mod. Phys. **A17**, 846 (2002).

[34] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and its Application* (Clarendon Press, Oxford, 1997); K.A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy* (World Scientific, Singapore, 2001); M. Bordag, U. Mohideen and V.M. Mostepanenko, Phys. Reports **353**, 1 (2001).

[35] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984); *Quantum Fields in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).

[36] S.A. Fulling et. al., *How Does Casimir Energy Fall?* arXiv:hep-th/070209.

[37] E. Elizalde, S. Nojiri and S.D. Odintsov, Phys. Rev. **D70**, 043539 (2004); E. Elizalde, S. Nojiri, S.D. Odintsov and P. Wang, Phys. Rev. **D71**, 103504 (2005).

[38] S. Capozziello, V.F. Cardone and A. Troisi, J. Cosmol. Astropart. Phys. **08**, 001 (2006).

[39] W. Hu and I. Sawicki, arXiv:0705.1158[astro-ph].

[40] S.A. Appleby and R.A. Battye, arXiv:0705.3199[astro-ph]; A. Starobinskii, JETP Lett.**86** 157 (2007).

[41] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. **D77**, 046009 (2008); arXiv:0712.4141[hep-th]; E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70** (2004) 043539 [arXiv:hep-th/0405034]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502** (2005) 010 [arXiv:hep-th/0501096]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D **73** (2006) 084007 [arXiv:hep-th/0601008]; E. Elizalde, S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D **67** (2003) 063515 [arXiv:hep-th/0209242]; E. Elizalde and J. Quiroga Hurtado, Mod. Phys. Lett. **A19**, 29 (2004) [GR-QC 0310128]; E. Elizalde, S. Naftulin, S.D. Odintsov, Phys. Rev. **D49**, 2852 (1994).