Riemann and Partial Differential Equations. A road to Geometry and Physics

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"Jornada Riemann", Barcelona, February 2008

Outline

1. **Mathematics, Physics and PDEs**
   - Origins of differential calculus
   - XVIII century
   - Modern times
2. **G. F. B. Riemann**
3. **Riemmann, complex variables and 2-D fluids**
4. **Riemmann and Geometry**
5. **Riemmann and the PDEs of Physics**
   - Picture gallery

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Differential Equations. The Origins

- The Differential World, i.e, the world of derivatives, was invented / discovered in the XVII century, almost at the same time that Modern Science (then called *Natural Philosophy)*, was born. We owe it to the great Founding Fathers, Galileo, Descartes, Leibnitz and Newton. Motivation came from the desire to understand Motion, Mechanics and Geometry.
- Newton formulated Mechanics in terms of ODEs, by concentrating on the movement of particles. The main magic formula is

$$m\frac{d^2\mathbf{x}}{dt^2} = F(t, \mathbf{x}, \frac{d\mathbf{x}}{dt})$$

  though he would write dots and not derivatives Leibnitz style.
- Newton thought about fluids, in fact he invented Newtonian fluids, and there you need dependence on space and time simultaneously, x as well as t. This means Partial Differential Equations (=PDEs). But his progress was really small if you compare with the rest. *We conclude that there was not much time for PDEs from from Big Bang to 1700 AD.*

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Origins of PDEs

- In the XVIII century, PDEs appear in the work of Jean Le Rond D'Alembert about string oscillations: there a *set of particles* moves together due to *elastic forces*, but every one of the infinitely many *solid elements* has a different motion, $u = u(x,t)$.

  This is one of the first instances of *continuous collective dynamics*. *PDEs are the mode of expression of such CCD.*

- Johann and Daniel Bernoulli and then Leonhard Euler lay the foundations of Ideal Fluid Mechanics (1730 to 1750), in Basel and StPetersburg. This is PDEs of the highest caliber: it is a system,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \qquad \nabla \cdot \mathbf{u} = 0.$$

  The system is nonlinear; it does not fit into one of the 3 types that we know today (elliptic, parabolic, hyperbolic); the main pure-mathematics problem is still unsolved (existence of classical solutions for good data; Clay Problems, year 2000).

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PDEs in the XIX Century

- The new century confronts revolutions in the concept of heat and energy, electricity and magnetism, and what is space. You may add a lesser revolution, real fluids.
- All of these fields end up mathematically in PDEs:

  (i) heat leads to the heat equation, $u_t = \Delta u$, and the merit goes to J. Fourier.

  (ii) electricity leads to the Coulomb equation in the Laplace-Poisson form: $-\Delta V = \rho$. This equation also represents gravitation!

  (iii) electromagnetic fields are represented by the Maxwell system. The vector potential satisfies a wave equation, the same as D'Alembert's, but it is vector valued and in several dimensions.

  (v) Real fluids are represented by the Navier Stokes equations. Sound waves follow wave equations, but they can create shocks (Riemann).

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PDEs cont. XIX Century

- (iv) Geometry was transformed from the Euclides tradition plus Cartesian Algebra to the spirit of PDEs by G. Gauss and B. Riemann. The spirit is condensed in a number of key words. Space is determined by its metric which is a local object which has tensor structure. The connection from point to point is a new object called covariant derivative, the curvature is a second order operator, a nonlinear relative of the Laplace operator.
- After these people, in particular Riemann, reality is mainly continuous and its essence lies in the physical law, that is a law about a field or a number of fields. In symbols, we have $\Phi(x, y, z, t)$ and its (system of laws)

$$L\Phi = \mathbf{F},$$

where F is the force field (a tensor).

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XXth century. Summing Up

- In the XXth century General Relativity and Quantum Mechanics take this form. *Space, matter and interactions become fields.*
- A main variant is Statistical Mechanics, a thread that leads to Brownian motion (Einstein, Smoluchowski), abstract probability (Kolmogorov, Wiener), stochastic calculus and stochastic differential equations (Itō).
- *The main (technical) task of the Mathematician working in Mathematical Physics is to understand the world of Partial Differential Equations, linear and nonlinear.*
  The same is true nowadays for geometers *(go to the CRM semester!).*
  The main abstract tool is Functional Analysis.
- The *combination* of Functional Analysis, PDES and ODEs, Geometry, Physics and Stochastic Calculus is one of the Great Machines of today's research, a child of the XXth century.

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Georg Friedrich Bernhard Riemann (1826-1866)

Reference to his life and work :

Wikipedia, MacTutor, Encyclopaedia Britannica and

- Detlef Laugwitz, Bernhard Riemann (1826-1866): Turning Points in the Conception of Mathematics, Birkhäuser (1999)
- M. Monastyrsky, Riemann, Topology, and Physics Birkhäuser, 2nd ed., 1999.

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Let us do Math! Complex Variables (Euler, Cauchy, Gauss, Riemann, Weierstrass)

- You start with a function

$$\boxed{u(x,y)=u(z)}, \qquad \boxed{z=x+iy=(x,y)}$$

that is supposed to be a good function of two real variables.

- good function of two real variables means (could mean) $u\in C^1(\Omega)$ for some Ω subdomain of $\mathbb{R}^2$.
- Therefore, it has a gradient, $\nabla u(z_0)=(u_x,u_y)$
- But, what is a good function of one complex variable?
- First of all, to keep the symmetry, there must be two real functions of two real variables:

$$\boxed{u=u(x,y), \quad v=v(x,y)}$$

which we write as $\boxed{f=f(z)}$ with $\boxed{f=u+iv,}$ and $\boxed{z=x+iy.}$

- The question is: Do we ask that $f\in C^1$ and that is all?

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- You start with a function

$$\boxed{u(x,y) = u(z)}, \qquad \boxed{z = x + iy = (x,y)}$$

  that is supposed to be a good function of two real variables.
- good function of two real variables means (could mean) $u \in C^1(\Omega)$ for some Ω subdomain of $\mathbb{R}^2$.
- Therefore, it has a gradient, $\nabla u(z_0) = (u_x, u_y)$
- But, what is a good function of one complex variable?
- First of all, to keep the symmetry, there must be two real functions of two real variables:

$$\boxed{u = u(x,y), \quad v = v(x,y)}$$

  which we write as $\boxed{f = f(z)}$ with $\boxed{f = u + iv,}$ and $\boxed{z = x + iy.}$
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Complex Variables II

- The answer is no and this is a consequence of algebra.
- Let us explain why: very nice real functions of one variable are polynomials, and
  very nice complex functions of one complex variable should also be polynomials.
- Now, polynomials are easy to define, for instance $f(z) = z^2$ means

$$u = x^2 - y^2, \qquad v = 2xy$$

- $f(z) = z^3$ means

$$u = x^3 - 3xy^2, \qquad v = 3x^2y - y^3.$$

- Can the reader do $f(z) = z^n$ by heart? Euler could! In fact, Euler and Moivre could see the whole trigonometry.
- Can you see something special in these pairs of functions, u and v ? Cauchy and Riemann could! They saw the whole theory of complex holomorphic functions.

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Complex Variables: the PDE code, called CRE

- What they saw is this hidden symmetry:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and they are called Cauchy-Riemann's equations for complex variables. They are one of the most important examples of a PDE system with extraordinary geometric and analytic consequences.

- We see that $\nabla u = (a, -b)$ is orthogonal to $\nabla v = (b, a)$. Consequence: the level lines $u = c_1$ and $c = c_2$ are orthogonal sets of curves.
- The linear algebra of infinitesimal calculus at every point is not 4-dimensional but two-dimensional. In fact, the system

$$du \sim adx - bdy, \quad dv \sim bdx + ady,$$

can be written together in the complex form $df \sim Jf(z)\, dz$, where Jf is the Jacobian matrix that we begin to call $f'(z) = a + ib$. *We will assume from such glorious moment on that this is the correct derivative of a 2-function of 2 variables that is a candidate to be a good complex differentiable function.*

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Complex Variables, analysis and geometry

- Hence, we know some magic formulas

$$f'(z) = a + bi = f_x, \quad f_y = -b + ai = if'(z).$$

hence $df = f'dx$ along the x-axis, $df = if'dy$ along the y axis. Good.

- Let us write the Jacobian

$$Jf(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aE + bJ.$$

This is an orthogonal matrix with determinant

$$det(Jf) = a^2 + b^2 = u_x^2 + u_y^2 = u_x^2 + v_x^2 = v_x^2 + v_y^2 = u_y^2 + v_y^2$$

which can be written as $|Jf| = |f'(z)|^2 = \|f_x\|^2 = \|f_y\|^2 = \|\nabla u\|^2 = \|\nabla u\|^2$. The infinitesimal transformation preserves the angles (of tangent curves) and amplifies the size by $Jf = |f'(z)|^2$.

- If the 2-2 function f is CR, then it defines a conformal transformation of the part of the plane where $f'(z) \neq 0$. Riemann's geometric theory of one CV is based on this idea. ♠

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Complex Variables and PDEs. Solving the equations. Potential theory

- How to find pairs of functions satisfying CR?
- Of course, real and imaginary parts of algebraic complex functions satisfy that. Also the Taylor Series of Weierstrass satisfies that.
- But PDE people want their way. Here is the wonderful trick:

$$\Delta u = u_{xx} + u_{yy} = v_{yx} + (-v_x)_y = 0.$$

  Idem $\Delta v = 0$. Solutions of this equation are harmonic functions, and they count amount the most beautiful C^∞ functions in analysis and the most important in physics, where solving $\Delta u = -\rho$ means finding the potential of ρ.
- Potential functions live in all dimensions . But in $d = 2$ they produce complex holomorphic functions. Given u we find v, its conjugate pair, by integration of the differential form

$$dv = Pdx + Qdy, \quad \text{with } P = v_x = -u_y, Q = v_y = u_x;$$

  this is an exact differential thanks to CR.

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Complex Variables and ideal fluids in $d = 2$

- We will recall that Riemann was a friend of Weber, the famous physicist.
- The velocity of a 2-D fluid is a field $\mathbf{v} = (v_1(x,y), v_2(x,y))$. Irrotational means that $\nabla \times \mathbf{v} = 0$. Incompressible means $\nabla \cdot \mathbf{v} = 0$. For the PDE person this is easy:

$$v_{2,x} - v_{1,y} = 0, \quad v_{1,x} + v_{2,y} = 0.$$

- Does it look like previous lesson? Yes, Combine both to get $\Delta v_1 = \Delta v_2 = 0$.
- Is v_2 harmonic conjugate to v_1? No, but in fact, $-v_2$ is.
- Idea to eliminate sign problems. Go to the scalar potential of the vector field $\mathbf{v}$:

$$d\Phi = v_1 dx + v_2 dy$$

(it is exact by irrotationality). Take the harmonic conjugate Ψ and define the complex potential of the flow as $F = \Phi + i\Psi$, a chf. In that case

$$F'(z) = F_x = \Phi_x + i\Psi_x = Phi_x - i\Phi_y = \overline{\mathbf{v}}.$$

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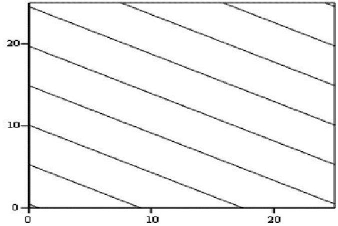
Some Pictures of 2D glory



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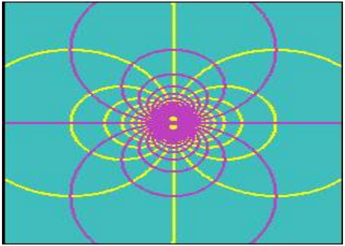
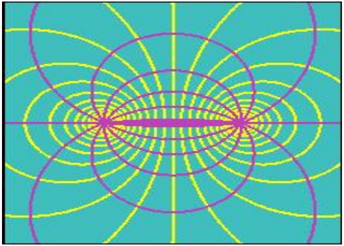
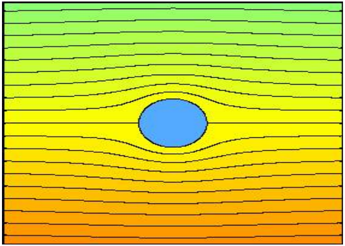
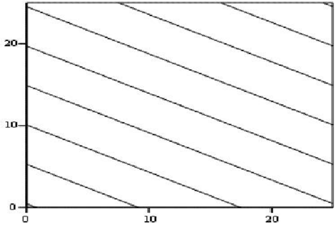


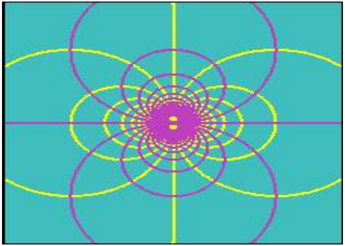


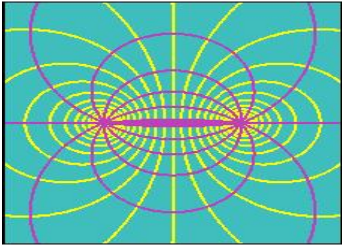


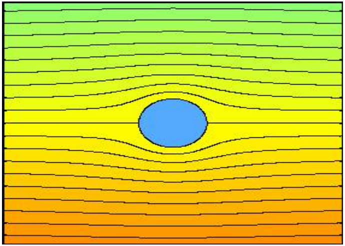
Some Pictures of 2D glory



ϕfreestream







Summary

THE BIG PICTURE IN 2D

- There is an equivalence between holomorphic complex variable theory $\Leftrightarrow$ conformal geometry $\Leftrightarrow$ harmonic functions $\Leftrightarrow$ ideal fluids.
- Any two dimensional, ideal fluid generates an analytic function and back, and it is a conformal mapping and back. ♠
- The complex derivative of the complex potential is just the conjugate of the velocity field.
- The stream function Ψ indicates the lines of current via $\Psi = c$.
- What happens when $F'(z) = 0$, i.e., when $\mathbf{v} = \mathbf{0}$?

These are singular points, called in physics the stagnation points. Many things can happen on a singularity, essentially one thing may happen on a regular point ($\Rightarrow$ the implicit function theorem). Riemann was an expert in singular points.

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Outline

1. Mathematics, Physics and PDEs
   - Origins of differential calculus
   - XVIII century
   - Modern times
2. G. F. B. Riemann
3. Riemmann, complex variables and 2-D fluids
4. **Riemmann and Geometry**
5. Riemmann and the PDEs of Physics
   - Picture gallery

From 2D to 3D

Riemann was able to understand very well the Two-Dimensional Space with its functions, analysis, geometry and physics.

- It is not as easy as it seems because complex holomorphic functions *try to follow their name and be globally defined,* actually they have analytic continuation.But they may have singularities blocking their way to global (global is called here *entire*).
- BR's main contribution to 2D analysis+geometry is the concept of Riemann surface with the curious branching points. A simple Riemann surface may be a part of $\mathbb{R}^3$ but more complicated RS live in a very strange situation, *a different world.*
- But we want now to forget 2D and remember that we live in 3D. Thinking about the geometry of 3D is old pastime, masterfully encoded by Euclides of Alexandria (325 BC - 265 BC).
- The 3D world is much more complicated that 2D and no part of the equivalence between analysis, Taylor series, elementary PDEs, conformal geometry and ideal Physics survives

What is Geometry according to BR

Let us follow the Encyclopædia Britannica article on BR.

- In 1854 Riemann presented his ideas on geometry for the official postdoctoral qualification at Göttingen; the elderly Gauss was an examiner and was greatly impressed.
- Riemann argued that the fundamental ingredients for geometry are a space of points (called today a manifold) and a way of measuring distances along curves in the space.
- He argued that the space need not be ordinary Euclidean space and that it could have any dimension (he even contemplated spaces of infinite dimension). Nor is it necessary that the surface be drawn in its entirety in three-dimensional space.
- It seems that Riemann was led to these ideas partly by his dislike of the (Newton's) concept of action at a distance in contemporary physics and by his wish to endow space with the ability to transmit forces such as electromagnetism and gravitation.
- A few years later this inspired the Italian mathematician Eugenio Beltrami to produce just such a description of non-Euclidean geometry, the first physically plausible alternative to Euclidean geometry. More italians influenced by BR: Ricci, Levi-Civita, Bianchi.

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Habilitationsvortrag, 1854. Riemannian Geometry

- Space around only has a definite sense locally around the place.
- The basic tool to do geometry is the metric, which is given by

$$ds^2 = \sum g_{ij} dx^i dx^j$$

It is local since it works on local entities, tangent vectors. Forget Pithagoras but remember $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ on the sphere.

- The metric field changes from point to point, $g_{ij}(x)$, x is locally a set like $\mathbb{R}^d$. He says for space of functions $d = \infty$.
- So there is no sense in principle of parallel vectors. We can instead define the derivative of a tangent vector $X = \sum_i a_i \mathbf{e_i}$ when we move along another vector $Y = \sum_j b_j \mathbf{e_i}$. This is the famous covariant derivative ∇.

$$\nabla_Y X = \sum_i Y(a_i)\mathbf{e_i} + \sum_{ijk} a_i b_j \Gamma^k_{ij} \mathbf{e_k}.$$

For the correct covariant derivative, called Levi-Civita connection, the Christoffel symbols are

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Objects with several indices are usually tensors. Note that although the Christoffel symbols have three indices on them, they are not tensors. Sorry, coordinates are intuitive but messy!!

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Curvatures at the center of geometry

- **Curvature tensor.** The Riemann curvature tensor is given by

$$R(X,Y,Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

or the other sign. In that case we have

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^s_{jk}\Gamma^i_{sl} - \Gamma^s_{jl}\Gamma^i_{sk}.$$

Contraction gives the low tensor $R_{ijkl} = g_{im} R^m_{jkl}$. Wikipedia gives

$$R_{iklm} = \frac{1}{2}\left(\partial^2_{kl} g_{im} + \partial^2_{im} g_{kl} - \partial^2_{km} g_{il} - \partial^2_{il} g_{km} +\right) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} + \Gamma^n_{km}\Gamma^p_{il}).$$

- **Ricci curvature.** R.c. for g is a contraction of the general curvature tensor:

$$R_{ij} = \sum_j R^s_{isj} = \sum_{s,m} g^{sm} R_{isjm}$$

The Ricci tensor has the same type $(0,2)$ (twice covariant) of the metric tensor. In coordinates we have (Nirenberg's sign, Wikipedia)

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The Laplacian operator in such geometries

- **Laplace-Beltrami operator.** Here is the definition of the geometer's Laplacian

$$\Delta_g(u) = -g^{ij}(\partial_{ij}u - \Gamma^k_{ij}\partial_k u) = -\frac{1}{|g|^{1/2}}\partial_i(|g|^{1/2}g^{ij}\partial_j u)$$

This is minus the contraction of the second covariant derivative tensor

$$(\nabla^2 u)_{ij} = \partial_{ij}u - \Gamma^k_{ij}\partial_k u$$

A coordinate chart (x^k) is called **harmonic chart** iff $\Delta_g x^k = 0$ for all i. Note that

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Therefore, (x^i) is harmonic iff $g^{ij}\Gamma^k_{ij} = 0$ for all k.

- The Laplacian is correct for analysis because the formula

$$\int_M (\Delta_g u)\, v\, d\mu + \int_M \langle \nabla_g u, \nabla_g v\rangle d\mu = 0.$$

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Yamabe problem. Ricci flow

- Let g be a metric in the conformal class of g_0 with its Levi-Civita connection denoted by D. We denote by $R = R_g$ and R_0 the scalar curvatures of the metrics g, g_0 resp. Write Δ_0 for the Laplacian operator of g_0. Then we can write

$$g = u^{4/(n-2)} g_0$$

locally on M for some positive smooth function u. Moreover, we have the formula

$$R = -u^{-N} Lu \quad \text{on } M,$$

with $N = (n+2)/(n-2)$ and

$$Lu = \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n+2}.$$

Note that $\Delta_0 - \frac{n-2}{4(n-1)} R_0$ is the conformal Laplacian relative to the background metric. Write equivalently, $R_g u^N = R_0 u - \kappa \Delta_0 u$.

- Yamabe Problem: given g_0, R_0, R_g find u.
  This is a nonlinear elliptic equation for u.
  You can now continue with Hamilton and Perelman.

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General relativity. Einstein equation and tensor

- Riemann's ideas went further and turned out to provide the mathematical foundation for the four-dimensional geometry of space-time in Einstein's theory of general relativity.
- The Einstein tensor T is a 2-tensor defined over Riemannian manifolds which is defined in index-free notation as

$$\mathbf{G} = \mathbf{R} - \frac{1}{2} R \mathbf{g}$$

where $\mathbf{R}$ is the Ricci tensor, $\mathbf{g}$ is the metric tensor and R is the Ricci scalar (or scalar curvature). In components, the above equation reads

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System of second order partial differential equations in 4 variables. The stress-energy tensor T is a tensor quantity in relativity. It describes the flow of energy and momentum and is therefore sometimes referred to as energy-momentum tensor.

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Outline

1 Mathematics, Physics and PDEs
- Origins of differential calculus
- XVIII century
- Modern times

2 G. F. B. Riemann

3 Riemmann, complex variables and 2-D fluids

4 Riemmann and Geometry

5 **Riemmann and the PDEs of Physics**
- Picture gallery

Riemann's interest in Physics

- The influence of the famous experimental physicist W. Weber was important in his view of mathematics.
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"Ueber die Fortpflanzung...", 1860. The equations of gas dynamics

- One-dimensional isentropic gas flow is a mathematical abstraction described by the system of differential equations

$$\begin{cases} u_t + u\,u_x + p_x/\rho = 0, \\ \rho_t + (\rho\,u)_x = 0 \end{cases} \tag{1}$$

plus the algebraic equation $p = p(\rho)$.

- In the application module, x is interpreted length along a tube, whose transversal dimensions are supposed to be irrelevant, u is interpreted as *fluid particle speed* and ρ as *density*. The last law is called State Lawand for ideal gases it takes the form $p = C\rho^\gamma$ where $\gamma = 1,4$.
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Hyperbolic systems

In order to continue we do linear algebra, calculating the eigenvectors and eigenvalues of the matrix A. We obtain

$$\lambda_1 = u + c, \lambda_2 = u - c$$

where $c^2 = p'(\rho)$ is called the speed of sound. Note that $\lambda(u, \rho)$ so it changes with (x, t) depending on the flow you solve at this time.

- If $\rho \neq 0$ then $c \neq 0$ and we have two different eigenvalues and we are entering with Riemann into the theory of NLHDS (Nonlinear Hyperbolic Differential Systems), still frightening today. Peter Lax, Courant Institute, Abel Prize winner, is a world leader in the topic.
- We now get a map from (x, t) into (u, ρ), with two nice directions for the linearization of the evolution equation,

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The eigenvectors of the system are

$$\mathbf{U}_1 = (c^2/\rho, 1), \quad \mathbf{U}_2 = (c^2/\rho, 1)$$

- Now Riemann tells us to find the characteristic lines: if we think that the solution is known, then solve the ODE Systems

$$\frac{dx}{dt} = \lambda(x, t, \mathbf{U})$$

- He tells you then to find functions F_1, F_2, called the Riemann invariants, which are independent and constant along the corresponding characteristics. In the gas example they are

$$F_{\pm} = u \pm \int \frac{c(\rho)}{\rho} d\rho.$$

- Since these functions are constant on the characteristics, they allow to see what the characteristics do and this says what the flow does at any moment. Replace (u, ρ) by F_1, F_2 and try to see something.

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Shocks

- The theory BR develops allows to solve the system in a classical way iff the characteristics of the same type for different points do not cross. In that case the invariant takes two values, a shock appears.
- Shocks appear in the examples even in $d = 1$, which is the Burger's equation

$$u_t + u\, u_x = 0$$

- Since it happens in the simplest nontrivial mathematics, BR concludes that you cannot avoid shock formation, and that a theory of solutions with discontinuities that propagate in some magical way is needed. This is today the theory of shocks and discontinuous solutions of conservation laws.
- Very soon the physical community recognizes this work as fundamental new insight into the complexity inherent to compressible fluids.
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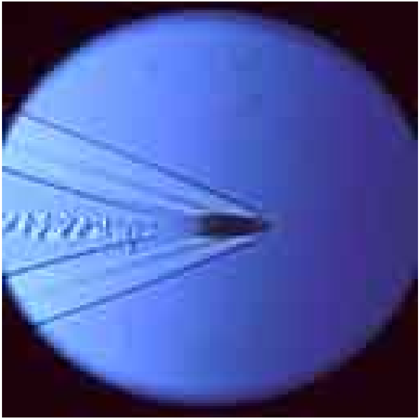
Some shock waves in Nature



(Left) **Schlieren Image – Convection Currents and Shock Waves**, *Steve Butcher, Alex Crouse, and Loren Winters – August, 2001*

The projectiles were 0.222 calibre bullets fired with a muzzle velocity of 1000 m/s (Mach 3).

The Schlieren lighting technique used for these images makes density gradients in fluids visible.

Color filtration provides false color images in which the colors provide information about density changes.

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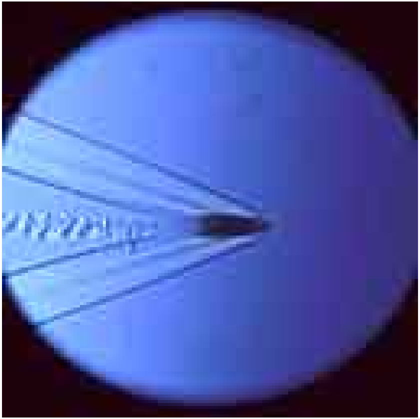

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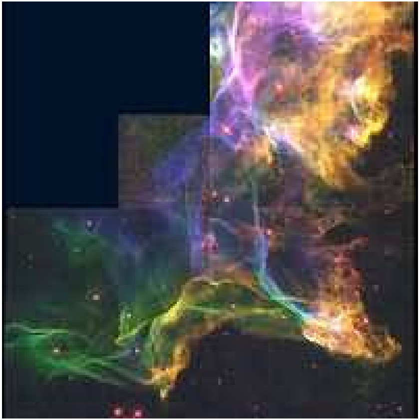

(Left) This Hubble telescope image shows a small portion of a nebula called the "Cygnus Loop."

This nebula is an expanding blast wave from a stellar cataclysm, a supernova explosion, which occurred about 15,000 years ago.

The supernova blast wave, which is moving from left to right across the picture, has recently hit a cloud of denser-than-average interstellar gas.

This collision drives shock waves into the cloud that heats interstellar gas, causing it to glow.

- Sandia Releases New Version of Shock Wave Physics Program: *http://composite.about.com/library/PR/2001/blsandia1.htm*

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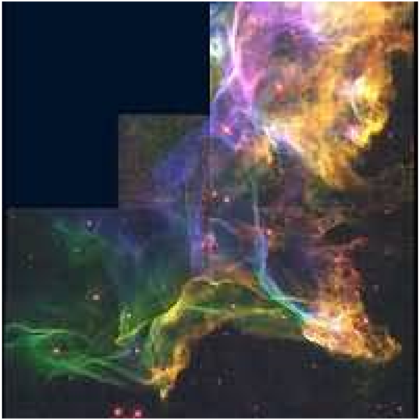

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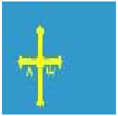

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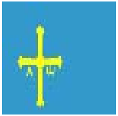

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