Bernhard Riemann (1826-1866)

Brief biography

- Riemann was born in Breselenz (Kingdom of Hanover) in 1826; his father was a Lutheran minister
- From 1840 to 1842 he attended the Gymnasium in Hanover, where he lived with his grandmother
- After his grandmother’s death, he moved to the Gymnasium of Lüneburg, very close to Quickborn where in the meantime his family had moved. He had his first health problems

- In 1846 he began to study at the University of Göttingen at the Faculty of Philology and Theology. Very soon he moved to the Faculty of Philosophy, which also included mathematics. Gauss was among his teachers
- In 1847 he moved to Berlin, where he knew Jacobi, Dirichlet, Steiner and Eisenstein, who deeply influenced his future research
- In 1848 there was a democratic uprising in Berlin. Riemann was among the conservatory students who opposed it
- In 1849 he went back to the University of Göttingen in agreement with his father’s hopes. He began to attend the seminar of Gauss and Weber on mathematical physics

- In 1851: Inauguraldissertation (on complex analysis)
- In 1853: Habilitationsschrift (on real analysis) (pub.1868)
- In 1854: Habilitationsvortrag (on differential geometry) (publ. 1868)
- In 1855 Gauss died; Dirichlet moved to the Univ. of Göttingen
- In 1857 Riemann was appointed extraordinary prof. at the Univ. of Göttingen; he published his celebrated paper on the theory of Abelian functions
- In 1859 he visited Paris where he knew Bertrand, Biot, Bouquet, Hermite, Puiseaux and Serret. He was appointed ordinary professor at the Univ. of Göttingen. He published his famous paper on number theory
- In 1862 he married Elise Koch
- From 1863 to 1865 he was in Pisa with Enrico Betti
- In 1866 he went to Italy again; he died on July 20 in Selasca

Great influence of Riemann’s work

- Complex theory of functions (Riemann surfaces, Cauchy-Riemann conditions…)
- Real analysis (Riemann integral, counterexamples,…)
- Topology (connection, “Querschnitte”,…)
- Differential geometry (Riemannian manifold, Riemannian curvature tensor)
- Number theory (Riemann’s conjecture)
- PDE
- Theory of algebraic curves
- Theory of Abelian functions

Main aims:

1. To show some connections among Riemann’s works in mathematics and also physics

2. To show that his “philosophical” papers – written about 1853 and where Riemann explained his model of the ether as well as his *Naturphilosophie* – can help to give a wider and deeper understanding of his works in mathematics and in physics

1851 Inauguraldissertation

1. "Cauchy –Riemann conditions": Riemann considered a complex function w=u+iv of a variable z=x+iy such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(which is nowadays called a "holomorphic function".)
From this equations he found that $\Delta u=0$, $\Delta v=0$, which are basic for investigating the properties of u and v.

Remark
Riemann's complex function theory is connected with potential theory in two dimensions – a theory which Riemann knew very well. In fact, he had followed Weber's lectures in 1849 and the following year he partecipated in the physics seminar led by Weber himself and Gauss.

2. The Dirichlet problem

Both Gauss and Dirichlet – who were Riemann's teachers in Göttingen and Berlin – studied the so-called "Dirichlet problem":

"To find a function u with continuous first partial derivatives on a given bounded domain, which satisfies the Laplace equation ($\Delta u=0$) within the domain and has given values on the boundary"

Gauss was led to the Dirichlet problem by studying the distribution of masses or electric charges on a closed surface S, assuming the potential constant on S.

Dirichlet published an important book on potential theory in 1876 (Vorlesungen ueber die im umgekehrten Verhältnisse des Quadrats der…).

3. Conformal mapping:
the Riemann Theorem

"Two plane and simply connected surfaces S and S′ can be transformed conformally one into the other (by a continuous and bijective mapping)."

Riemann remarked that the assumption "simply connected" surfaces can be removed; then, the Riemann theory is valid also for Riemannian surfaces.

Remark

Riemann proved his theorem by solving a special Dirichlet problem.

4. The “Riemann surface” was introduced by Riemann in order to study multi-valued functions – such as algebraic functions and their integrals. The Riemann surface associated to a function is composed of as many sheets as are the branches of the function, connected in a particular way – so that continuity is preserved and a single-valued function on the surface is obtained.

Therefore, he gave an abstract conception of the space of complex variables by using a geometrical formulation. In addition, an algebraic function has now a geometrical meaning, being interpreted as a Riemann surface.

5. “Querschnitte”: Riemann made the surface simply connected with suitable transversal cuts and studied the behaviour of the function in the neighbourhood of the singularities.

Remark

The idea of transversal cut on a surface struck Riemann after a long discussion with Gauss on a mathematical-physical problem (letter by Riemann to Betti). The origin of the ideas of Riemann surfaces and Querschnitte can be found in his note on a problem of electrostatic or thermical equilibrium on the surface of a cylinder with transversal cuts (“Gleichgewicht der Electricitaet ….”, 1876). Here, he was led to consider a Dirichlet problem on a simply connected and simple sheeted surface.

Remark

1. Complex analysis and mathematical physics (potential theory) are strictly connected
2. Connections with PDE. In fact, the Laplace equation represents a PDE of the elliptic kind. Green (1828) showed that it can be solved by using the so-called Green function. Unfortunately the solution can be explicitly deduced only for special cases.

Generalizations of the Green function

Many mathematicians of the 19th century – such as Helmholtz, Lipschitz, Betti, Carl and Franz Neumann, and Riemann himself – deduced functions similar to Green's function in order to solve problems in acoustics, electrodynamics, magnetism, theory of heat, and elasticity. In a paper ("Ueber die Fortpflanzung ebener Luftwelle von endlicher Schwingungsweite") published in 1860, Riemann applied the method of Green's function in order to integrate the differential equation of hyperbolic type describing the diffusion of acoustic waves. He introduced a function, which plays the same role as Green's function did for the Laplace equation and is today called "Green's function for the hyperbolic problem".

Naturphilosophie and the local-global approach

In an undated note, written after the completion of his Inauguraldissertation, Riemann wrote that his “main work” involved “a new interpretation of the known laws of nature – whereby the use of experimental data concerning the interaction between heat, light, magnetism, and electricity would make possible an investigation of their interrelationships. I was led to this primarily through the study of the works of Newton, Euler and, on the other side, Herbart” (publ. 1876).

Remark

Herbart’s psychology inspired both Riemann’s model of the ether (the elastic fluid filling all the universe) and his principles of *Naturphilosophie*.

Herbart:

The “psychic act” (or “representation”) is an act of self-preservation with which the “ego” opposed the perturbations coming from the external world. A continuous flow of representations went from the ego to the conscious and back. Herbart studied the connections between different representations in mechanical terms as compositions of forces.

Remark

Riemann followed Herbart’s psychology in a paper drafted in March 1853 (“Neue mathematische Principien der Naturphilosophie”) and in other notes on Naturphilosophie which he intended to publish (letter to the brother Wilhelm in December 1853) (all of them publ. in 1876).

Riemann, Neue math. Prinzipien der Naturph.

- The universe is filled with a substance (*Stoff*) flowing continually through atoms and there disappearing from the material world (*Körperwelt*). From this obscure assumption, Riemann tried to build a mathematical model of the space surrounding two interacting particles of substance: if a single particle of substance is concentrated at the point O at time t and at the point O' at time t', then he considered the two homogeneous forms:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2, \quad ds'^2 = dx'^2_1 + dx'^2_2 + dx'^2_3$$

where (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) are the coordinates of O and O'.

- Riemann considered an appropriate new basis and compared the two forms associated to the "particle of substance" at times t and t′. The difference between the two forms is given by

$$\delta(ds) = ds'^2 - ds^2 = (G_1^2 - 1)ds_1^2 + (G_2^2 - 1)ds_2^2 + (G_3^2 - 1)ds_3^2$$

where ds_1, ds_2, ds_3 is the new basis.

His result can be interpreted in terms of the classical theory of elasticity. Since ether is an elastic, homogeneous and isotropic substance, then one can consider an infinitesimal displacement u and deduce that the difference δ(ds) depends on the strain tensor due to the displacement. Riemann supposed that this variation δ(ds) produced a force able to modify the particle in such a way that the same particle, by opposing this deformation, would propagate the physical forces through space.

- If δ(ds)=0, that is to say that the particle does not change its form from the time t to the time t′, then the particle does not propagate any force since space is not submitted to deformation by a force.
- On the contrary, if δ(ds) is different from zero, a physical phenomenon is propagated through space.
- Riemann tried to connect this formula with the different forces, such as with gravity, and with heat and light propagation.

Riemann could not explicitly show these connections. He limited himself to state that gravity, and light and heat propagation can be explained by assuming that every particle of the homogeneous substance filling space has a direct effect only on its neighbourhood and the mathematical law according to which this happens is due to:

"1) the resistance with which a particle opposes a change of its volume, and

2) The resistance with which a physical line element opposes a change of length.

Gravity and electric attraction and repulsion are founded on the first part; light, heat propagation, electrodynamic and magnetic attraction and repulsion on the second part."

Remark

Riemann's *Naturphilosophie* is connected both to some of his physical concepts on electricity and electromagnetism and to his ideas on differential geometry.

1. Connections to physics

- Kohlrausch's experiment:

  in a Leyden jar (the first capacitor) which had been charged, then discharged and left for some time, a residual charge appeared.

  In September 1854, Riemann tried to explain Kohlrausch's experiment; for this purpose, he developed a physical explanation of the electromotive force and of electric propagation through a body on the basis of the model of ether expressed in his notes on N*aturphilosophie.* The paper was addressed to *Ann. der Physik*, but Kohlrausch – the editor of the journal – asked for so many changes that Riemann retracted the paper.

Riemann proposed a new theory of electricity by assuming that the electric currrent was caused by a reaction of the body opposed to the change of its own electric state. This reaction is proportional to the charge density, and it decreased or increased the electric density according as the body contained positive or negative electricity.

Therefore the transmission of electricity could not be instantaneous but electricity moves “against ponderable bodies” with a certain speed.

Riemann, “Ein Beitrag zur Electrodynamik”, 1858

He developed a new theory of electromagnetism, by assuming that electric phenomena travel with the velocity of light and that the differential equations for the electric force are the same as those valid for light and heat propagation. (Influence of Gauss and Weber)

Riemann, *Partielle Differentialgleichungen und deren Anwendungen auf physikalische Fragen*, 1876 (ed. by Hattendorff)

He tried to describe the ether surrounding two interacting electric particles. For this purpose, he assumed that the ether possesses physical properties which guarantee electric propagation. He actually deduced a differential equation expressing the flux of ether in space, by using classical Lagrangian mechanics.

2. Connections to differential geometry
Riemann, Habilitationsvortrag, 1854

Riemann tried to generalize the ideas contained in his *Naturphilosophie*: he extended the "local" investigation of particles of ether to the "global" analysis of n-dimensional spaces.

Remark

If one considers that an ether fills all space, then a deformation of space is linked to a force which has to be propagated. Force and curvature of space are then closely connected; it is space which propagates forces by changing its curvature.

- A deeper analysis on the connection between n-dimensional manifolds and their curvatures is also developed in Riemann's 1861 paper ("Commentatio Mathematica…").

- This ideas were shared by many other mathematicians of the 19th century (see Beltrami, Lobachevskij, Clifford). Clifford (The common sense of exact sciences, 1885) asked the question "whether physicists might not find it simpler to assume that space is capable of a varying of curvature, and of a resistance to that variation…" and that this resistance was the responsible of the propagation of phenomena.

The idea of being in a curved space which, thanks to its changes of curvature, transmitted physical forces was at the basis of many mathematical and physical reflections during the 19th century, long before Einstein's theory of relativity.

This investigations led to deduce equations and results in a curved space, which is mathematically represented by a Riemannian manifold. Tensor calculus is the more natural formalism for doing this (tensorial equations do not change under coordinate changes).

It is not by chance that tensor calculus is the theory ad hoc in Einstein's General Theory of Relativity.