RIEMANN'S INFLUENCE IN GEOMETRY, ANALYSIS AND NUMBER THEORY

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ABSTRACT. The purpose of this article is to describe some ways in which Zeta functions enter geometry, and their relation to the theory of Riemann surfaces.

Riemann's collected works take one small volume, but every contribution to this volume was very original work that supplied foundations for the mathematics of the next century.

To give an idea of his range of interests, and of his influence up to recent times, I will first describe briefly three areas which bear his name:

(1) Riemann Zeta function (number theory)
(2) Riemann surfaces (algebraic geometry-topology)
(3) Riemannian metric (fundamental in differential geometry)

Then I will review three examples where these areas interact. They are three examples of work in the 20th century which have their roots in the ideas of Riemann. The first one will be a theorem of the 1930's which involves the interaction of (2) and (3) on this list. Next one will be a theorem of the 1950's, also involving the interation of algebraic topology/geometry with differential geometry. And the last one will be a theorem of the 1980's which involves in fact all three, including number theory.

In summary, the main points will be:

- Review the three topics (1), (2) and (3) above.
- A theorem of 1930's involving (2) and (3).
- A theorem of 1950's involving (2) and (3).
- A theorem of 1980's involving (1), (2) and (3).

Riemann Zeta function

It is the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

This function converges for $\mathrm{Re}(s) > 1$ and has analytic continuation to whole complex s-plane with a simple pole at $s = 1$. For $s = 0$ we have $\zeta(0) = 1/2$.

Functional equation. If we write

$$\xi(s) = \frac{s(s-1)}{2}\Gamma(s/2)\pi^{-s/2}\zeta(s)$$

then

$$\xi(s) = \xi(1-s).$$

Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

where the product ranges over the primes p.

Riemann Hypothesis (10^6 dollar prize). It is the most famous unsolved problem since Riemann's time and it is about the zeros of the Riemann zeta function.

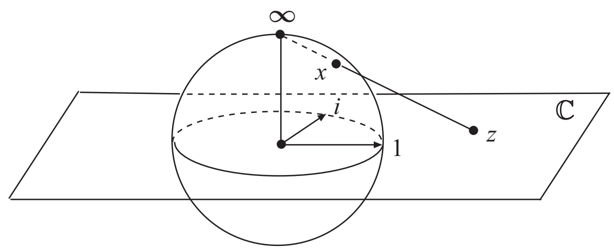
The "trivial zeros" of $\zeta(s)$ occur at $s = -2, -4, ...$ (even negative integers).

The Riemann hypothesis states that *all non-trivial zeros of* $\zeta(s)$ *lie on the line* $\mathrm{Re}(s) = 1/2$.

It has very important consequences for the detailed distribution of prime numbers.

Riemann surfaces

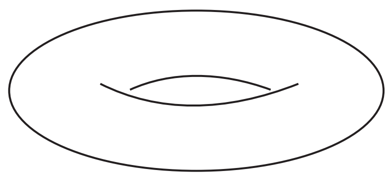
Adding the point at infinity (∞) to the complex plane $\mathbb{C}$ we get the *Riemann sphere.*



The complex solutions (x, y) of the equation

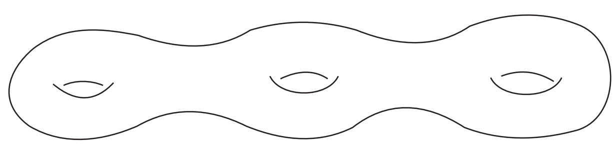
$$y^2 = f(x), \quad f \text{ quartic polynomial}$$

(including ∞) form a *complex torus*. It is a double covering of the sphere with 4 branch points at the 4 roots of $f(x) = 0$ (assumed distinct).



The sphere leads to the theory of *rational functions* and the torus to the theory of *elliptic functions*.

Higher degree polynomial equations $p(x, y) = 0$ lead to a theory of functions involving surfaces of higher genus.[1] These are called *Riemann surfaces*. Here is a picture of a genus 3 surface:



The genus is a topological invariant. It is $B_1/2$, where B_1 is the first Betti number, or the rank of the first homology group, which agrees with the number of independent 1-cycles that can be drawn on the surface. For a torus, for example, two independent cycles are one going

[1]The genus g of a compact orientable surface is the number of its handles or holes. Thus $g = 0$ for the sphere and $g = 1$ for the torus.

around the hole and one around the tube. In general, for each hole we have one more pair, which means $2g$ in all.

On the Riemann surface we have a relationship with analysis. We can write down what are called *holomorphic differentials*. If z is a local coordinate on the surface, a holomorphic differential has (locally) the form

$$f(z)dz,$$

where f is a holomorphic function. If we make a change of variable from one region to another one, say $z = z(u)$, then $f(z)dz$ becomes $f(z(u))z'(u)du$. Now *the genus g is the dimension of the space of holomorphic differentials*. For example, if $g = 0$ there are no non-zero holomorphic differentials. The key fact is that if z is the usual coordinate on the complex plane, then $u = 1/z$ is a local coordinate at ∞ and $dz = -du/u^2$ has a pole there. On the other hand on the torus, thought of as the quotient $\mathbb{C}/L$ of $\mathbb{C}$ by a lattice L, dz is a holomorphic differential, and this differential is unique up to scalar factor. For higher genus there are more complicated formulas giving the g independent holomorphic differentials.

Thus we have a fundamental link between the topology (the number of holes) and the analysis (the description of the holomorphic differentials). Holomorphic differentials occur as integrands of integrals along curves. In fact, that is how the theory of elliptic integrals first arose.

The Riemannian metric

The Riemannian metric is the foundation of modern differential geometry. It has an interesting beginning, because when Riemann was to submit his thesis for a doctoral degree the custum in Germany of the time was that the examiners would ask him some minor topic outside of the main topic of interest. Gauss asked Riemann to investigate the foundations of geometry. I think he knew that Riemann had ideas, and Riemann's answer was to produce the foundations of differential geometry –as a side issue to the main thesis.

On an n-dimensional manifold, which locally looks like an Euclidean space with n real coordinates $x_1, \ldots, x_n$, a *Riemannian metric*, which is a way of measuring distances, is a symmetric positive definite quadratic form

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j,$$

with the g_{ij} varying smoothly. That gives the element of length, ds, and the length of a curve is then given by integrating ds along it.

What Riemann discovered was that the most essential and important object that can be associated with the Riemannian metric is what is called the *Riemann curvature tensor*: An expression R_{ijkl}, given by a quite simple formula involving the g_{ij} and their derivatives, which describes all the fundamental ways in which geometry is curved.

Note that a Riemann surface is at the same time, a 1-dimensional complex manifold and a 2-dimensional Riemannian manifold, where the real coordinates are the real and imaginary part of the complex coordinate.

Examples of Riemannian manifolds

(1) Take an ordinary smooth surface in 3-dimensional space, as for instance a sphere, and restrict to it the ordinary Euclidean metric. This gives a way of measuring lengths of curves on the surface.

(2) We can do similar things in higher dimensions. Consider a non-singular algebraic curve in the the complex projective plane $\mathbb{C}\mathbf{P}_2$, which has real dimension 4. It is given by an equation

$$f_d(x, y, z) = 0,$$

where f_d is a homogeneous polynomial of degree d, and its genus $g = \frac{1}{2}(d-1)(d-2)$. This curve acquires a natural Riemannian metric, namely, the metric induced on it by the standard metric in $\mathbb{C}\mathbf{P}_2$.

(3) Similarly, a non-singular algebraic surface $f_d(x, y, z, t) = 0$ in $\mathbb{C}\mathbf{P}_3$ with the induced metric.

Note. The Riemannian curvature tensor has four indices i, j, k, l in all dimensions, but it reduces to a scalar function in dimension 2, the Gauss scalar curvature, and it is essentially a 2-tensor in dimension 3 (the Ricci tensor). Dimension 4 is the first dimension in which the most general form of the curvature tensor is required. It is the dimension of Einstein's space-time, in which Riemannian ideas, and especially Riemann's curvature tensor, played a very important role. It is also the dimension of Simon Donaldson's theory, in which he applied ideas of Riemannian geometry and got many fantastic results showing that 4 real dimensions in some ways are quite unique amongst all dimensions.

Hodge theory (1930's)

Consider an n-dimensional real manifold, compact and oriented, with a Riemannian metric. Hodge considered the exterior differential forms Ω^q, the natural integrands of q-dimensional integrals ($q = 0, 1, ..., n$). They are skew-symmetric q-tensors. Thus Ω^0 are the scalar functions $f(x)$, and Ω^1 the forms of degree 1 (locally, $\sum_i f_i(x)dx_i$). Higher degree forms Ω^q involve wedge products (skew symmetric) of any q of the $dx_1, \ldots, dx_n$. For $q = n$ there is only one such product (a volume form). There is also the exterior differentiation operator, $d : \Omega^{q-1} \to \Omega^q$, which gives rise to the *De Rham complex*:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n$$

The metric defines a dual or orthogonal form $*\omega \in \Omega^{n-q}$ associated to a q-form $\omega \in \Omega^q$. In particular, if f is a function, $*f$ is a volume form.

In Hodge theory, a form ω is called harmonic if $d\omega = 0$ and $d * \omega = 0$. One of Hodge's motivations for introducing this concept was the fact that the electromagnetic field can be represented by a 2-form ω in Minkowski space and that Maxwell's equations are then equivalent to $d\omega = 0$ and $d * \omega = 0$.

Another equivalent way of defining harmonic forms is the following. Define

$$d^* : \Omega^q \to \Omega^{q-1}$$

as the adjoint of d (if n is even, $d^* = - * d *$) and let

$$\Delta = dd^* + d^*d : \Omega^q \to \Omega^q.$$

This operator is the *Hodge Laplacian* and $\Delta\varphi = 0$ is equivalent to saying that φ is harmonic. This fact explains why they are called 'harmonic', as $\Delta\varphi$ is the ordinary Laplacian for a function φ in Euclidean space.

The fundamental theorem of Hodge was a simple relationship between the harmonic forms and the topology:

Theorem. *The space of harmonic q-forms is isomorphic the real cohomology space* H^q (the map is given $\omega \mapsto \int_\gamma \omega$, γ any q-cycle).

The expression $\int_\gamma \omega$ is called the *period* of ω along γ, and the theorem says that every harmonic form is determined by its periods and that every 'period' can actually occur.

For example, on a Riemann surface the harmonic 1-forms, for an appropriate metric, are the real and imaginary parts of the holomorphic forms. Note that this implies that $\dim H^1 = 2g$.[2]

Now a big problem was going from algebraic curves to higher dimensions (algebraic surfaces, for example) and replacing the holomorphic differentials. Hodge's great idea was using a Riemannian metric and taking harmonic forms. That was the beginning of the foundations of modern algebraic geometry.

Signature of a 4-manifold

For $n = 2$ (surface) the only topological invariant is the genus g (or $B_1 = 2g$).

For $n = 4$ we have

$$B_1 = B_3 \quad \text{and} \quad B_2.$$

But there is a further invariant on H^2 (or H_2): the *intersection form* (or intersection matrix), $H^2 \times H^2 \to \mathbb{R}$. This pairing is a non-degenerate quadratic form. Now a quadratic form over the real numbers has another invariant, aside from the rank, namely the signature. When diagonalized, it has B_2^+ positive terms and B_2^- negative terms, where $B_2 = B_2^+ + B_2^-$ (no zeros, as it is non-degnerate), and the signature is $B_2^+ - B_2^-$. This signature is called the *signature* of the manifold, and it is another topological invariant.

[2] Another way of saying this is that the complex valued harmonic forms on the surface are the holomorphic and antiholomorphic forms.

Note. There is no analogue of the signature in dimension 2, since the intersection pairing is skew-symmetric: (21) is an odd permutation while (3412) is an even permutation.

This signature invariant for quadratic forms was known for a long time, but its applications to manifolds topology were first pointed out by Hermann Weyl, another of my heroes in mathematics. Interestingly enough for this audience, it was published in a Spanish journal, in Spanish, in an old paper that is not very well known.

Let's connect this with the work of Hodge in differential geometry. On a 4-manifold M, the $*$ operator acts on Ω^2 and $*^2 = 1$. The corresponding eigenspaces H^2_+ and H^2_- have dimensions B_2^+ and B_2^-, respectively. Hence

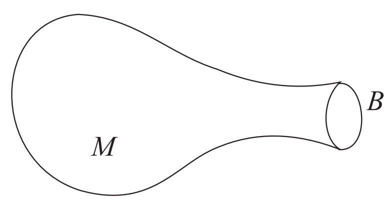
$$\text{signature} = \dim H^2_+ - \dim H^2_-.$$

By a famous theorem of Hirzebruch (1950's),

$$\text{signature} = \int_M f(R),$$

where f is a polynomial in the curvature. This beautiful theorem connects topology on one side with differential geometry on the other and it is the first of a large family of theorems of this kind. It is part of a big development in the 1950's. Hirzebruch proved that the theorem also holds in dimensions $n = 4k$ (with f depending on k).

Hirzebruch's result was for closed manifolds. So the natural question now is to ask what happens for manifolds with boundary.



Choose a Riemannian metric on M which is isometric to the product $B \times \mathbb{R}$ near the boundary:

$$ds_M^2 = ds_B^2 + dr^2.$$

Then $f(R) = 0$ on $B \times \mathbb{R}$ and therefore $\int_M f(R)$ is independent of the length of the boundary region.

The signature is still defined on $H^2(M, B)$, but the (diagonal) matrix has some zeros. Count only non-zero diagonal elements.

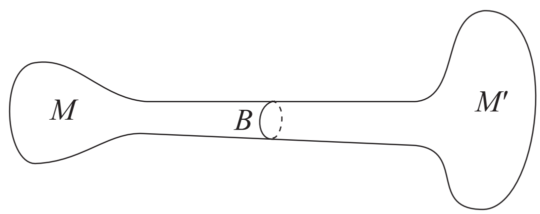
Question. Is signature$(M) = \int_M f(R)$?

Answer. No.

But we have:

Proposition. *The difference between the* signature *and* $\int_M f(R)$*, which is called the* signature defect*, depends only on the boundary.*

Proof. If $B = \partial M$ and $B = \partial M'$, stick M and M' together along B to get a manifold X without boundary.



Then we have

$$\text{signature}(X) = \text{signature}(M) + \text{signature}(M'),$$
$$\int_X f(R) = \int_M f(R) + \int_{M'} f(R),$$

and it is enough to subtract and use Hirzebruch's theorem for X (a sign change occurs for M' since its boundary is B with the opposite orientation). $\square$

Problem. What kind of invariant of the Riemannian 3-manifold B is the signature defect?

Note. The signature defect changes sign when we reverse the orientation of B.

To solve the problem, introduce the 1st order elliptic differential operator A acting on

$$\Omega^{\text{ev}} = \Omega^0 \oplus \Omega^2,$$
$$A = \pm(*d - d*) \quad [- \text{ on } \Omega^2, \quad + \text{ on } \Omega^0].$$

It is self-adjoint ($A^* = A$) and[3]

$$A^2 = \Delta^0 \oplus \Delta^2 \qquad \text{(Hodge Laplacian).}$$

The eigenvalues λ of A are discrete and λ^2 are the eigenvalues of $\Delta^0 \oplus \Delta^2$. The eigenvalue $\lambda = 0$ gives the harmonic forms $H^0 \oplus H^2$. Define

$$\eta(s) = \sum_{\lambda \neq 0} \frac{\operatorname{sign}(\lambda)}{|\lambda|^s}.$$

The function $\eta(s)$ converges for $\operatorname{Re}(s) > 3$ (because $3 = \dim(B)$) and has an analytical continuation with no pole at $s = 0$ (so $\eta(0)$ is well defined).

Theorem (Atiyah–Patodi–Singer, 1970's)**.**

$$\operatorname{signature}(M) = \int_M f(R) - \eta(0).$$

See [APS] and [Atiyah].

Note. 1) Reversing orientation changes

$$\begin{array}{ccc} * & \text{to} & -* \\ A & \text{to} & -A \\ \lambda & \text{to} & -\lambda \end{array}$$

If follows that $\eta(0)$ changes to $-\eta(0)$, and this behaviour checks with the theorem.

2) In classical notation for $\mathbb{R}^3$, and identifying a 2-form with a vector field,

$$A = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & \operatorname{curl} \end{pmatrix}$$

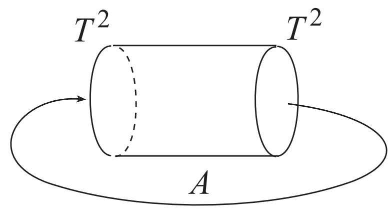
The theorem shows that the signature defect is a spectral invariant. It measures the degree to which the positive eigenvalues and the negative eigenvalues differ. If they are the same (for example if the manifold has an orientation reversing isometry), then the signature defect is 0, as the positive eigenvalues pair up one by one with the negative eigenvalues. In general they are not the same, and we say that the signature defect measures the *spectral asymetry*.

[3] A^2 is the square of the operator A, while Δ^2 simply refers to the Hodge Laplacian on 2-forms.

Example involving number theory

Here we will consider an example of an interesting application of the APS theorem.

Let B be the 2-torus bundle over a circle defined by an automorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, i.e., an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2,\mathbb{Z})$.



Note that $SL(2,\mathbb{Z})$ gives linear automorphisms of $\mathbb{R}^2$ that map the lattice $\mathbb{Z}^2$ onto itself. We have $\det(A) = ad - bc = 1$ and we will assume that $a + d = \operatorname{trace}(A) > 2$. This implies that the eigenvalues of A, namely the roots λ and $\lambda' = 1/\lambda$ of

$$\lambda^2 - (a+d)\lambda + 1 = 0,$$

are real and positive, as its discriminant $\Delta = (a+d)^2 - 4 > 0$.

This situation arises in number theory from the real quadratic field

$$K = \mathbb{Q}(\sqrt{\Delta}).$$

Let us examine more carefully the matrix A. Over $\mathbb{R}$, it is conjugate to the diagonal matrix $\begin{pmatrix} \lambda & \\ & \lambda' \end{pmatrix}$ and we have a 1-parameter group $A^t \sim \begin{pmatrix} \lambda^t & \\ & (\lambda')^t \end{pmatrix}$. The orbits of A^t in the (x,y) plane are (branches of) hyperbolas whose asymptotes are given by the equation of the eigenvectors (N is the norm):

$$N(y,x) = 0,$$

where

$$N(y,x) = cy^2 + (d-a)xy - bx^2 = c(y - \alpha x)(y - \beta x),$$

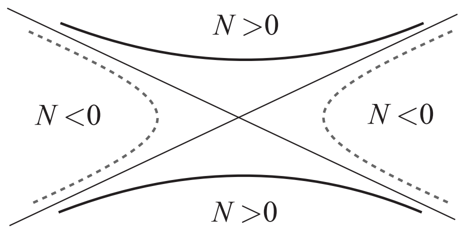
with

$$\alpha = \frac{(a-d)+\sqrt{\Delta}}{2c}, \quad \beta = \frac{(a-d)-\sqrt{\Delta}}{2c}$$

and

$$\Delta = (d-a)^2 + 4bc = (a+d)^2 - 4.$$

Assume $c < 0$.



Since $A \in SL(2,\mathbb{Z})$, it preserves each branch and maps integer points to integer points, so branches that have one integer point actually have infinitely many. Since the norm is constant along a branch, N is an integer for branches containing an integer point.

Zeta function of $K = \mathbb{Q}(\sqrt{(}\Delta))$:

$$\zeta_A(s) = \sum_{(\gamma)} \frac{1}{|N(\gamma)|^s},$$

where $\gamma = (m,n)$ is an integer lattice point and we sum over the non-zero orbits (γ) of $\{A\}$ (the group generated by A).

ζ_A converges for $\mathrm{Re}(s) > 1$, has analytic continuation, with no pole at $s = 0$, and has an Euler product expansion.

L**-function of** $K = \mathbb{Q}(\sqrt{(}\Delta))$:

$$L_A(s) = \sum \frac{\mathrm{sign}(N(\gamma))}{|N(\gamma)|^s}$$

over the non-zero integer orbits of $\{A\}$. It has similar properties to $\zeta_A(s)$.

$\zeta_A(0)$ is well-defined (actually it is a rational number).

Theorem (Atiyah–Donnelly–Singer, 1980's)**.**

$$L_A(0) = \eta_A(0)$$

where $A \in SL(2,\mathbb{Z})$.

Notes. 1) $L(s)$ and $\eta(s)$ are very different analytic functions:

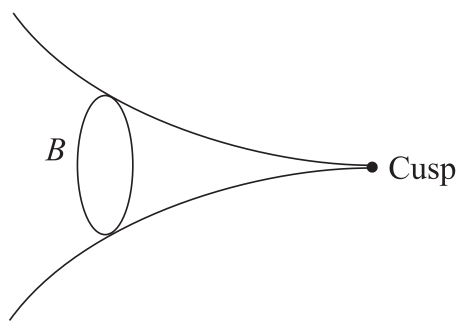
- $L(s)$ converges for $\mathrm{Re}(s) > 1$
- $\eta(s)$ converges for $\mathrm{Re}(s) > 3$.

But the theorem can be proved by using Fourier series on T^2, an approach that was initiated by Carl L. Siegel.

2) The theorem generalizes to any totally real field K of any degree:

- B becomes a T^r bundle over T^{r-1}, where r is the rank of the integers in K (and $r-1$ the rank of the group of units of K).

3) The theorem is also related to algebraic geometry. Hirzebruch proved [Hirz] that the signature defect of a 3-manifold B is $L_A(0)$ by a direct method related to the periodic continued fraction of $\sqrt{\Delta}$ and the resolution of the cusp singularities of an algebraic surface. This provided



one motivation for the ADS theorem, as it led to a conjectured generalization to higher degree fields. See [ADS].

References

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