HILBERT REVISITED

DAVID BUCHSBAUM

1. Introductory comments

The seeds of a good deal of the material I'll talk about today may be found in the beautiful 1890 article by Hilbert: On the theory of algebraic forms (Über die Theorie der algebraische Formen), [13]. When I was a graduate student, people spoke of that paper as the one that killed off invariant theory. I hope the irony of this perception will become evident as we proceed with this talk.

It was in that paper that the famous Basis Theorem and Syzygy Theorem were proved; the first, a fundamental result for the development of commutative algebra as we know it; in fact, the basis of one of the definitions of "Noetherian," and the second, the nineteenth century form of the statement that the global dimension of the polynomial ring in n variables is n.

Hilbert was concerned with fundamental problems of invariant theory: given a linear group, G, acting linearly on the ring of polynomials, $S = K[X_1, \ldots, X_N]$, we let S^G be the subring of invariants. Is S^G finitely generated as an algebra over the field, K, and if so, what are its generators? Assuming it is finitely generated, that is, that $S^G = K[Y_1, \ldots, Y_{N'}]/I$, is it the case that I is finitely generated as an ideal in $K[Y_1, \ldots, Y_{N'}]$? (The ideal, I, is called **the ideal of relations on the invariants**.) To answer the latter question, Hilbert proved his famous *Basis Theorem*, undeniably one of the cornerstones of commutative algebra.

Because G acts linearly on the variables of S, it's clear that the ring S^G is graded and that the graded piece of degree ν, $(S^G)_\nu$, is a finite-dimensional vector space over the field, K. A question of prime concern

was to determine the dimension of this space for all ν, that is, to evaluate the function $\chi(\nu) = \dim_K((S^G)_\nu)$ for all ν (what we today call the *Hilbert function*), and to determine its growth. What Hilbert saw was that this could be done if one could write down a finite free resolution of S^G over the polynomial ring, $K[Y_1, \ldots, Y_{N'}]$ (this gives an explicit computation of the dimension, and also shows that the function is polynomial; we'll see an example of this in the next section). He then proceeded to prove the *Syzygy Theorem,* clearly a fundamental result in homological algebra.[1]

So, starting with a basic problem in invariant theory, Hilbert was led to prove two of the building blocks of commutative and homological algebra. If, indeed, he did "kill" invariant theory, he certainly atoned for it with a wealth of reparations. But we'll get back to representation theory again later.

2. An example

To illustrate the ideas mentioned above, let us consider the variables $\{U_{ij}, V_{jk}\}$ over a field, K, with $i = 1, 2, 3; j = 1, 2; k = 1, 2$, and the polynomial ring, $S = K[U_{ij}, V_{jk}]$. The general linear group of 2×2 non-singular matrices, $G = GL_2(K)$, operates on S in the following way: A matrix, A, operates on S by multiplying the matrix (U_{ij}) on the right by A^{-1}, and by multiplying the matrix (V_{jk}) on the left by A. It's fairly easy to show that the ring of invariants, S^G, is generated by all six elements of the form $Z_{ik} = \sum_{j=1,2} U_{ij}V_{jk}$; the ideal of relations on the invariants is, by the Hilbert Basis Theorem, finitely generated, and it is again not difficult to prove that the 2×2 minors of the matrix (Z_{jk}) generate this ideal, which we will call I_2. Thus, our ring S^G is the graded ring, $K[\{Z_{ik}\}]/I_2$.

Suppose, now, that we want to calculate the Hilbert function of S^G. Let $x_i = Z_{i1}$ and $y_i = Z_{i2}$, for $i = 1, 2, 3$. Our ideal is generated by the

[1]The Syzygy Theorem did more than provide a way to evaluate the function, $\chi(\nu)$; it actually showed that for large values of ν, the function is a polynomial function, that is, it agrees with a polynomial for sufficiently large ν.

minors of order two of the matrix:

$$\Delta = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3' \end{pmatrix},$$

and we set R to be the graded polynomial ring, $K[x, y]$. We get the exact sequence (the exact sequence associated to the syzygies of the R-module, R/I_2):

$$0 \to R^2 \xrightarrow{\beta} R^3 \xrightarrow{\alpha} R \to R/I_2 \to 0$$

with $\beta : R^2 \to R^3$ the map whose 2×3 matrix is $\begin{pmatrix} x_1 & -x_2 & x_3 \\ y_1 & -y_2 & y_3 \end{pmatrix}$ and $\alpha : R^3 \to R$ the map whose 3×1 matrix is $\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}$, where Δ_i is the 2×2 minor obtained from Δ by eliminating the i^{th} row.

It's now easy to compute the νth degree of our factor ring as

$$\binom{6+\nu-1}{\nu} - 3\binom{6+\nu-3}{\nu-2} + 2\binom{6+\nu-4}{\nu-3} =$$

$$3\binom{\nu}{3} + 7\binom{\nu}{2} + 5\binom{\nu}{1} + 1.$$

This is clearly a polynomial function of degree three.

3. Homology and commutative ring theory

Here we'll elaborate a bit on the notions of homological dimension, global dimension, and other ways that homology ties in with algebra.

One very simple illustration of the relationship between homology and algebra is the following. Suppose R is a commutative ring, and x an element of R. One is immediately led (if one is so inclined) to consider the complex:

$$0 \to R \xrightarrow{x} R \to 0.$$

Its zero-dimensional homology is the module, $R/(x)$, and its one-dimensional homology is the ideal, $(0) : x = \{y \in R / xy = 0\}$. Thus, the vanishing of the first tells us that the element, x, is a unit, and the

vanishing of the second tells us that x is not a zero-divisor in R. If we replaced R by the R-module, M, and considered the complex

$$0 \to M \xrightarrow{x} M \to 0,$$

the vanishing of the zero-dimensional homology would tell us that every element of M is divisible by x, and the vanishing of the one-dimensional homology would say that x is a regular element for M, that is, that there is no non-zero element of M that is annihilated by x. We can extend the notion of regular element to the notion of **regular sequence of elements** for a module, M: the sequence, $\{x_1, \ldots, x_n\}$, of elements of R is said to be regular for M, if for each $i = 0, \ldots, n-1$, the element x_{i+1} is regular for $M/(x_1, \ldots, x_i)M$. When $i = 0$, this just means that x_1 is regular for M. One also adds the condition that $(x_1, \ldots, x_n)M \neq M$.

Now, if we have a complex, $\mathbf{C}$, of R-modules, and an element $x \in R$, multiplication by the element x on $\mathbf{C}$ yields a map of the complex into itself. The mapping cone of this map of complexes is another complex which we can denote by $\mathbf{C}(x)$.[2] Given a sequence of elements, $\{x_1, \ldots, x_n\}$, we can iterate this procedure[3] and obtain the complex, $\mathbf{C}(x_1, \ldots, x_n)$. In particular, given a module, M, and the sequence of elements $\{x_1, \ldots, x_n\}$, we can form the complex, $M(x_1, \ldots, x_n)$; this is known as the **Koszul complex over M associated to the sequence, $\{x_1, \ldots, x_n\}$.**

A very nice relationship between this homological construction and the notion of regular sequence is the following:

If R is a local ring, M an R-module, and $\{x_1, \ldots, x_n\}$ a sequence of elements in the maximal ideal of R such that $M/(x_1, \ldots, x_n)M \neq 0$, then $\{x_1, \ldots, x_n\}$ is a regular M-sequence if and only if

$$H_1(M(x_1, \ldots, x_n)) = 0.^4$$

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[2] If $\mathbf{A}$ and $\mathbf{B}$ are complexes, and $f : \mathbf{A} \to \mathbf{B}$ is a map (of degree 0), the mapping cone of f is the complex, $\mathbf{Y}$ defined by: $Y_n = A_{n-1} \bigoplus B_n$, and with the boundary map, $d_f : \mathbf{Y} \to \mathbf{Y}$ given by $d_f(a, b) = (d_{\mathbf{A}}(a), d_{\mathbf{B}}(b) + (-1)^{n-1} f(a))$.

[3] Here we are using the fact that R is commutative; if R weren't commutative, we could still iterate this procedure if we assumed that the elements x_i commuted among themselves.

[4] The vanishing of H_1 is equivalent to the vanishing of H_i for all $i > 0$ in the case of a local ring.

For simplicity, we'll assume from now on that our rings, R, are all local rings.

Regular sequences have played a pretty important role in local ring theory. In fact, they are fundamental in defining a sort of arithmetic notion of "size" of an ideal and of a local ring.

In noetherian ring theory, if we are given an ideal, I, we use the rank or height of I as a measure of its "size" (really tied up with a geometric notion of dimension). If we're given a ring, R, we use its Krull dimension as a measure of its "size." I say they are a geometric type of measure since they are connected with the lengths of nested sequences of varieties. We say that the **depth** of an ideal, I, is the length of the longest R-regular sequence contained in I, and the **codimension** of a local ring is the depth of its maximal ideal. Since the regularity of a sequence is a more or less arithmetic idea, I think of depth and codimension as an "arithmetic measure" of size. In a Cohen-Macaulay ring, these two measures coincide; for that reason, the application of algebraic techniques to Cohen-Macaulay varieties is so effective.

For a local ring, then, we have two integers connected with its size: its codimension, and its dimension (or Krull dimension). There is also another well-known integer associated with a local ring: its **embedding dimension**, written $\operatorname{edim}(R)$. This number is the smallest number of elements required to generate the maximal ideal of the ring.

The Krull Principal Ideal Theorem tells us that the embedding dimension of a local ring is always greater than or equal to its dimension, and it's easy to show that the dimension of a local ring is always greater than or equal to its codimension. The classical definition of a **regular local ring** is that a local ring is regular if its embedding dimension is equal to its dimension. A classical question about regular local rings, unanswered until the advent of homological tools, was the following: If R is a regular local ring, and $\mathfrak{p}$ a prime ideal of R, is the localization, $R_{\mathfrak{p}}$, again regular?

To see how the introduction of homological tools helped to solve this problem, we define a few more constants associated with a local ring.

The first is **projective dimension of a module** (and this idea goes straight back to the Hilbert paper, [13]). If

$$\mathbf{P} : \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

is a complex of projective modules (P is **projective** if it is the direct summand of a free module), we say it is a **projective resolution of the module,** M, if $H_0(\mathbf{P}) = M$ and $H_i(\mathbf{P}) = 0$ for all $i > 0$. Every module has a projective resolution and, since over a local ring, all projective modules are free, the notions of a free resolution (the only ones considered by Hilbert) and a projective one, coincide.

We say that such a resolution is finite if there is an integer, m, such that $P_i = 0$ for all $i > m$, and we define the length of such a resolution to be the smallest such integer m. We say that a module, M, has finite projective dimension if it has a finite projective resolution, and in that case we define its projective dimension, $\mathrm{pd}_R(M)$, to be the smallest of the lengths of its projective resolutions. If a module has no finite resolution, we say its projective dimension is infinite.

If R is a local ring, we define the **global dimension of** R, denoted by $\mathrm{gldim}(R)$, to be $\sup_M \mathrm{pd}_R(M)$, where M runs over all R-modules. We define the **finitistic global dimension of** R, denoted by $\mathrm{fgldim}(R)$, to be $\sup_N \mathrm{pd}_R(N)$, where N runs over all R-modules such that $\mathrm{pd}_R(N) < \infty$.

It was known almost from the inception of homological algebra that if R is a regular local ring, then $\mathrm{fgldim}(R)$ is finite (and equal to its dimension). This is pretty much the Hilbert Syzygy Theorem for local rings. If it could be shown that finite global dimension characterized regular local rings, then the problem of the localization by a prime ideal would have an affirmative answer. For it's known that the global dimension of the localization of a ring, R, is always less than or equal to that of R. Therefore, it was pretty evident that this homological characterization of regularity would be quite useful.

Much of the foregoing discussion, as well as the proof of the following result, can be found in [4].

THEOREM

If R is a local ring, the following inequalities always hold:

$$\text{fgldim}(R) = \text{codim}(R) \leq \dim(R) \leq \text{edim}(R) \leq \text{gldim}(R).$$

An immediate corollary of this is the following:

COROLLARY

A local ring, R, is regular if and only if $\text{gldim}(R)$ is finite.

Another classical problem that defied solution without the use of homological tools was to show that every regular local ring is factorial. One clue that factoriality and homology are related is that, for a commutative ring, R, factoriality is equivalent to $\text{pd}_R R/(x,y) \leq 2$ for every pair of elements, x, y, in R. The proof that every regular local ring is factorial is a bit long for this talk, but it can be found in [4].

4. Further ramblings in local ring theory

We spoke earlier about the Hilbert function, or Hilbert polynomial, but didn't relate it to local ring theory directly. In 1947, Pierre Samuel [14] used a generalization of the Hilbert function to define the dimension of a local ring, as well as its multiplicity. The scheme was this: we let R, as usual, be a local ring with maximal ideal, $\mathfrak{m}$. Then for every integer ν, the module $R/\mathfrak{m}^\nu$ has finite length as an R-module, and we define $\chi(\nu) = \text{length}(R/\mathfrak{m}^\nu)$. The degree, d, of this polynomial is the dimension of R, and the leading coefficient of this polynomial divided by $d!$ is the multiplicity of R. This, and the study of intersection multiplicities, made the Hilbert polynomial a basic tool in algebraic geometry and ring theory.

Samuel used this approach to define a more general theory of intersection multiplicity than existed at that time, and in around 1957 Serre proposed a definition that involved the functors Tor (this, too, is discussed in more detail in [4]). While Serre's definition was shown to be "right" for unramified regular local rings, there were many questions left standing for the ramified case. Here, A. Grothendieck stepped in with a suggestion. Since every ramified regular local ring, R, is of the

form $S/(x)$, where S is an unramified regular local ring and x an element of S, he pointed out that the difficulties in the ramified case could be resolved if one could solve the "Lifting Problem." This problem can be stated this way.

Let S be a local ring, x an element of S which is regular in S, $R = S/(x)$, and let M be an R-module. Is there an S-module, $\bar{M}$, such that i) $\bar{M}/x\bar{M} \cong M$ and ii) x is regular for $\bar{M}$? Clearly item ii) is the difficult one to satisfy, since we could always take $\bar{M}$ to be M itself, if not for that condition.

A fairly straightforward way to proceed might be this. We let

$$\mathbf{F} : \cdots \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0$$

be a free resolution of M over R.

By choosing bases for the free modules of the resolution, the maps may be described by matrices. Now let

$$\bar{\mathbf{F}} : \cdots \to \bar{F}_k \to \bar{F}_{k-1} \to \cdots \to \bar{F}_1 \to \bar{F}_0$$

be a "lifting" of the complex $\mathbf{F}$. This means that the barred modules are free of the same rank as the corresponding non-barred free modules, and the maps are matrices whose entries in S are representatives of the entries in the corresponding matrices over R. (That is, if $a_{ij} \in R$ is a matrix entry, we let $\bar{a}_{ij} \in S$ be an element which, modulo (x), is equal to a_{ij}.) In short, $\bar{\mathbf{F}}/x\bar{\mathbf{F}} = \mathbf{F}$.

Since x is regular in S, we have the short exact sequence:

$$0 \to \bar{\mathbf{F}} \xrightarrow{x} \bar{\mathbf{F}} \to \mathbf{F} \to 0,$$

which, if it were a sequence of complexes, would imply that the barred complex is acyclic, and its 0-dimensional homology would be a lifting of M.

But the problem is how to lift the matrices in $\mathbf{F}$ in such a way that the sequence of modules and maps forming $\bar{\mathbf{F}}$ is a complex.

In the case of a module, M, of projective dimension 0 or 1, this is clearly not a problem. The first case that must be seriously considered, then, is that of $\mathrm{pd}_R(M) = 2$.

To make a long story short, it is possible to show in this low-dimensional case that a lifting is possible,[5] but a number of counterexamples have been produced to show that in this generality, lifting is not always possible in higher dimensions. However, it is important to note that the essential step used in lifting the two-dimensional module was to start with the matrix at the "tail" of the resolution (hence very heavy reliance on the finite-dimensionality of the module, M), and to transport the information it carried, forward to the other terms of the resolution. In fact, this kind of analysis of a finite free resolution led Eisenbud and me to work on the structure of finite free resolutions in general (see [6], [7] and [8]).

The study of matrices and determinantal ideals was essential in the work on the Lifting Problem, but questions about determinantal ideals had been percolating for a long time before that. Because of the success of the application of the Koszul complex to so many of the classical algebraic problems: characterization of regularity; multiplicity and depth; characterization of Cohen-Macaulay rings, and because of the existence of the generalized Cohen-Macaulay Theorem, it seemed natural to try to find a generalization of that complex that would be associated to the presentation of a (finitely presented) module, rather than just to a cyclic one (which is what the Koszul complex does). This little problem took more effort and time than one would have originally thought it should, but the project ended up with joint work with D. S. Rim, namely [5] and [11]. A little before the work with Rim appeared, there was a paper by Eagon and Northcott [12], which associated a complex to the ideal generated by the maximal minors of a matrix. The family of complexes in [5] and [11] included one of the Eagon-Northcott type (it was much "fatter"), but also included a whole family all related to each other via the ideal of maximal minors of the presentation matrix of a module. These new complexes did yield a proof of the generalized Cohen-Macaulay Theorem, as well as a generalized multiplicity, which

[5]This was first done using a theorem I had proven about determinantal ideals, which I later learned had also been proven independently by a number of other mathematicians for a number of different reasons. The first person in this list was a mathematician named L. Burch. However Kaplansky discovered that a form of this theorem had originally been proven by Hilbert, so the theorem is now known as the Hilbert-Burch Theorem.

was left in the dust for many years, but then resuscitated in the work of Kirby, Rees and Tierney, and expanded by Kleiman and Thorup. References to this work, and some discussion of this material, can be found in [4].

After the work with Eisenbud on finite free resolutions, we turned our attention briefly to these generalized Koszul complexes, and tried to "slim them down" so that we could recover the Eagon-Northcott complex in some systematic way. This project culminated in the paper [9].

The "slimming-down" process that was used led into the realm of Weyl modules, and thus into representations of the general linear group. The work was fairly technical, but because it was this development that led us back to representation theory and Hilbert, it may be worth taking a detour into some more personal anecdotes. First, let me clear up a matter of notation: why do many authors use the letter K to stand for Weyl modules instead of, say, W? After all, the Schur modules had classically been denoted by the letter S. Furthermore, the same authors who use the letter K for Weyl modules tend to use the letter L for Schur modules. How come?

This state of affairs was brought about by a combination of serendipity and ignorance. It was clear that the slimming down process we were looking for depended upon looking at the kernels of some maps whose domains were the terms in our fat complexes. As I never was much of a cherisher of notation (or terminology, for that matter), I simply denoted these kernels by K (what else?), with suitable indexing to indicate what these were the kernels of.

The ignorance factor resides in the fact that at the time, neither Eisenbud nor I had any serious knowledge of Weyl and Schur modules, so that we didn't recognize these Ks for the Weyl modules they were. However, their duals eventually became equally important to us, so the natural thing to do was to call them Ls. These Ls are what the world knows as the Schur modules of classical representation theory.

In any event, serendipity and ignorance have often been behind what can turn out to be interesting material, as is the case here.

One surprising thing is that, although the paper [9] was written in 1975, no one thought to rigourously compare the old fat complexes with the

new slim ones. It wasn't until Boffi and I were writing the book, [4], that we realized there was this gap, so we wrote [3] to fill it. It turns out that these two classes are homotopically equivalent, as we may all have expected.[6] It also became apparent from the proof why it was that so much time had elapsed between the establishment of these classes and the proof that they were essentially the same: the proofs involve fairly heavy use of representation theory and combinatorics, topics that neither Eisenbud nor I had much command of in those earlier years.

5. Determinantal ideals

During the search for a generalized Koszul complex, it became clear that it would be useful to have not only a complex that was associated to the maximal minors of a matrix, but also complexes associated to the minors of arbitrary order of a matrix. A good deal of time was spent on this problem, but little headway was made. However, after the work on [9], Eisenbud went off to Paris for the year. There, through Verdier, he met A. Lascoux, who asked him if he could suggest some application of his combinatoric and representation-theoretic results to something in commutative algebra. Eisenbud mentioned to him that certain representation modules apparently found their way into resolutions that we had been considering, so he suggested that this might also be the case with resolutions of determinantal ideals of any size. I should mention that as early as 1966, A. Andreotti and I had concluded that, "philosophically," the terms of such a resolution should be representation modules of $Gl(F) \times Gl(G)$, if the matrix represented a map from F to G, but we just couldn't see how to convert that philosophy into reality. Well, ten years later, at the suggestion of Eisenbud, Lascoux found the way to do just that. The methods Lascoux used were very much those of characteristic zero, so that, while his resolutions could be of real interest to geometers (especially differential geometers), there was still the nagging question of how to make this approach universal.

[6] As far as I know, no one has gone back to the paper, [11], to see if the definition and properties of the generalized multiplicity are easier to handle with this new class of thin complexes.

Over the next few years, two of my students, K. Akin and J. Weyman, and I, found a convenient generalization of the classical representation theory to the characteristic-free case. I say "convenient" because it had to be a theory with a sufficiently large class of modules to allow the introduction of homological technique (such as filtrations and exact sequences) that could replace the combinatorics and counting methods that are so powerful in characteristic zero. This meant that our class of shapes had to consist of far more than partitions, or even skew-partitions. In fact, a new class of shapes came into play (almost skew-shapes), and that set into motion a whole new interaction of homological methods with classical algebra, in particular with representation theory. And so we've completed a circle, and returned to the area that Hilbert purportedly "killed off."

6. Stepping further into representation theory

In fact, no sooner had Akin, Weyman and I defined these characteristic-free analogues of Weyl and Schur modules than we saw some lovely problems arise in connection with $\mathbb{Z}$-forms. What happened was that with the new definitions, it first of all became clear that the terms and maps of the Lascoux resolutions would not work as resolutions of generic determinantal ideals; a simple aping of these terms produced torsion homology, rather than trivial homology. Part of the problem, which we understood with some degree of insight, was that there are different $\mathbb{Z}$-forms of the same rational representation. For example, over the rationals, $\mathbb{Q}$, there is the representation S_n, the n-th symmetric power. However, over the integers, one has both S_n and the inequivalent n-th divided power, D_n. When you tensor over the rationals, the two become equivalent; hence the representation, D_n, is called a $\mathbb{Z}$-form of S_n.

In the case of the resolution of the ideal of submaximal minors, [2], Akin, Weyman and I found that certain $\mathbb{Z}$-forms of the hook representations obtruded, and we had to find ways to handle these so that we would rid the complex we were constructing of torsion homology.

What we had to do was prove and use the rather strange fact that the complex

$$(1)\quad 0 \to \Lambda^k \to D_1 \otimes \Lambda^{k-1} \to \cdots \to D_l \otimes \Lambda^{k-l} \to \cdots \to D_k \to 0,$$

in which the maps entail diagonalizing the exterior power and multiplying in the divided power, is exact from Λ^k up to $l = [\frac{k}{2}]$; that is, exact till halfway up. Thus, the kernels (or images) in those dimensions are universal $\mathbb{Z}$-free representations of Gl_n which, when tensored with the rationals, $\mathbb{Q}$, are isomorphic to the Schur modules of the hooks in the corresponding dimensions (since the Schur hooks are the kernels of the corresponding complex where the divided power, D, is replaced by the symmetric power, S). This is a less trivial example of what we're calling $\mathbb{Z}$-forms of the same rational representation of Gl_n.

To illustrate how to construct non-isomorphic $\mathbb{Z}$-forms and their relation to resolutions of Weyl modules, consider the following.

We have the short exact sequence

$$0 \to D_{k+2} \to D_{k+1} \otimes D_1 \to K_{(k+1,1)} \to 0$$

where $K_{(k+1,1)}$ is the Weyl module associated to the hook partition $(k+1,1)$. If we take an integer, t, and multiply D_{k+2} by t, we get an induced exact sequence and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & D_{k+2} & \to & D_{k+1} \otimes D_1 & \to & K_{(k+1,1)} & \to & 0 \\ & & \downarrow t & & \downarrow & & \downarrow & & \\ 0 & \to & D_{k+2} & \to & E(t; k+1, 1) & \to & K_{(k+1,1)} & \to & 0, \end{array}$$

where $E(t; k+1, 1)$ stands for the cofiber product of D_{k+2} and $D_{k+1} \otimes D_1$. Each of these modules is a $\mathbb{Z}$-form of $D_{k+1} \otimes D_1$, but for t_1 and t_2, two such are isomorphic if and only if $t_1 \equiv t_2 \bmod k+2$ (see[1]). In fact, one can easily show that $\mathrm{Ext}^1_A(K_{(k+1,1)}, D_{k+2}) = \mathbb{Z}/(k+2)$, where A stands for the Schur algebra of appropriate degree (namely, $k+2$).

We now see how such forms are related to resolutions of Weyl modules. Consider the partition $(k,2)$, the associated Weyl module, $K_{(k,2)}$, and its resolution (over the integers):

$$0 \to D_{k+2} \to \{D_{k+2} \oplus D_{k+1} \otimes D_1\} \to D_k \otimes D_2 \to K_{(k,2)} \to 0.$$

The map of D_{k+2} into the indicated direct sum is the usual diagonalization into the second summand (or the place polarization ∂_{21}),[7] but is multiplication by 2 into the first. The map from D_{k+2} to $D_k \otimes D_2$ is the second divided power of ∂_{21} or $\partial_{21}^{(2)}$. As a result, we see that the cokernel of this map is what we called above, $E(2; k+1, 1)$. This shows how these $\mathbb{Z}$-forms show up in resolutions of Weyl modules.

Now the problem of resolving Weyl modules at first seemed to be easy; at least it was in the case of two-rowed shapes, where our approach required a proof of exactness of a certain fundamental sequence, but did not compel us to introduce a class of shapes beyond the skew-shapes. However, once we left the two-rowed situation, everything seemed to hit the fan: the short exact sequence we had used to such advantage before required us to introduce what are now called **almost skew-shapes**; the combinatorics of overlapping rows was far more complicated; the proof of the exactness of our sequences seemed to require an ungainly spectral sequence (where the terms of the n-th filtration, F_n, were described in terms of the dyadic expansion of n). By the end of the summer of 1982, I had calculated the resolution of the skew-shape with two three-fold overlaps, and realized that the tools at our command were insufficient to proceed intelligently with this project. Akin and I put this problem aside, and turned, instead, to the study of intertwining numbers (a topic we will not discuss here, but can be found in [1], [10] and [4]). These numbers arise in modular representation theory.

In 1990, Rota and I decided to get together to study this resolution problem and, after a few months of getting to understand each other's language, Rota introduced me to letter-place techniques and, in particular, place polarizations. A rather complete description of these notions, and the combinatorical proofs of the fact that these letter-place algebras have a basis consisting of double standard double tableaux filled by "letters" and "places," can be found in the later chapters of [4]; here I will just illustrate with an example or two. In any event, using these notions, a good deal of the incomprehensible morass of the 1982 calculations was cleared away, and much progress could again be made on the problem of resolving Weyl modules. "Much progress" doesn't mean that the problem is solved; it's still open. But at least we have

[7] We'll discuss these place polarizations later in this section.

a description of all the terms that appear in these resolutions, for all almost skew-shapes of any number of rows ([4]).

Let's look now at some letter-place ideas and place polarizations.

An element $w \otimes w' \in D_p \otimes D_q$ would be written, in letter-place algebra, as $(w|1^{(p)})(w'|2^{(q)})$ to indicate that it is the tensor product of a basis element of degree p in the first factor, and one of degree q in the second. This is then collected in double tableau form as

$$\left(\begin{array}{c|c} w & 1^{(p)} \\ w' & 2^{(q)} \end{array} \right).$$

If we further agree that the symbol $(v|1^{(p)}2^{(q)})$ means $\sum v(p) \otimes v(q) \in D_p \otimes D_q$, where v is an element of degree $p+q$ and the sum represents the diagonalization of v in $D_p \otimes D_q$, then we can also talk about the double tableau

$$\left(\begin{array}{c|c} w & 1^{(p)}2^{(k)} \\ w' & 2^{(q-k)} \end{array} \right),$$

which means $\sum w(p) \otimes w(k)w'$. Ordering the basis elements of the underlying free module, we can now talk about 'standard' and 'double standard' double tableaux. A major result on letter-place algebra is that the set of double standard double tableaux form a basis for $D_p \otimes D_q$ ([4]).

In general, one could talk about $D_{p_1} \otimes D_{p_2} \otimes \cdots \otimes D_{p_n}$ in letter-place terms, where the "places" run from 1 to n.

To illustrate the basis theorem: suppose $p < q$, and we have the element $a^{(p)} \otimes b^{(q)} \in D_p \otimes D_q$. Then, although $\left(\begin{array}{c|c} a^{(p)} & 1^{(p)} \\ b^{(q)} & 2^{(q)} \end{array} \right)$ is a basis element of $D_p \otimes D_q$, it isn't a double standard tableau (even assuming $a < b$ and $1 < 2$) since $p < q$.

To write $\left(\begin{array}{c|c} a^{(p)} & 1^{(p)} \\ b^{(q)} & 2^{(q)} \end{array} \right)$ as a linear combination of standard tableaux, we clearly must have

$$\left(\begin{array}{c|c} a^{(p)} & 1^{(p)} \\ b^{(q)} & 2^{(q)} \end{array} \right) = \sum_{l=0}^{p} c_l \left(\begin{array}{c|c} a^{(p)}b^{(q-p+l)} & 1^{(p)}2^{(q-p+l)} \\ b^{(p-l)} & 2^{(p-l)} \end{array} \right),$$

and we want to determine the coefficients c_l. Rewriting the above, we get

$$a^{(p)} \otimes b^{(q)} = \sum_{l=0}^{p} c_l \sum_{k=0}^{p} \binom{q-k}{p-l} a^{(p-k)} b^{(k)} \otimes a^{(k)} b^{(q-k)};$$

we want the c_l to be such

$$\sum_{l=0}^{p} c_l \binom{q-k}{p-l} = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ otherwise} \end{cases}.$$

Clearly, if we set $c_l = \binom{p-q}{l}$, then for $k = 0$, the sum above is

$$\sum_{l=0}^{p} \binom{p-q}{l} \binom{q}{p-l} = \binom{p}{p} = 1,$$

while for $k > 0$, we get

$$\sum_{l=0}^{p} \binom{p-q}{l} \binom{q-k}{p-l} = \binom{p-k}{p} = 0$$

as we wanted.

To illustrate a place polarization, let's look at the map

$$\square : D_{p+k} \otimes D_{q-k} \to D_p \otimes D_q$$

given by the composition

$$D_{p+k} \otimes D_{q-k} \to D_p \otimes D_k \otimes D_{q-k} \to D_p \otimes D_q,$$

where the first map is diagonalization, and the second is multiplication. From a letter-place perspective, let's write an element of $D_{p+k} \otimes D_{q-k}$ as $\left(\begin{array}{c|c} w & 1^{(p+k)} \\ w' & 2^{(q-k)} \end{array} \right)$, and let's "polarize" the place, 1, to 2, k times, that is let's send $1^{(p+k)}$ to $1^{(p)}2^{(k)}$. Then our element $\left(\begin{array}{c|c} w & 1^{(p+k)} \\ w' & 2^{(q-k)} \end{array} \right)$ goes to $\left(\begin{array}{c|c} w & 1^{(p)}2^{(k)} \\ w' & 2^{(q-k)} \end{array} \right)$. If we write out what this means in terms of tensor products of divided powers, we see that this last element is precisely the element in $D_p \otimes D_q$ that is the image of the map, $\square$, applied to $w \otimes w'$.

Recognizing that the maps Akin and I had been working with were these polarizations, and then proving some Capelli-like identities for these maps, we were able to express all the inexplicable identities of the 1982 work in a coherent way, and thus able to proceed to consideration of the general case.

While all of this may seem terribly *ad hoc*, these operations can all be described in the conventional terms that algebraists are accustomed to. Again I refer the reader to [4] for the details.

7. Weyl-Schur complexes

The above discussion dealt only with the letter-place aspect of tensor products of divided powers; in this case, the letters and places are all considered to be positive. One may deal with letters and places of negative sign, as well as of mixed sign, and corresponding place polarizations. When one takes this to the extreme (that is, letter and place "alphabets" of mixed signs), one can generalize these Weyl modules to complexes, called Weyl-Schur complexes. When we do this (see [4]), we find that the fundamental exact sequences that I mentioned earlier, can be proven exact without recourse to any spectral sequence argument.

References

[1] K. Akin and D.A. Buchsbaum, *Characteristic-free representation theory of the general linear group, II. Homological considerations*, Adv. Math. **72** (1988), 171–210.

[2] K. Akin, D.A. Buchsbaum and J. Weyman, *Resolutions of determinantal ideals: The submaximal minors*, Adv. Math. **39** (1981), 1–10.

[3] G. Boffi and D.A. Buchsbaum, *Homotopy equivalence of two families of complexes*, Trans. Am. Math. Soc. **356** (2004), 3077–3107.

[4] G. Boffi and D.A. Buchsbaum, *Threading Homology Through Algebra: Selected Patterns*, Oxford University Press, 2006

[5] D.A. Buchsbaum, *A generalized Koszul complex, I*, Trans. Am. Math. Soc. **111** (1964), 183–196.

[6] D.A. Buchsbaum and D. Eisenbud, *What makes a complex exact?*, J. Alg. **25** (1973), 259-268.

[7] D.A. Buchsbaum and D. Eisenbud, *Some structure theorems for finite free resolutions.* Adv. in Math. **12** (1974), 84-139.

[8] D.A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447-485.

[9] D.A. Buchsbaum and D. Eisenbud, *Generic free resolutions and a family of generically perfect ideals.* Adv. in Math. **18** (1975), 245-301.

[10] D.A. Buchsbaum and D. Flores de Chela, *Intertwining numbers, the three-rowed case*, J. Algebra **183** (1996), 605–635.

[11] D.A. Buchsbaum and D.S. Rim, *A generalized Koszul complex, II. Depth and Multiplicity*, Trans. Am. Math. Soc. **111** (1964), 197–224.

[12] J. Eagon and D.G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Royal Soc. A **269** (1962), 188–204.

[13] D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473-534.

[14] P. Samuel, *Une gnralisation des polynomes de Hilbert* C. R. Acad. Sci. Paris 225, (1947). 1111–1113.