

ON THE COSMOLOGICAL CONSTANT, THE VACUUM ENERGY, AND DIVERGENT SERIES¹

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Abstract

The cosmological constant was considered by Albert Einstein as the ‘silliest mistake of his life’ (“die grösste Eselei meines Lebens”). Quite on the contrary, nowadays it has turned out to be an absolutely necessary term in order to have the possibility to explain the observed acceleration in the expansion of the universe. Such term is unavoidably related with the energy of the quantum vacuum, from which it cannot be easily disentangled in the observational result: both contributions go together and are to be added in the final value. The concept of vacuum energy of a quantum system is most fundamental and appears in very different situations. In particular it becomes manifest in the so called Casimir force. When computing the corresponding physical quantities one has to deal, from the very beginning, with divergent series and determinants, which must be regularized through procedures that neither Einstein himself nor Paul Dirac (among other much distinguished physicists) could ever admit –but which have given the most outstanding approximation (to the fourteenth order) between any known physical theory and experiment to date. In mathematics this does not seem to be such a problem: some centuries ago Euler already maintained the idea that one should be able to assign to any given series a certain number, in some reasonable way. A method extraordinarily elegant and useful in order to do this (at least for a large family of divergent series) starts from the consideration of the ζ function associated with the Hamiltonian operator of the quantum system in question. It was Hawking who definitely introduced this method into physics, through a famous paper published in *Communications on Mathematical Physics* in 1975.

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1 Introduction

The question, phrased by Eugene Wigner as that of *the unreasonable effectiveness of mathematics in the natural sciences* [1] is an old and intriguing one. It goes back to Pythagoras and his school (*ca. 550 BC*, “*all things are numbers*”), even probably to the Sumerians and Babylonians, and maybe to more ancient cultures, which left no trace. In more recent times, Immanuel Kant said that *the eternal mystery of the world is its comprehensibility* and Albert Einstein adhered also to this same idea in an essay written in 1936, “*Physics and Reality*”.³ More to the point, *mathematical simplicity, and beauty*, have remained for many years crucial ingredients when having to choose among different plausible theories. However, it should be pointed out that we scientists no more use such tall words as ‘comprehend’ or ‘understand’ but the much more humble ones ‘describe,’ ‘characterize’ or ‘model.’ Let’s bring up just one example: the fact that Newton’s law for the attraction of two bodies is both so extraordinarily simple and general, covering such a large number of scales from the terrestrial bodies to the solar systems does not mean at all that we *understand* the universe but only that we are able to describe it in very simple and general terms. Of course, after these considerations the deep mystery above still remains unchanged: why is the cosmos so easy to model or describe by the minute human mind?

An example of *unreasonable effectiveness* is provided by the regularization procedures in quantum field theory (QFT) based upon analytic continuation in the complex plane (dimensional, heat-kernel, zeta-function regularization, and the like). That one obtains a physical, experimentally measurable, and extremely precise result after these weird mathematical manipulations is, if not utterly unreasonable, certainly very mysterious. More one highly honorable physicist (including Einstein himself and also Paul Dirac, who was a genius in the application of mathematics into physics) always considered those to be illegal practices. Such methods are nowadays fully justified and blessed with Nobel Prizes, but more because of the many and very precise experimental checkouts (the *effectiveness*) than for their intrinsic *reasonableness*.

The fact that the infinite series

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \tag{1}$$

has the value $s = 1$ is nowadays clear to any school student. It was not so, even to well learned men, for many centuries, as we can recall from *Zeno of Elea’s paradox* (or Zeno’s paradox of the tortoise and Achilles), transmitted by Aristotle and based on the pretended impossibility to do an *infinite* number of summations (or recurrent ‘jumps’ or steps of any kind, in a *finite* amount of time). In fact there are still modern versions of the Zeno paradox (e.g. the quantum Zeno paradox) which pop up now and then. [2] We shall not

³With such a success that many consider him now as the author of this deep and beautiful sentence. By the way, this essay has been freely distributed, both in English and German, with the occasion of the Year of Physics 2005.

discuss those here, but rather assume that the following process is clear to the reader: by taking the first term, $1/2$, to the left what remains on the r.h.s. is just one half of the original series (i.e., $1/2$ is a common factor), so that

$$s - \frac{1}{2} = \frac{s}{2} \implies s = 1. \quad (2)$$

Thus the conclusion follows that when to one half of an apple pie we add a quarter of it and then an eighth, and so on, what we get in the end is the whole pie.

Now, something more difficult: what is the sum of the following series?

$$s = 1 + 1 + 1 + 1 + \dots \quad (3)$$

Again, any of us will answer immediately: $s = \infty$. In fact, whatever ∞ is, everybody recognizes in this last expression the definition itself of the concept of infinity, e.g. the piling of one and the same object, once and again, without an end. Of course, this idea is absolutely true, but it is at the same time of little use to modern Physics. Being more precise, since the advent of Quantum Mechanics and Quantum Field Theory. In fact, calculations there are plagued with divergent series, and it is of no use to say that: look, this series here is divergent, and this other one is also divergent, and the other there too, and so on. One gets non-false but also non-useful information in this way, and actually we *do not observe* these many infinities in Nature. Thus it was discovered in the 30's and 40's that something very important was missing in the formulation or mathematical modelization of quantum physical processes.

Within the mathematical community, for years there was the suspicion that one could indeed give sense to divergent series. This has now been proven experimentally (with 10^{-14} accuracy in some cases) to be true in Physics, but it were the mathematicians —many years before— who first realized such possibility. In fact, Leonard Euler (1707-1783) was convinced that *“To every series one can assign a number”* [3] (that is, in a reasonable, consistent, and possibly useful way, of course). Euler was unable to prove this statement in full, but he devised a technique (Euler's summation criterion) in order to ‘sum’ a large family of divergent series. His statement was however controverted by some other great mathematicians, as Abel, who said that *“The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever”*. [4] There is a classical treatise due to G.H. Hardy and entitled simply *Divergent series* [5] that can be highly recommended to the reader.

Actually, regularization and renormalization procedures are essential in present day Physics. Among the different techniques at hand in order to implement these processes, zeta function regularization is one of the most beautiful. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system, which could, e.g., contribute to the cosmic force leading to the present acceleration of the expansion of our universe. The zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically (from a mathematical point of view) to the computation of determinants of elliptic pseudodifferential

operators (Ψ DOs) [6]. It is thus no surprise that the preferred definition of determinant for such operators is obtained through the corresponding zeta function (see, e.g., [7, 8]).

2 Some basic considerations on divergent series

As usual in modern Mathematics, one starts the attack on divergent series by invoking a number of ‘reasonable’ axioms, like [5]

1. If $a_0 + a_1 + a_2 + \dots = s$, then $ka_0 + ka_1 + ka_2 + \dots = ks$.
2. If $a_0 + a_1 + a_2 + \dots = s$, and $b_0 + b_1 + b_2 + \dots = t$, then $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + \dots = s + t$.
3. If $a_0 + a_1 + a_2 + \dots = s$, then $a_1 + a_2 + \dots = s - a_0$.

A couple of examples.

1. Using the third axiom we obtain that for the series $s = 1 - 1 + 1 - 1 + \dots$, we have $s = 1 - s$, and therefore $s = 1/2$. This value is easy to justify, since the series is oscillating between 0 and 1, so that $1/2$ is the more ‘democratic’ value for it.
2. Using now the second axiom, we obtain that for the series $t = 1 - 2 + 3 - 4 + \dots$, it turns out by subtracting it term by term from the former one that $s - t = t$, and therefore $t = s/2 = 1/4$. Such result is already quite difficult to swallow. This is in common with most of the finite values that are obtained for infinite, divergent series.

But, what to do about our initial series $1 + 1 + 1 + \dots$? This one is most difficult to tame, and the given axioms do not serve to this purpose. But there is more to the axioms, which are only intended as a humble starting point. By reading Hardy’s book [5] one learns about a number of different methods that have been proposed and is good to know. They are due to Abel, Euler, Cesàro, Bernoulli, Dirichlet, Borel and some other mathematicians.⁴ The most powerful of them involve analytic continuation in the complex plane, as is the case of the so called zeta regularization method.

Thus, for instance, a series

$$a_0 + a_1 + a_2 + \dots \tag{4}$$

will be said to be *Cesàro summable*, and its sum to be the number s , if the limit of partial sum means exists and gives s , namely

$$\exists \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} \frac{A_n}{n} = s, \quad A_n \equiv \sum_{j=1}^n a_j. \tag{5}$$

⁴Padé approximants should in no way be forgotten in this discussion. [9]

This criterion can be extended and gives rise to a whole family of criteria for *Cesàro summability*. On the other hand, the series before will be said to be *Abel summable*, and its sum to be the number s ,

$$\sum_{n=0}^{\infty} a_n = s, \quad (6)$$

if the following function constructed as a power series

$$f(x) \equiv \sum_{n=0}^{\infty} a_n x^n \quad (7)$$

is well defined for $0 < x < 1$ and the limit when x goes to 1 from the left exists and gives s , namely,

$$\exists \lim_{x \rightarrow 1^-} f(x) = s. \quad (8)$$

And similarly for the rest of the criteria, [5] which are not equivalent, as can easily be checked.

2.1 The zeta function as a summation method

The method of zeta regularization evolved from the consideration of the Riemann zeta function as a ‘series summation method’. The zeta function, on its turn, was actually introduced by Euler, from considerations of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots, \quad (9)$$

which is logarithmically divergent, and of the fact that, putting a real exponent s over each term,

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots, \quad (10)$$

then for $s > 1$ the series is convergent, while for $s \leq 1$ it is divergent. Euler called this expression, as a function of s , the ζ -function, $\zeta(s)$, and found the following important relation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (11)$$

which is crucial for the applications of this function in Number Theory. By allowing the variable s to be complex, Riemann saw the relevance of this function (that now bears his name) for the proof of the prime number theorem⁵, and formulated thereby the *Riemann hypothesis*, which is one of the most important problems (if not *the* most) in the history of Mathematics. More to that in the excellent review by Gelbart and Miller. [10]

⁵Which states that the number $\Pi(x)$ of primes which are less than or equal to a given natural number x behaves as $x/\log x$, as $x \rightarrow \infty$. It was finally proven, using Riemann’s work, by Hadamard and de la Vallée-Poussin.

For the Riemann $\zeta(s)$, the corresponding complex series converges absolutely for the open half of the complex plane to the right of the abscissa of convergence $\text{Re } s = 1$, while it diverges on the other side, but it turns out that it can be analytically continued to that part of the plane, being then everywhere analytic and finite except for the only, simple pole at $s = 1$ (Fig. 1).⁶ In more general cases, namely corresponding to the Hamiltonians

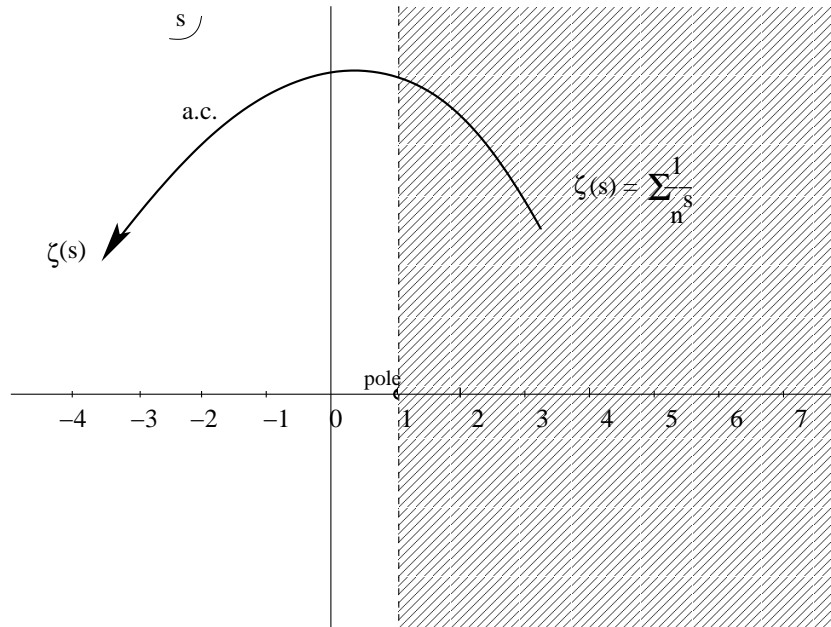


Figure 1: The zeta function $\zeta(s)$ is defined in the following way, on the whole complex plane, $s \in \mathbf{C}$. On the open half of the complex plane lying on the r.h.s of the abscissa of convergence $\text{Re } s = 1$, ζ is defined as the absolutely convergent series: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. In the rest of the s -complex plane, $\zeta(s)$ is defined as the (unique) analytic continuation of the preceding function, which turns out to be meromorphic. Specifically, it is analytic everywhere except for one simple pole, at the point $s = 1$, with residue equal to 1 (it corresponds to the logarithmically divergent harmonic series).

which are relevant in physical applications, [11, 12, 13] the situation is in essence quite similar, albeit in practice it can be rather more involved. A mathematical theorem exists, which assures that under very general conditions the zeta function corresponding to a Hamiltonian operator will be also meromorphic, with just a discrete number of possible poles, which are simple and extend to the negative side of the real axis.⁷

The above picture already hints towards the use of the zeta function as a summation method. Let us consider two examples.

⁶Where it yields the harmonic series: there is no way out for this divergence.

⁷Although there are some exceptions to this general behavior, they correspond to rather twisted situations, and are outside the scope of this brief presentation. [14]

1. We interpret our starting series

$$s_1 = 1 + 1 + 1 + 1 + \dots \quad (12)$$

as a particular case of the Riemann zeta function, e.g. for the value $s = 0$. This value is on the left hand side of the abscissa of convergence (Fig. 1), where the series as such diverges but where the analytic continuation of the zeta function provides a perfectly finite value:

$$s_1 = \zeta(0) = -\frac{1}{2}. \quad (13)$$

So this is the value to be attributed to the series $1 + 1 + 1 + 1 + \dots$.

2. The series

$$s_2 = 1 + 2 + 3 + 4 + \dots \quad (14)$$

corresponds to the exponent $s = -1$, so that

$$s_2 = \zeta(-1) = -\frac{1}{12}. \quad (15)$$

A couple of comments are in order.

- In a short period of less than a year, two distinguished physicists, A. Slavnov and F. Yndurain, gave seminars in Barcelona, about different subjects. It was remarkable that, in both presentations, at some point the speaker addressed the audience with these words: “*As everybody knows, $1 + 1 + 1 + \dots = -1/2$ ”.*⁸
- That positive series, as the ones above, can yield a *negative* result may seem utterly nonsensical. However, it turns out that the most precise experiments ever carried out in Physics do confirm such results. More precisely: models of regularization in QED built upon these techniques lead to final numbers which are in agreement with the experimental values up to the the 14th figure. [15] In recent experimental proofs of the Casimir effect [16] the agreement is also quite remarkable (given the difficulties of the experimental setup). [17]
- The method of zeta regularization is based on the analytic continuation of the zeta function in the complex plane. Now, how easy is to perform that continuation? Will we need to undertake a fashionable complex-plane computation every time? It turns out that this is not so. The result is immediate to obtain, in principle, once you know the appropriate reflection formula (also called functional equation) that your zeta function obeys: in the case of the Riemann zeta $\xi(s) = \xi(1 - s)$, $\xi(s) \equiv \pi^{-s/2} \Gamma(s/2) \zeta(s)$. In practice these formulas are however not optimal for actual calculations, since they are ordinarily given in terms of power

⁸Implying maybe: *If you do not know this it is no use to continue listening.* Remember by the way the lemma of the Pythagorean school: *Do not cross this door if you do not know Geometry.*

series expansions (as the Riemann zeta itself). Fortunately, sometimes there are more clever expressions, that can be found, which converge exponentially fast, as the Chowla-Selberg [18] formula and some others. [19, 20] Those give real power to the method of zeta regularization. More about this point to come below, where a number of such expressions will be explicitly found and used in some applications.

3 The zeta function of a Ψ DO and its associated determinant

The existence conditions of the zeta function of a pseudodifferential operator and the definition of determinant thereby obtained are reviewed, as well as the concept of multiplicative anomaly associated with the determinant and its calculation by means of the Wodzicki residue.[21]

3.1 Pseudodifferential operator (Ψ DO)

A *pseudodifferential operator* A of order m on a manifold M_n is defined through its symbol $a(x, \xi)$, which is a function belonging to the space $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ of \mathbf{C}^∞ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha, \beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}. \quad (16)$$

The definition of A is given, in the distribution sense, by

$$Af(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} a(x, \xi) \hat{f}(\xi) d\xi, \quad (17)$$

where f is a smooth function, $f \in \mathcal{S}$; remember that

$$\mathcal{S} = \left\{ f \in C^\infty(\mathbf{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \forall \alpha, \beta \in \mathbf{R}^n \right\}, \quad (18)$$

\mathcal{S}' being the space of tempered distributions and \hat{f} the Fourier transform of f . When $a(x, \xi)$ is a polynomial in ξ one gets a differential operator. In general, the order m can be complex. The *symbol* of a Ψ DO has the form

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_{m-j}(x, \xi) + \cdots, \quad (19)$$

being $a_k(x, \xi) = b_k(x) \xi^k$.

Pseudodifferential operators are useful tools, both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem [22] and also for the proof of the Atiyah-Singer index formula [23]. In quantum field theory they appear in any analytical continuation process (as complex powers of differential operators, like the Laplacian) [24]. And they constitute nowadays the basic starting point of any rigorous formulation of quantum field theory through microlocalization, a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the invention of distributions [25].

3.2 The zeta function

Let A a positive-definite elliptic Ψ DO of positive order $m \in \mathbf{R}$, acting on the space of smooth sections of E , an n -dimensional vector bundle over M , a closed n -dimensional manifold. The *zeta function* ζ_A is defined as

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} \equiv s_0. \quad (20)$$

where $s_0 = \dim M / \text{ord } A$ is called the *abscissa of convergence* of $\zeta_A(s)$. Under these conditions, it can be proven that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane \mathbf{C} (regular at $s = 0$), provided that the principal symbol of A (that is $a_m(x, \xi)$) admits a *spectral cut*:

$$L_\theta = \{\lambda \in \mathbf{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}, \quad \text{Spec } A \cap L_\theta = \emptyset \quad (21)$$

(Agmon-Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut L_θ . The only possible singularities of $\zeta_A(s)$ are *simple poles* at $s_k = (n - k)/m$, $k = 0, 1, 2, \dots, n - 1, n + 1, \dots$. M. Kontsevich and S. Vishik have managed to extend this definition to the case when $m \in \mathbf{C}$ (no spectral cut exists) [26].

3.3 The zeta determinant

Let A a Ψ DO operator with a spectral decomposition: $\{\varphi_i, \lambda_i\}_{i \in I}$, where I is some set of indices. The definition of determinant starts by trying to make sense of the product $\prod_{i \in I} \lambda_i$, which can be easily transformed into a “sum”: $\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i$. From the definition of the zeta function of A : $\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s}$, by taking the derivative at $s = 0$: $\zeta'_A(0) = -\sum_{i \in I} \ln \lambda_i$, we arrive to the following definition of determinant of A [27]:

$$\det_\zeta A = \exp[-\zeta'_A(0)]. \quad (22)$$

An older definition (due to Weierstrass) is obtained by subtracting in the series above (when it is such) the leading behavior of λ_i as a function of i , as $i \rightarrow \infty$, until the series $\sum_{i \in I} \ln \lambda_i$ is made to converge. The shortcoming is here —for physical applications— that these additional terms turn out to be *non-local* and, thus, are non-admissible in any renormalization procedure.

In algebraic QFT, in order to write down an action in operator language one needs a functional that replaces integration. For the Yang-Mills theory this is the Dixmier trace, which is the *unique* extension of the usual trace to the ideal $\mathcal{L}^{(1, \infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum: $\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N)$, $\mu_0 \geq \mu_1 \geq \dots$. The definition of the Dixmier trace of T is: $\text{Dtr } T = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sigma_N(T)$, provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \rightarrow \infty$ (remember that: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$). Then, the Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ (see Connes [28]): $\text{Dtr } T = \lim_{s \rightarrow 1^+} (s - 1) \zeta_{T^{-1}}(s)$.

3.4 The Wodzicki residue

The Wodzicki (or noncommutative) residue [29] is the *only* extension of the Dixmier trace to the Ψ DOs which are not in $\mathcal{L}^{(1,\infty)}$. It is the *only* trace one can define in the algebra of Ψ DOs (up to a multiplicative constant), its definition being: $\text{res } A = 2 \text{Res}_{s=0} \text{tr}(A\Delta^{-s})$, with Δ the Laplacian. It satisfies the trace condition: $\text{res}(AB) = \text{res}(BA)$. A very important property is that it can be expressed as an integral (local form) $\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x, \xi) d\xi$ with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue).

If $\dim M = n = -\text{ord } A$ (M compact Riemann, A elliptic, $n \in \mathbf{N}$) it coincides with the Dixmier trace, and one has $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1}$. The Wodzicki residue continues to make sense for Ψ DOs of arbitrary order and, even if the symbols $a_j(x, \xi)$, $j < m$, are not invariant under coordinate choice, their integral is, and defines a trace. All residua at poles of the zeta function of a Ψ DO can be easily obtained from the Wodzicki residue [30].

3.5 The multiplicative anomaly and its implications

Given A , B and AB Ψ DOs, even if ζ_A , ζ_B and ζ_{AB} exist, it turns out that, in general, $\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$. The multiplicative (or noncommutative, or determinant) anomaly is defined as:

$$\delta(A, B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0). \quad (23)$$

Wodzicki's formula for the multiplicative anomaly [29, 31]:

$$\delta(A, B) = \frac{\text{res} \left\{ [\ln \sigma(A, B)]^2 \right\}}{2 \text{ord } A \text{ord } B (\text{ord } A + \text{ord } B)}, \quad \sigma(A, B) := A^{\text{ord } B} B^{-\text{ord } A}. \quad (24)$$

At the level of Quantum Mechanics (QM), where it was originally introduced by Feynman, the path-integral approach is just an alternative formulation of the theory. In QFT it is much more than this, being in many occasions *the* actual formulation of QFT [6]. In short, consider the Gaussian functional integration

$$\int [d\Phi] \exp \left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \dots \right] \right\} \longrightarrow \det(\quad)^\pm, \quad (25)$$

and assume that the operator matrix has the following simple structure (being each A_i an operator on its own):

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}, \quad (26)$$

where the last expression is the result of diagonalizing the operator matrix. A question now arises. What is the determinant of the operator matrix: $\det(AB)$ or $\det A \cdot \det B$? This has been very much on discussion [32]. There is agreement in that: (i) In a situation where a superselection rule exists, AB has no sense (much less its determinant), and then the answer must be $\det A \cdot \det B$. (ii) If the diagonal form is obtained after a change of basis (diagonalization process), then the quantity that is preserved by such transformations is the value of $\det(AB)$ and *not* the product of the individual determinants (there are counterexamples supporting this viewpoint [33]).

3.6 A word on determinants

Many fundamental calculations of QFT reduce, in essence, to the computation of the determinant of some operator. One could even venture to say that, at one-loop order, any such theory reduces to a theory of determinants. The operators involved are ‘differential’ ones, as the normal physicist would say. In fact, properly speaking, they are pseudodifferential operators (Ψ DO), that is, in loose terms ‘some analytic functions of differential operators’ (such as $\sqrt{1+D}$ or $\log(1+D)$, but *not!* $\log D$). This is explained in detail in Refs.[20, 22, 25]

Important as the concept of determinant of a differential or Ψ DO may be for theoretical physicists (in view of what has just been said), it is surprising that this seems not to be a subject of study among function analysts or mathematicians in general. This statement must be qualified: I am specifically referring to determinants that involve in its definition some kind of regularization, very much related to operators that are not trace-class. This piece of calculus —always involving regularization— falls outside the scope of the standard disciplines and even many physically oriented mathematicians know little or nothing about it. In a sense, the subject has many things in common with that of *divergent series* but has not been so deeply investigated and lacks any reference comparable to the very beautiful book of Hardy,[5] already mentioned. Actually, from this general viewpoint, the question of regularizing infinite determinants was already addressed by Weierstrass in a way that, although it has been pursued by some theoretical physicists with success, is not without problems —as a general method— since it ordinarily leads to non-local contributions that cannot be given a physical meaning in QFT. We should mention, for completion, that there are, since long ago, well established theories of determinants for degenerate operators, for trace-class operators in the Hilbert space, Fredholm operators, etc.[34] but, again, these definitions of determinant do not fulfill all the needs mentioned above which arise in QFT.

3.7 On the method of zeta-function regularization

Hawking introduced this method [35, 36] as a basic tool for the regularization of infinities in QFT in a curved spacetime.[6, 37, 38] The idea is the following.[35] One could try to tame Quantum Gravity using the canonical approach, by defining an arrow of time and working on the space-like hypersurfaces perpendicular to it, with equal time commutation

relations. Reasons against this:

1. There are many topologies of the space-time manifold that are not a product $\mathbf{R} \times M_3$.
2. Such non-product topologies are sometimes very interesting.
3. What does it mean ‘equal time’ in the presence of Heisenberg’s uncertainty principle?

One thus turns naturally towards the path-integral approach:

$$\langle g_2, \phi_2, \mathcal{S}_2 | g_1, \phi_1, \mathcal{S}_1 \rangle = \int \mathcal{D}[g, \phi] e^{iS[g, \phi]}, \quad (27)$$

where g_j denotes the spacetime metric, ϕ_j are matter fields, \mathcal{S}_j general spacetime surfaces ($\mathcal{S}_j = M_j \cup \partial M_j$), \mathcal{D} a measure over all possible ‘paths’ leading from the $j = 1$ to the $j = 2$ values of the intervening magnitudes, and S is the action:

$$S = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x, \quad (28)$$

R being the curvature, Λ the cosmological constant, g the determinant of the metric, and L_m the Lagrangian of the matter fields. Stationarity of S under the boundary conditions

$$\delta g|_{\partial M} = 0, \quad \vec{n} \cdot \vec{\partial} \delta g|_{\partial M} = 0, \quad (29)$$

leads to Einstein’s equations:

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab}, \quad (30)$$

T_{ab} being the energy-momentum tensor of the matter fields, namely,

$$T_{ab} = \frac{1}{2\sqrt{-g}} \frac{\delta L_m}{\delta g^{ab}}. \quad (31)$$

The path-integral formalism provides a way to deal ‘perturbatively’ with QFT in curved spacetime backgrounds.[37] First, through a rotation in the complex plane one defines an Euclidean action:

$$iS \longrightarrow -\hat{S}. \quad (32)$$

One can also easily introduce the finite temperature formalism by the substitution $t_2 - t_1 = i\beta$, which yields the partition function

$$Z = \sum_n e^{-\beta E_n}. \quad (33)$$

If one now adheres to the principle that the Feynman propagator is obtained as the limit for $\beta \rightarrow \infty$ of the thermal propagator, we have shown, some time ago,[39] that

the usual principal-part prescription in the zeta-function regularization method (to be described below) need *not* be imposed any more as an additional assumption, since it beautifully follows from, and thus can actually be replaced, by this more general (and natural) principle.[39]

Next comes the stationary phase approach (also called one-loop, or WKB), for calculating the path integral, which consists in expanding around a fixed background:

$$g = g_0 + \bar{g}, \quad \phi = \phi_0 + \bar{\phi}, \quad (34)$$

what leads to the following expansion in the Euclidean metric:

$$\hat{S}[g, \phi] = \hat{S}[g_0, \phi_0] + S_2[\bar{g}, \bar{\phi}] + \dots \quad (35)$$

This is most suitably expressed in terms of determinants (for bosonic, resp. fermionic fields) of the kind (here A, B are the relevant (pseudo-)differential operators in the corresponding Lagrangian):

$$\Delta_\phi = \det \left(\frac{1}{2\pi\mu^2} A \right)^{-1}, \quad \Delta_\psi = \det \left(\frac{1}{2\mu^2} B \right). \quad (36)$$

For its application in practice [40, 41], the zeta function regularization method relies on the existence of quite simple formulas that give the analytic continuation of the zeta function, $\zeta(s)$, from the region of the complex plane extending to the right of the abscissa of convergence, $\text{Re } s > s_0$, to the rest of the complex plane [7, 11, 12, 42]. These are not only the reflection formula of the corresponding zeta function in each case, but also some other, very fundamental expressions, as the Jacobi theta function identity, Poisson's and Plana's resummation formulas, and the Chowla-Selberg formula. However, some of these powerful expressions are often restricted to specific zeta functions, and their explicit derivation is usually quite involved. For instance, until very recently, the Chowla-Selberg (CS) formula was only known to exist for the homogeneous, two-dimensional Epstein zeta function. Also, all these formulas make use of the fact that the sum over the index is done over a full lattice in \mathbf{R} or \mathbf{R}^n , e.g., extending from $-\infty$ to $+\infty$, and they do not survive in the case of truncated sums (where one gets much more involved, asymptotic expressions only) [11, 12].

A fundamental property shared by all zeta functions is the existence of a reflection formula. For the Riemann zeta function:

$$\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s). \quad (37)$$

For a generic zeta function, $Z(s)$, we may write it as: $Z(\omega - s) = F(\omega, s)Z(s)$. It allows for its analytic continuation in a very easy way —what is, in simple cases, the whole story of the zeta function regularization procedure. But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type [43] (actually the same that the original series had, on its convergence domain).

Some years ago, S. Chowla and A. Selberg found a formula, for the Epstein zeta function in the two-dimensional case [18], that yields *exponentially quick convergence everywhere*, not just in the reflected domain. They were very proud of that formula. In Ref. [19], a first attempt was done in order to try to extend this expression to inhomogeneous zeta functions (very important for physical applications, see [44]), but remaining always in *two* dimensions, for this was commonly believed to be an insurmountable restriction of the original formula (see, for instance, Ref. [45]). Later, extensions to an *arbitrary* number of dimensions [20, 46, 14], both for the homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases were constructed. However, some of the new formulas (remarkably the ones corresponding to the zero-mass case, e.g., the original CS framework) were not explicit, since they involved solving a rather non-trivial recurrence (what may also explain why the CS formula had not been extended to higher-dimensional Epstein zeta functions before). In [14] the recurrence was solved and *explicit* formulas were obtained. Aside from this explicit quadratic case, which corresponds to the Epstein zeta function and generalizations thereof, the linear case is also very important (and difficult too) for its many physical applications (think just of a system of harmonic oscillators or a multidimensional oscillator). The most general linear zeta function studied to date is the Barnes' one. Here again many explicit expressions are missing, as for its derivative in the general case.

3.8 Zeta regularization in a nutshell

As advanced already, the regularization and renormalization procedures are essential issues of contemporary physics —without which it would simply not exist, at least in the form we now know it. [47] Among the different methods, zeta function regularization —which is obtained by analytic continuation in the complex plane of the zeta function of the relevant physical operator in each case— is maybe the most beautiful of all. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of any kind, in principle). Assume the corresponding Hamiltonian operator, H , has a spectral decomposition of the form (think, as simplest case, in a quantum harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i \in I}$, being I some set of indices (which can be discrete, continuous, mixed, multiple, ...). Then, the quantum vacuum energy is obtained as follows: [11]

$$E/\mu = \sum_{i \in I} \langle \varphi_i, (H/\mu) \varphi_i \rangle = \text{Tr}_\zeta H/\mu = \sum_{i \in I} \lambda_i/\mu = \sum_{i \in I} (\lambda_i/\mu)^{-s} \Big|_{s=-1} = \zeta_{H/\mu}(-1), \quad (38)$$

where ζ_A is the zeta function corresponding to the operator A , and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class).⁹ Note that the formal sum over the eigenvalues is usually ill defined, and

⁹The reader should be warned that this ζ -trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained by the author elsewhere [48]. Some colleagues are unaware of this fact, which has led to important mistakes and erroneous conclusions too often.

that the last step involves analytic continuation, inherent with the definition of the zeta function itself. Also, the unavoidable regularization parameter with dimensions of mass, μ , appears in the process, in order to render the eigenvalues of the resulting operator dimensionless, so that the corresponding zeta function can indeed be defined. We shall not discuss these important details here, which are just at the starting point of the whole renormalization procedure. The mathematically simple-looking relations above involve very deep physical concepts (no wonder that understanding them took several decades in the recent history of quantum field theory).

3.9 The Casimir energy

In fact things do not turn out to be so simple. One cannot assign a meaning to the *absolute* value of the zero-point energy, and any physical effect is an energy difference between two situations, such as a quantum field in curved space as compared with the same field in flat space, or one satisfying BCs on some surface as compared with the same in its absence, etc. This difference is the Casimir energy: $E_C = E_0^{BC} - E_0 = \frac{1}{2} (\text{tr } H^{BC} - \text{tr } H)$.

But here a problem appears. Imposing mathematical boundary conditions (BCs) on physical quantum fields turns out to be a highly non-trivial act. This was discussed in much detail in a paper by Deutsch and Candelas a quarter of a century ago [49]. These authors quantized em and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress-energy tensor, to find out that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite *physical* energy was obtained if the mathematical BCs were to be fulfilled. However, the authors argued that surfaces have non-zero depth, and its value could be taken as a handy (dimensional) cutoff in order to regularize the infinities. Just two years after Deutsch and Candelas' work, Kurt Symanzik carried out a rigorous analysis of QFT in the presence of boundaries [50]. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure and point splitting to be applied. In any case, the issue was proven to be perfectly meaningful within the domains of renormalized QFT. In this case the BCs and the hypersurfaces themselves were treated at a pure mathematical level (zero depth) by using (Dirac) delta functions.

Recently, a new approach to the problem has been postulated [51]. BCs on a field, ϕ , are enforced on a surface, S , by introducing a scalar potential, σ , of Gaussian shape living on and near the surface. When the Gaussian becomes a delta function, the BCs (Dirichlet here) are enforced: the delta-shaped potential kills *all* the modes of ϕ at the surface. For the rest, the quantum system undergoes a full-fledged QFT renormalization,

as in the case of Symanzik’s approach. The results obtained confirm those of [49] in the several models studied albeit they do not seem to agree with those of [50]. They are also in clear contradiction with the ones quoted in the usual textbooks and review articles dealing with the Casimir effect [52], where no infinite energy density when approaching the Casimir plates has been reported [53].

4 Observing the Universe: large scale structure

4.1 A landmark in observational cosmology

Redshift surveys of galaxies, being three-dimensional, do not suffer from projection effects of two-dimensional maps on the sky surface and are much more appropriate to obtain the true large scale structure of our Universe. However, not all the contribution to the redshift comes from the cosmological expansion (which defines the third dimension, along the line of sight), since there are also additional contributions coming from the peculiar velocities of the galaxy in question (attraction of other galaxies in a cluster, displacement of the cluster itself, etc.):¹⁰

$$Cz_{observed} = Cz_{cosmological} + v_{peculiar}. \quad (39)$$

This can originate artifacts such as the “finger of God” effect in which clusters of galaxies appear as long fingers pointing radially towards the observer. It is not easy to correct for these effects, so one must be careful when trying to make sense of such structures and of three-dimensional maps in general.

The CfA redshift survey of de Lapparent, Geller and Huchra (1986, 1988) was a landmark.[54] It was just “*A slice of the Universe*”, but for the first time, in a map of black dots where each dot corresponded to a whole galaxy, the large scale structure of our Universe (a map of our world) appeared in front of our eyes, for the very first time ever (Fig. 2). That survey, and the corresponding Southern Sky Redshift Survey (da Costa et al., 1988),[55] showed the by now familiar filaments and walls surrounding voids:[56] the “bubble-like” textures of the galaxy distribution, on scales where the galaxy-galaxy correlation function is negligible. The impact was immediate, and influenced a large amount of physicists working in different subjects, who tried to explain, modelize and even reconstruct the point distribution of the slice, in terms of more or less fundamental theories. A paper by J. Ostriker, C. Thomson, and E. Witten[57] tried to explain the voids and other structures as a consequence of string theory. I tried to address the much more technical (but in my view not less important) issue of how to perform the comparison of two point distributions. They would be, in the case under study, the observed galaxy distribution for the slice geometry and any point distribution obtained, say, from a simulation of a theoretical model which would pretend to yield ‘the same’ or a good approximation to the observed point map. The simple (but very difficult) question to be answered is just: how

¹⁰This formula is just an approximation, since as space is curved, for non-near galaxies the distance/redshift relation is non-linear.

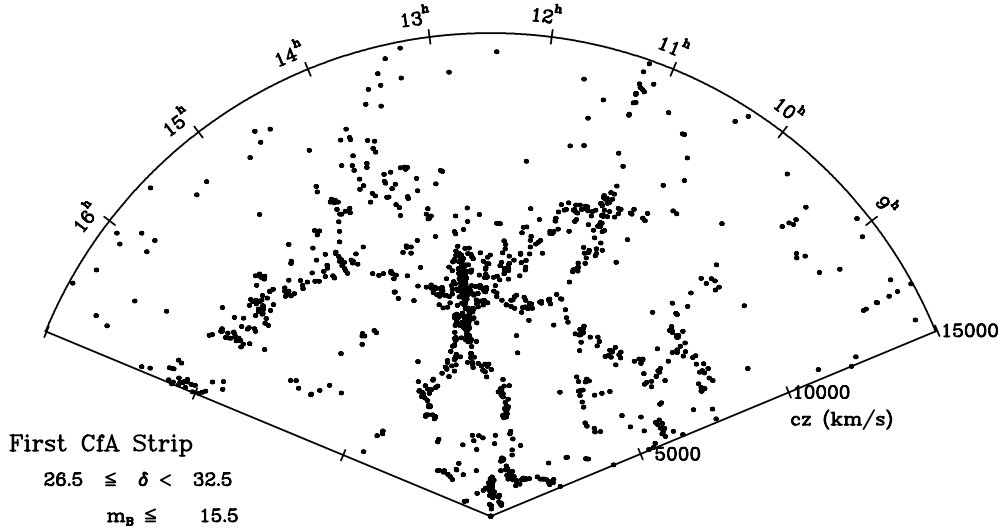


Figure 2: The first CfA redshift survey caused an immediate impact on the scientific community. It clearly showed that the distribution of galaxies in space was anything but random, with galaxies actually appearing to be distributed on surfaces, almost bubble like, surrounding large empty regions, or “voids.” V. de Lapparent, M. Geller and J.P. Huchra, *Astrophys. J. Lett.* **302**, L1 (1986) Smithsonian Astrophysical Observatory.

close are the observed map and the one obtained from a model? Of course, a rigorous answer can be given, from Statistics, in terms of the 2-point, 3-point, ..., n -point correlation functions of both point distributions. But it turns out that in practice higher order correlation functions are very difficult to compute for a large sample of points, and one has to find alternative, much more direct and optimized routes: the highest possible unbiased information from the lowest number of moments of the distribution. One of methods we considered are the so-called *counts in cells*. I suggested this as the starting point for a PhD Thesis to Enrique Gaztañaga, who had come to me at the appropriate time in search of a subject to work on. We did quite nice work together on these matters, later extended by Pablo Fosalba and Jose Barriga with considerable success.¹¹

Two important redshift surveys have been based on the IRAS catalogue (the IRAS survey and QDOT). The original surveys have now been extended to other slices and the original few thousands of galaxies of CfA are now transformed into the several millions monitored by the 2 Degree Field survey, 2dF, the Sloan Digital Sky Survey, SDSS, etc.

¹¹This was the seed and the beginning of our cosmology group at the IEEC/CSIC Institute in Barcelona. Presently we are involved in PLANCK’s[58] High and Low Frequency Instruments,[59] the Sloan SDSS,[60] the APM,[61] the 2dF,[62] WMAP,[63] etc.

The most recent results of the 2dF survey can be seen in [62]. Comparison with Fig. 2 of the first CfA redshift survey shows the enormous progress in observational cosmology in the last 15 years.

The observations seem to conclude that the structures of the large scale galaxy point distribution are essentially sheet-like, and that the scale of the sheets is limited only by the scale of the survey. The most remarkable feature, the so-called “great wall” (Geller and Huchra, 1989)[64] has been seen to be enhanced by the selection function for the sample, but it also appears in other deep wide angle surveys. In any case, there has been much discussion about this issue. Great walls do not bound great voids, but seem to surround collections of smaller voids that are themselves bounded by not-so-great walls. It could be that the great walls are picked out and correlated by our own eyes to build a larger structure.[65] Looking at N-body models suggests this kind of effect because it is easy for the brain to identify coherent structures on scales where there is no physical mechanism for generating structure.[56]

4.2 The Universe is homogeneous and isotropic

The reader should not confuse homogeneity and isotropy. The pattern of a red brick wall (like the beautiful ones in Boston’s Beacon Hill) is an homogeneous but not isotropic one. On the contrary, the pattern of light rays emitted in any direction by a shining light in darkness is isotropic but not homogeneous.

Direct evidence for statistical homogeneity in the distribution of matter at sufficiently large scales came from the first accurate measurement of the galaxy two-point correlation function.[66] Totsuji and Kihara (1969)[67] solved this long-standing problem. In their own words: “*The correlation function for the spatial distribution of galaxies in the universe is determined to be $(r_0/r)^{1.8}$, r being the distance between galaxies. The characteristic length r_0 is 4.7 Mpc. This determination is based on the distribution of galaxies brighter than the apparent magnitude 19 counted by Shane and Wirtanen (1967). The reason why the correlation function has the form of the inverse power of r is that the universe is in a state of ‘neutral’ stability.*” Deep physical insight into the gravitational many-body problem—usually a good way to short-circuit complicated mathematical formalism—led Totsuji and Kihara to their conclusion. Previous guesses at an exponential or Gaussian form for the correlation had been intensively discussed. With the new results, galaxy clustering could be considered to be a phase transition from a Poisson distribution to a correlated distribution, slowly developing on larger and larger scales as the universe expands.

The isotropic and homogeneous Universe case became much stronger after Penzias and Wilson announced the discovery of the Cosmic Microwave Background in 1965. In fact, as we now know, the deviations from homogeneity in the CMB radiation are of about a part in 10^5 , what makes it *too* homogeneous and creates a severe problem when one wants to find some inhomogeneities, to serve as seeds for star and galaxy formation. Beautiful maps of the whole universe showing the temperature fluctuations of the CMB have been produced by WMAP.[63]

The following plot (Fig. 3) shows that our universe approaches homogeneity (as measured now from the matter distribution) as big enough regions of the same are considered, of about or larger than 100 Mpc.

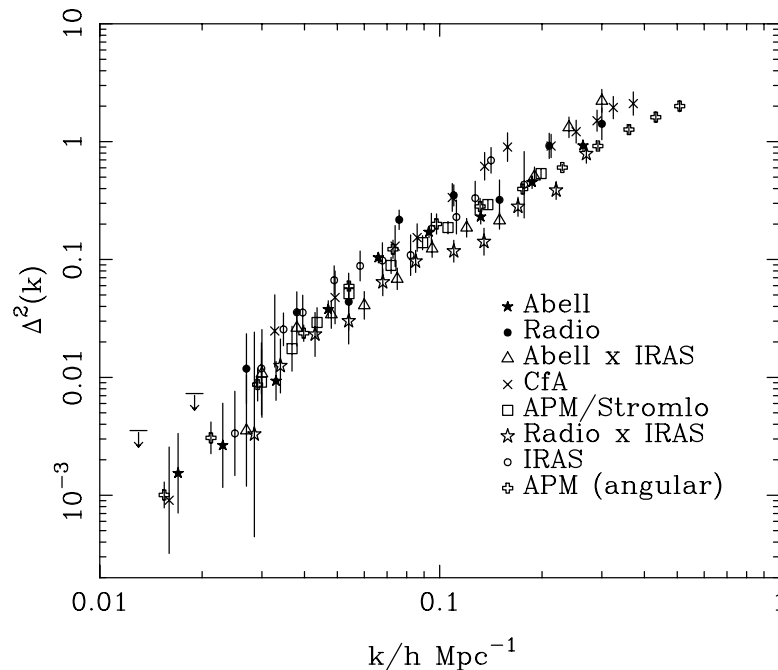


Figure 3: For 100 Mpc regions the Universe is smooth to within several percent. From J.A. Peacock and S.J. Dodds, *Mon. Not. Roy. Astron. Soc.*, 267 (1994) 1020.

4.3 Short summary of inflation

Basic to cosmological observations, as those that already lead to the Big Bang model, is the consideration of a scale factor, $a(t)$, to be taken e.g. as the distance between any pair of comoving objects (e.g. two distant galaxies), or even the curvature of the universe itself, if it is non-vanishing. The scale factor grows by an amount $1 + Hdt$ during a time interval dt , that is:

$$D_G(t) = a(t)D_G(to), \quad (40)$$

with $D_G(to)$ being the distance to the galaxy G right now, and $a(t)$ a universal scale factor that applies to all comoving objects. This law had to be changed for the description of the very beginning of the cosmos, owing to the serious problems of the original Big Bang model.

A. Starobinsky and A. Guth offered a solution to the flatness-oldness problem and to the horizon (or causality) problem of the old Big Bang theory, that was absolutely unable to explain them (together with some other, as the present absence of magnetic monopoles). In 1980, Alan Guth proposed a modification to the Big Bang theory, by suggesting that

in the first moments of its life our universe inflated as if it had been the soapy membrane of a small bubble, that become gigantic in a small fraction of a second.[68] Inflation is in fact a modification of the conventional Big Bang theory, proposing that the expansion of the universe was propelled by a repulsive gravitational force generated by an exotic form of matter. Although Guth's initial proposal was flawed, this was soon overcome by the invention of "new inflation," by Andrei Linde and independently by Andreas Albrecht and Paul Steinhardt. "*The bang was there, but it was not big,*" said at some occasion A. Linde.[69]

Nowadays no self-respecting theory of the Universe is complete without a reference to inflation. But there is in the meantime such a large variety of versions that it would be impossible here to provide a minimally consistent account even of the basic ideas to encompass all them, thus the reader is referred to the excellent bibliography by the creators of the theory.[70]

Inflationary theory does not replace Big Bang theory, but adds an extra stage: before the Big Bang, the universe went through a period of extremely rapid expansion, growing by 30 orders of magnitude in a fraction of a second. It is difficult to imagine something becoming this large this quickly. In the words of Guth: "*To picture a pea expanding to the size of the Milky Way more quickly than the blink of an eye.*" When he came up with the theory of cosmic inflation, Guth was a 34-year old physicist at the Stanford Linear Accelerator Center, in the ninth year of a seemingly interminable career as a postdoctoral fellow. He was working on the problem of magnetic monopoles: the Big Bang model predicts an abundance of magnetic monopoles but none have ever been found.

A few years earlier, Linde had suggested that, in its early stages, the universe had undergone a series of phase transitions, accompanied by supercooling. Supercooling is seen quite often in phase transitions from one form of matter to another, such as water cooling to ice. In supercooling, water will remain liquid as it cools below 0, but at the slightest disturbance it will immediately freeze. Guth and Tye were working on the problem of how supercooling in the early universe would affect the production of magnetic monopoles. "*So I went home one night and did that calculation and discovered that it would have a dramatic effect on the evolution of the universe,*" Guth said once.[68] The supercooled matter would cause gravity to reverse direction, so that objects would repel each other, resulting in exponential inflation. This would also make magnetic monopoles exceedingly rare. The impact of the theory was immediate.

One major puzzle solved by inflation is the fact that it explains the extreme homogeneity and isotropy of the universe, as observed by COBE and now with much greater precision by WMAP and several balloons. This is a highly improbable state viewed from Big Bang theory. In the inflationary scenario, however, stretching out a tiny, uniform universe exponentially results in a similarly uniform larger universe. Inflation also explains why parallel lines don't cross —something everyone learns in school as a basic principle of Euclidean geometry. But other types of geometry are possible. The density of the universe determines whether it is open or closed. Theoretical calculations show that a universe coming from the usual Big Bang should be very curved, whereas scientific obser-

vations show the universe as flat and Euclidean. This “flatness” problem is also solved by inflation. For some interesting reference books see e.g. Ref. [71]

One of the intriguing consequences of inflation is that quantum fluctuations in the early universe can be stretched to astronomical proportions, providing the seeds for the large scale structure of the universe. The predicted spectrum of these fluctuations was calculated in 1982. One thinks of vacuum as empty and massless (with a density $< 10^{-30}$ g/cc). Now, as we know from Quantum Field Theory (QFT), the vacuum is not empty but filled with virtual particles. These quantum fluctuations, once enormously enlarged by inflation, can be seen today as ripples in the cosmic background radiation, but the amplitude of these faint ripples is only about one part in 10^5 . Nonetheless, these ripples were detected by the COBE satellite in 1992, and they have now been measured to much higher precision by the WMAP satellite and several balloons (like MAXIMA, DASI and BOOMERanG). The properties of the radiation are found to be in excellent agreement with the predictions of the simplest models of inflation. Also, according to Guth and Farhi,[72] with quantum tunneling it might be theoretically possible to ignite inflation in a hypothetical laboratory, thereby creating a new universe. The new universe, if it can be created, would not endanger our own universe. Instead it would slip through a wormhole and rapidly disconnect completely. And yet another intriguing feature of inflation is that almost all versions of inflation are eternal: once inflation starts, it never stops completely. Inflation has ended in our part of the universe, but very far away one expects that inflation is continuing, and will continue forever. Is it possible, then, that inflation is also eternal into the past? Recently Guth, Vilenkin and Borde[73] have shown that the inflating region of spacetime must have a past boundary, and that some new physics, perhaps a quantum theory of creation, would be needed to understand it.

The increasing precision of cosmological data sets is opening up new opportunities to test predictions from cosmic inflation. The impact of high precision constraints on the primordial power spectrum is expected to be important and the new generation of observations could provide real tests of the slow-roll inflation paradigm, as well as produce significant discriminating power among different slow-roll models. In particular, proposed next-generation measurements of the CMB temperature anisotropies, and specially polarization, as well as new Lyman- α measurements could become practical in the near future. Relationships between the slope of the power spectrum and its first derivative are nearly universal among existing slow-roll inflationary models, and therefore these relationships can be tested on several scales with new observations. Among other things, this provides additional motivation for the measure of CMB polarization, to be accomplished with the PLANCK mission, in which our group is participating.

4.4 On the topology and curvature of space

The Friedmann-Robertson-Walker (FRW) model, which can be derived as the *only* family of solutions to the Einstein’s equations compatible with the assumptions of *homogeneity* and *isotropy* of space, is the generally accepted model of the cosmos. But, as we surely

know, the FRW is a family with a free parameter, k , the curvature, that can be either positive, negative or zero (the flat or Euclidean case). This curvature, or equivalently the curvature radius, R , is not fixed by the theory and should be matched with cosmological observations. Moreover, the FRW model, and Einstein's equations themselves, can only provide local properties, not global ones, so they cannot tell about the overall topology of our world: is it closed or open? finite or infinite? Even being quite clear that it is, in any case, extremely large—and possibly the human species will never reach more than an infinitesimally tiny part of it—the question is very appealing to any of us. Note that all this discussion concerns only three dimensional space curvature and topology, time will not be involved.

4.4.1 On the curvature

Serious attempts to measure the possible curvature of the space we live in go back to Gauss, who measured the sum of the three angles of a big triangle with vertices on the peaks of three far away mountains (Brocken, Inselberg, and Hohenhagen). He was looking for evidence that the geometry of space is non-Euclidean. The idea was brilliant, but condemned to failure: one needs a much bigger triangle to try to find the possible non-zero curvature of space. Now cosmologists have recently measured the curvature radius R by using the largest triangle available, namely one with us at one vertex and with the other two on the hot opaque surface of the ionized hydrogen that delimits our visible universe and emits the CMB radiation (some 3 to 4×10^5 years after the Big Bang).[74] The CMB maps exhibit hot and cold spots. It can be shown that the characteristic spot angular size corresponds to the first peak of the temperature power spectrum, which is reached for an angular size of $.5^\circ$ (approximately the one subtended by the Moon) if space is flat. If it has a positive curvature, spots should be larger (with a corresponding displacement of the position of the peak), and correspondingly smaller for negative curvature.

The joint analysis of the considerable amount of data obtained during the last years by the balloon experiments (BOOMERanG, MAXIMA, DASI), combined also with galaxy clustering data, have produced a lower bound for $|R| > 20h^{-1}\text{Gpc}$, that is, twice as large as the radius of the observable universe, of about $R_U \simeq 9h^{-1}\text{Gpc}$.

4.4.2 On the topology

Let us repeat that GR does not prescribe the topology of the universe, or its being finite or not, and the universe could perfectly be flat and finite. The simplest non-trivial model from the theoretical viewpoint is the toroidal topology (that of a tyre or a donut, but in one dimension more). Traces for the toroidal topology and more elaborated ones, as negatively curved but compact spaces, have been profusely investigated, and some circles in the sky with near identical temperature patterns were identified.[75] And yet more papers appear from time to time proposing a new topology.[76] However, to summarize all these efforts and the observational situation, and once the numerical data are interpreted without bias

(what sometimes was not the case, and led to erroneous conclusions), it seems at present that available data point towards a very large (we may call it *infinite*) flat space.

5 From General Relativity to Cosmology

5.1 The cosmological constant

Our universe seems to be spatially flat and to possess a non-vanishing cosmological constant. Thus, Einstein's 'great mistake' may turn out ultimately to be a great discovery, a necessary ingredient in order to explain the acceleration of the universe. In any case, for elementary particle physicists it constitutes (in the words of J. Bjørken) a great embarrassment,[77] calculations there being off (when compared with physical facts) by the famous 120 orders of magnitude.

First, physicists tried to find a way to get rid of it (Coleman, Weinberg, Polchinski, ...),[78] in the hope that it could be proven to be zero, what was hard enough. But now it turns out that it is non-vanishing, albeit very small, indeed a very peculiar quantity.

The cosmological constant has to do with cosmology, of course (through Einstein's equations and the FRW universe obtained from them),[79] but it has to do also with the local structure of elementary particle physics as the stress-energy density μ of the vacuum

$$L_{cc} = \int d^4x \sqrt{-g} \mu^4 = \frac{1}{8\pi G} \int d^4x \sqrt{-g} \lambda. \quad (41)$$

In other words: two contributions appear, on the same footing

$$\frac{\Lambda c^2}{8\pi G} + \frac{1}{\text{Vol}} \frac{\hbar c}{2} \sum_i \omega_i. \quad (42)$$

5.2 On the meaning of Einstein's equations

Recall Einstein's equations (formulated in 1915-17), including a cosmological constant Λ :

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \quad (43)$$

with

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad R = R^\mu{}_\mu. \quad (44)$$

As we have seen before, these equations can be obtained from a variational principle, starting from an effective Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) + \int d^4x \sqrt{-g} \mathcal{L}_{mat}. \quad (45)$$

They have a very profound meaning.

On April last year I had to explain some couple of aspects of Special and General Relativity to the mixed TV audience of a Thursday evening. I decided to start with the most famous equation $E = mc^2$. “Look”, I said to the imposing camera in front of my face, “with this equation Einstein’s genius put on the same footing matter and energy, which had been always thought to be non-matching quantities. All of us have been told at primary school that apples and oranges don’t match: no way to add 3 apples and 4 oranges. Since, what the result would be? Seven, ... but seven what? However, we go to the grocer’s or the supermarket every week and we buy not only fruit, but all sort of different things, and the owner or the cashier just puts all items inside a bag and then ... he does it! precisely what we were told it was impossible to do: adds for us everything together and says ‘this makes 14.76 euros’. But this is exactly what Einstein did: to find a conversion factor for the different quantities, which in his case was the velocity of light squared and for the cashier it’s just the price per pound of every item. Easy, isn’t it? Einstein, as the grocer’s, was not constrained by what we learn at school. In Einstein’s case this opened the way to the possibility of converting matter into energy, and vice-versa, what was soon put to test.

And then I went on, “Now let’s turn to GR. This is actually rather more difficult to grasp and no shop owner or cashier on Earth would have guessed the answer this time. Look, in the language of Cosmology, Einstein’s equation reads: $\Omega_{\text{matter}} + \Omega_{\text{radiation}} + \Omega_{\Lambda} + \Omega_{\text{curvature}} = 1$. This equation is not so widely known as the previous one, but is in no way less important. At first sight, you would say —alas, that’s again the same as before, only more items appear say, a computer, a car, a cellular phone, a futon, a T-shirt, ..., of course one can buy them all together at the mall, no problem— But this wouldn’t be the whole truth. An important issue is missing from that argument, namely the last term, $\Omega_{\text{curvature}}$, which refers to the mathematical curvature of space-time itself. That’s very different from the rest of terms, since it means that the reference system itself gravitates, that there is no ‘outside reference system’. In other words and following with the same example as before, what Einstein did here was to put the grocer himself inside the bag together with the rest of the things we bought! Who will now do the sum for us? Actually the first to guess our Universe could behave in this remarkable way was Ernst Mach (1838-1916).¹² And Einstein found out the precise equations to try to confirm such extraordinary idea. This is what Gravity Probe B is going to confirm, with very good precision, so that there can be no doubt that our Universe does in fact behave this way.” Then I went on, to explain frame warping and frame dragging.

What Einstein did, specifically, when building his equations:

- Geometry (curvature), radiation energy, matter, the cc, all are on the same footing and can be equated together. This is here the particular mathematical concretion of Mach’s principle.
- $G_{\mu\nu}$ is a linear combination of the metric $g_{\mu\nu}$ and of first and second derivatives of the same.

¹²There is a nice essay from Frank Wilczek on that issue.[80]

- $T_{\mu\nu}$ is the energy-momentum tensor, and Λ a (possible) cosmological constant.

Actually Einstein didn't quite succeed in pinning down in his theory of GR the whole content of Mach's principle (see [80]); but there is no doubt that remarkable glimpses of it are to be found in Einstein's equations, namely frame warping and frame dragging by distributions of matter and rotating massive shells, respectively.

5.3 Gravity Probe B

The mission Gravity Probe B was launched by NASA on April 20, 2004, with the idea to try to see these two effects in great precision.[81]

1. *Frame warping* was proposed by deSitter in 1916, as the geodetic force a gyroscope would suffer in the presence of the space-time curvature induced by the presence of a mass. In the case of Gravity Probe B, which describes a polar orbit of 640 Km radius —the gyroscopes' axis having been oriented towards a convenient guide star (IM Pegasi)— the calculated effect is a displacement of the gyroscopes' orbit (it won't be exactly circular around the Earth) of 6.6 arcsec/year, with a expected error of less than 10^{-4} .
2. *Frame dragging* was discovered by Lense and Thirring in 1918 as a gravito-magnetic force.[82] It is produced, in the case of Gravity Probe B, by the Earth's (a massive body) rotation on the reference system, defined in this situation by the gyroscopes themselves. The orbit has been specifically chosen so that both effects on the gyroscopes are perpendicular. Frame dragging results in the rotation axis of the gyroscopes trying to approach the Earth's rotation axis by an amount of 42 milliarc-sec/year, with an estimated error of 10^{-2} (equivalent to the section of a human hair seen from 15 Km distance, while the effect amounts to seeing the same hair from 400 m. Reportedly, this precision is still unprecedented in experimental observations. More details about this important mission can be found in its web page,[81] where one can learn a lot about the physical meaning of GR, and this in the best possible way, namely, in the framework of an actual experiment.

Although the idea is very simple, and the first plans to launch such a satellite were met over 40 years ago, the technical difficulties involved are extraordinary and have postponed its final launch till a year ago.

If the results confirm what almost every physicist believes, i.e. the validity and accuracy of GR, then there will be no way out but to admit that the mere notion of the existence of an ideal reference frame in the cosmos is absolutely erroneous. Any mathematical reference will also 'gravitate', that is, it will be unavoidably subject to the influence of all the masses in our Universe, and their rotation. In plain words, *'the grocer himself will have to be put into the bag, indeed, and nobody will be able to do the sum for us.'* This is what we learn from looking at our Universe. If confirmed, these results will completely demolish Isaac Newton's original formulation of the concept of *absolute space*, that was more clear to him than *'the purest of waters'* (and also to more one reputed philosopher, as I. Kant).

5.4 Solutions to Einstein's equations: the Friedmann equation in cosmology

The Schwarzschild solution (1916)[83] of Einstein's equations reads

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (46)$$

It was soon realized that it could describe a black hole, but not a entire universe. However, the Friedmann-Lemaître-Robertson-Walker (1935-36) solution (FRW),[84] first found by A. Friedmann in 1922,

$$ds^2 = dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \quad (47)$$

with $k = 0, +1, -1$, not only can do so, but it is the only solution of Einstein's equations (up to the constant k , the curvature) that can satisfy the requirements of homogeneity and isotropy, that our universe is (observationally) known to possess, to a high degree of accuracy. Now, in Cosmology the Friedmann equation is always written under the equivalent form:

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_V + (1 - \Omega) \left(\frac{a_0}{a}\right)^2 \right], \quad (48)$$

with Ω_R being the radiation (or relativistic matter) content of our Universe, which satisfies the equation of state pressure = density/3, that is $p_R = \frac{1}{3}\rho_R$, being $\rho_R \propto a^{-4}$, and a a typical distance (say the mean intergalactic one), as the Universe expands; the ordinary (non-relativistic) matter is Ω_{NR} , and satisfies $p_{NR} = 0$, being $\rho_{NR} \propto a^{-3}$; the vacuum energy density, Ω_V is undistinguishable from the cosmological constant, with an equation of state $p_V = -\rho_V$, being here $\rho_V = \text{const.}$ Finally, the equation has been normalized to one, so that the sum of the different contributions equals this number; in other words, $1 - \Omega$ in the equation before (where $\Omega = \Omega_R + \Omega_{NR} + \Omega_V$) is the contribution of the geometry (of the curvature of the Universe, while Ω is 'physical' contribution), Ω_K , which behaves as $\rho_K \propto a^{-2}$, and

$$\Omega_R + \Omega_{NR} + \Omega_K + \Omega_V = 1. \quad (49)$$

Presently, the contribution of Ω_K can be neglected, since from observations we obtain that $\Omega_K \simeq 0.0 \pm 0.1$, so that, as we explained before, we seem to live in a flat universe. Thus, one gets the famous *cosmological triangle*, which is a simple and intuitive graphical representation where one can read the proportions of the different terms in any proposed model for the present universe. The most plausible ones at present (e.g. from WMAP), yield a mere 4% for the entire ordinary matter+energy content (i.e., barions+photons, with just some .05% for radiation), some 25% comes from dark or invisible matter (trapped mainly in galaxy clusters, and a fraction in galaxy halos),[85] and the biggest part, around 70%, is an absolutely unknown called dark energy (those values are for an $h \simeq .72$).

Concerning dark matter, Zwicky noticed in 1933 already, that the gravitational action of the luminous matter was not enough in order to hold galaxy clusters together (could explain kpc structures at most.) Different kind of cold matter particles, with magnitudes that can differ in almost 100 orders, have been invoked by existing models, from axions and neutrinos to large planets (or ‘Jupiters’), and also theories deviating from ordinary Newtonian physics. A portion of particle dark matter is sure to exist: the mass coming from neutrinos could be already as large as the mass in visible stars. The lack of Newtonian matter is seen to occur at an extense range of distances, from the less than 1 kpc corresponding to dwarf spirals to the more than 100 Mpc in large clusters of galaxies. Twenty years ago, Milgrom made the remarkable observation that the need for dark matter in galaxies only arises when the Newtonian acceleration is less than a value $a_0 \simeq .3cH_0$. This is called now Milgrom’s law, and has given rise to a theory, the Modified Newtonian Dynamics (MOND).[85] Of the known quantities only the cosmological constant has an equation of state which could contribute to the dark energy term (or, similarly, the proposed models of dynamic cc’s, as quintessence[86]).

6 Vacuum energy and the cosmological constant

The issue of the cosmological constant has got renewed thrust from the observational evidence of an acceleration in the expansion of our Universe, initially reported by two different groups.[87, 88, 89] There was some controversy on the reliability of the results obtained from those observations and on its precise interpretation, by a number of different reasons. Anyway, after new data has been gathered, there is presently reasonable consensus among the community of cosmologists that there is, in fact, an acceleration, and that it has the order of magnitude obtained in the above mentioned observations.[90, 91, 92] In support of this consensus, the recently issued analysis of the data taken by the BOOMERanG[93] and MAXIMA-1[94] balloons have been correspondingly crossed with those from the just mentioned observations, to conclude that the results of BOOMERanG and MAXIMA-1 can perfectly account for an accelerating universe and that, taking together both kinds of observations, one infers that we most probably live in a flat universe. As a consequence, many theoreticians have urged to try to explain this fact, and also to try to reproduce the precise value of the cosmological constant coming from these observations, in the available models.[95, 96, 97]

Now, as crudely stated by Weinberg in a review paper,[98] it is even more difficult to explain why the cosmological constant is so small but non-zero, than to build theoretical models where it exactly vanishes.[78] Rigorous calculations performed in quantum field theory on the vacuum energy density, ρ_V , corresponding to quantum fluctuations of the fields we observe in nature, lead to values that are over 120 orders of magnitude in excess of the values allowed by observations of the space-time around us.

6.1 Quantum fluctuations of the cosmological vacuum energy

Rather than trying to understand the fine-tuned cancellation of such enormous values at this *local* level (a very difficult question that we are going to leave unanswered, and even unattended, here), in this section we will elaborate on a quite simple and primitive idea (but, for the same reason, of far reaching, inescapable consequences), related with the *global* topology of the universe[99] and in connection with the possibility that a very faint, massless scalar field pervading the universe could exist. Fields of this kind are ubiquitous in inflationary models, quintessence theories, and the like. In other words, we do not pretend to solve the old problem of the cosmological constant, not even to contribute significantly to its understanding, but just to present an extraordinarily simple model which shows that the right order of magnitude of (some contributions to) ρ_V , in the precise range deduced from the astrophysical observations,[100, 87, 88] e.g. $\rho_V \sim 10^{-9}$ erg/cm³ $\sim 10^{-11}$ (eV)⁴, are not difficult to get. To say it in different words, we only address here what has been termed by Weinberg[98] the *new* cosmological constant problem.

In short, we shall assume the existence of a scalar field background extending through the universe and shall calculate the contribution to the cosmological constant coming from the Casimir energy density[16] corresponding to this field for some typical boundary conditions. The ultraviolet contributions will be safely set to zero by some mechanism of a fundamental theory. Another hypothesis will be the existence of both large and small dimensions (the total number of large spatial coordinates will be always three), some of which (from each class) may be compactified, so that the global topology of the universe will play an important role, too. There is by now a quite extensive literature both in the subject of what is the global topology of spatial sections of the universe[99] and also on the issue of the possible contribution of the Casimir effect as a source of some sort of cosmic energy, as in the case of the creation of a neutron star.[101] There are arguments that favor different topologies, as a compact hyperbolic manifold for the spatial section, what would have clear observational consequences.[102] Other interesting work along these lines was reported in Ref.[40, 41] and related ideas have been discussed very recently in Ref.[103]. However, we differ from all those in several respects. To begin, the emphasis is put now in obtaining the right order of magnitude for the effect, e.g., one that matches the recent observational results. At the present stage, in view of the observational precision, it has no sense to consider the whole amount of possibilities concerning the nature of the field, the different models for the topology of the universe, and the different boundary conditions possible, with its effect on the sign of the force.

At this level, from our previous experience in these calculations and from the many tables (see, e.g., Refs.[11, 12] where precise values of the Casimir effect corresponding to a number of different configurations have been reported), we realize that the range of *orders of magnitude* of the vacuum energy density for the most common possibilities is not so widespread, and may only differ by at most a couple of digits. This will allow us, both for the sake of simplicity *and* universality, to deal with a most simple situation, which is the one corresponding to a scalar field with periodic boundary conditions. Actually, as

explained in Ref.[104] in detail, all other cases for parallel plates, with any of the usual boundary conditions, can be reduced to this one, from a mathematical viewpoint.

6.2 Two basic space-time models

Let us thus consider a universe with a space-time of one of the two following types: $\mathbf{R}^{d+1} \times \mathbf{T}^p \times \mathbf{T}^q$, $\mathbf{R}^{d+1} \times \mathbf{T}^p \times \mathbf{S}^q, \dots$, which are actually plausible models for the space-time topology. A (nowadays) free scalar field pervading the universe will satisfy

$$(-\square + M^2)\phi = 0, \tag{50}$$

restricted by the appropriate boundary conditions (e.g., periodic, in the first case considered). Here, $d \geq 0$ stands for a possible number of non-compactified dimensions.

Recall now that the physical contribution to the vacuum or zero-point energy $\langle 0|H|0 \rangle$ (where H is the Hamiltonian corresponding to our massive scalar field and $|0 \rangle$ the vacuum state) is obtained on subtracting to these expression —with the vacuum corresponding to our compactified spatial section with the assumed boundary conditions— the vacuum energy corresponding to the same situation with the only change that the compactification is absent (in practice this is done by conveniently sending the compactification radii to infinity). As well known, both of these vacuum energies are in fact infinite, but it is its *difference*

$$E_C = \langle 0|H|0 \rangle|_R - \langle 0|H|0 \rangle|_{R \rightarrow \infty} \tag{51}$$

(where R stands here for a typical compactification length) that makes physical sense, giving rise to the finite value of the Casimir energy E_C , which will depend on R (after a well defined regularization/renormalization procedure is carried out). In fact we will discuss the Casimir (or vacuum) energy *density*, $\rho_C = E_C/V$, which can account for either a finite or an infinite volume of the spatial section of the universe (from now on we shall assume that all diagonalizations already correspond to energy densities, and the volume factors will be replaced at the end). In terms of the spectrum $\{\lambda_n\}$ of H :

$$\langle 0|H|0 \rangle = \frac{1}{2} \sum_n \lambda_n, \tag{52}$$

where the sum over n is a sum over the whole spectrum, which involves, in general, several continuum and several discrete indices. The last appear typically when compactifying the space coordinates (much in the same way as time compactification gives rise to finite-temperature field theory), as in the cases we are going to consider. Thus, the cases treated will involve integration over d continuous dimensions and multiple summations over $p + q$ indices (for a pedagogical description of this procedure, see Ref.[104]).

To be precise, the physical vacuum energy density corresponding to our case, where the contribution of a scalar field, ϕ in a (partly) compactified spatial section of the universe

is considered, will be denoted by ρ_ϕ (note that this is just the contribution to ρ_V coming from this field, there might be other, in general). It is given by

$$\rho_\phi = \frac{1}{2} \sum_{\mathbf{k}} \frac{1}{\mu} (k^2 + M^2)^{1/2}, \quad (53)$$

where the sum $\sum_{\mathbf{k}}$ is a generalized one (as explained above) and μ is the usual mass-dimensional parameter to render the eigenvalues adimensional (we take $\hbar = c = 1$ and shall insert the dimensionfull units only at the end of the calculation). The mass M of the field will be here considered to be arbitrarily small and will be kept different from zero, for the moment, for computational reasons —as well as for physical ones, since a very tiny mass for the field can never be excluded. Some comments about the choice of our model are in order. The first seems obvious: the coupling of the scalar field to gravity should be considered. This has been done in all detail in, e.g., Ref.[105] (see also the references therein; in Ref.[106] the general case of mixed boundary conditions has been considered). The conclusion is that taking it into account does not change the results to be obtained here. Of course, the renormalization of the model is rendered much more involved, and one must enter a discussion on the orders of magnitude of the different contributions, which yields, in the end, an ordinary perturbative expansion, the coupling constant being finally re-absorbed into the mass of the scalar field. In conclusion, we would not gain anything new by taking into account the coupling of the scalar field to gravity. Owing, essentially, to the smallness of the resulting mass for the scalar field, one can prove that, quantitatively, the difference in the final result is at most of a few percent.

Another important consideration is the fact that our model is stationary, while the universe is expanding. Again, careful calculations show that this effect can actually be dismissed at the level of our order of magnitude calculation, since its value cannot surpass the one that we will get (as is seen from the present value of the expansion rate $\Delta R/R \sim 10^{-10}$ per year or from direct consideration of the Hubble coefficient). As before, for the sake of simplicity, and in order to focus just on the essential issues of our argument, we will perform a (momentaneously) static calculation. As a consequence, the value of the Casimir energy density, and of the cosmological constant, to be obtained will correspond to the present epoch, and are bound to change with time.

The last comment at this point would be that (as shown by the many references mentioned above), the idea presented here is not entirely new. However, the simplicity and the generality of its implementation below are indeed new. The issue at work here is absolutely independent of *any* specific model, the only assumptions having been clearly specified before (e.g., existence of a very light scalar field and of some reasonably compactified scales, see later). Secondly, it will turn out, in the end, that the only ‘free parameter’ to play with (the number of compactified dimensions) will actually not be that ‘free’ but, on the contrary, very much constrained to have an admissible value. This will become clear after the calculations below. Thirdly, although the calculation may seem easy to do, in fact it is not so. Some reflection identities, due to the author, will allow to be performed analitically.

6.3 The vacuum energy density and its regularization

To exhibit explicitly a couple of the wide family of cases considered, let us write down in detail the formulas corresponding to the two first topologies, as described above. For a (p, q) -toroidal universe, with p the number of ‘large’ and q of ‘small’ dimensions:

$$\begin{aligned} \rho_\phi &= \frac{\pi^{-d/2}}{2^d \Gamma(d/2) \prod_{j=1}^p a_j \prod_{h=1}^q b_h} \int_0^\infty dk k^{d-1} \sum_{\mathbf{n}_p=-\infty}^\infty \sum_{\mathbf{m}_q=-\infty}^\infty \\ &\quad \left[\sum_{j=1}^p \left(\frac{2\pi n_j}{a_j} \right)^2 + \sum_{h=1}^q \left(\frac{2\pi m_h}{b_h} \right)^2 + M^2 \right]^{1/2} \\ &\sim \frac{1}{a^p b^q} \sum_{\mathbf{n}_p, \mathbf{m}_q=-\infty}^\infty \left(\frac{1}{a^2} \sum_{j=1}^p n_j^2 + \frac{1}{b^2} \sum_{h=1}^q m_h^2 + M^2 \right)^{(d+1)/2+1}, \end{aligned} \quad (54)$$

where the last formula corresponds to the case when all large (resp. all small) compactification scales are the same. In this last expression the squared mass of the field should be divided by $4\pi^2\mu^2$, but we have renamed it again M^2 to simplify the ensuing formulas (as M is going to be very small, we need not keep track of this change). We also will not take care for the moment of the mass-dim factor μ in other places –as is usually done– since formulas would get unnecessarily complicated and there is no problem in recovering it at the end of the calculation. For a $(p$ -toroidal, q -spherical)-universe, the expression turns out to be

$$\begin{aligned} \rho_\phi &= \frac{\pi^{-d/2}}{2^d \Gamma(d/2) \prod_{j=1}^p a_j b^q} \int_0^\infty dk k^{d-1} \sum_{\mathbf{n}_p=-\infty}^\infty \sum_{l=1}^\infty P_{q-1}(l) \\ &\quad \times \left[\sum_{j=1}^p \left(\frac{2\pi n_j}{a_j} \right)^2 + \frac{Q_2(l)}{b^2} + M^2 \right]^{1/2} \\ &\sim \frac{1}{a^p b^q} \sum_{\mathbf{n}_p=-\infty}^\infty \sum_{l=1}^\infty P_{q-1}(l) \left(\frac{4\pi^2}{a^2} \sum_{j=1}^p n_j^2 + \frac{l(l+q)}{b^2} + M^2 \right)^{(d+1)/2+1}, \end{aligned} \quad (55)$$

where $P_{q-1}(l)$ is a polynomial in l of degree $q-1$, and where the second formula corresponds to the similar situation as the second one before. On dealing with our observable universe, in all these expressions we assume that $d = 3 - p$, the number of non-compactified, ‘large’ spatial dimensions (thus, no d dependence will remain).

As is clear, all these expressions for ρ_ϕ need to be regularized. We will use zeta function regularization, taking advantage of the powerful equalities that have been derived by the author,[20, 39] and which reduce the enormous burden of such computations to the easy application of some formulas. For the sake of completeness, let us very briefly summarize how this works.[44, 104] We deal here only with the case when the spectrum of the

Hamiltonian operator is known explicitly. Going back to the most general expressions of the Casimir energy corresponding to this case, namely Eq. (53), we replace the exponents in them with a complex variable, s , thus obtaining the zeta function associated with the operator as:

$$\zeta(s) = \frac{1}{2} \sum_{\mathbf{k}} \left(\frac{k^2 + M^2}{\mu^2} \right)^{-s/2}. \quad (56)$$

The next step is to perform the analytic continuation of the zeta function from a domain of the complex s -plane with $\text{Re } s$ big enough (where it is perfectly defined by this sum) to the point $s = -1$, to obtain:

$$\rho_\phi = \zeta(-1). \quad (57)$$

The effectiveness of this method has been sufficiently described before (see, e.g., [11, 12]). As we know from precise Casimir calculations in those references, no further subtraction or renormalization is needed in the cases here considered, in order to obtain the physical value for the vacuum energy density (there is actually a subtraction at infinity taken into account, but it is of null value, and no renormalization, not even a finite one, very common to other frameworks, applies here).

Using the formulas[20] that generalize the well-known Chowla-Selberg expression to the situations considered above, Eqs. (54) and (55) —namely, multidimensional, massive cases— we can provide arbitrarily accurate results for different values of the compactification radii. However, as argued above we only aim here at matching the *order of magnitude* of the Casimir value and, thus, we shall just deal with the most simple cases of Eqs. (54) or (55), which yield the same orders of magnitude as the rest of them). Also in accordance with this observation, we notice that among the models here considered and which lead to the values that will be obtained below, there are in particular the very important typical cases of isotropic universes with the spherical topology. As all our discussion here is in terms of orders of magnitude and not of precise values with small errors, all these cases are included on *equal footing*. But, on the other hand, it has no sense to present a lengthy calculation dealing in detail with all the possible spatial geometries. Anyhow, all these calculations can indeed be done, and are very similar to the one here, as has been described in detail elsewhere.[40, 41, 11, 12]

For the analytic continuation of the zeta function corresponding to (54), we obtain:[20]

$$\begin{aligned} \zeta(s) = & \frac{2\pi^{s/2+1}}{a^{p-(s+1)/2} b^{q-(s-1)/2} \Gamma(s/2)} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sum_{h=0}^p \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \left(\frac{\sum_{j=1}^h n_j^2}{\sum_{k=1}^q m_k^2 + M^2} \right)^{(s-1)/4} \\ & \times K_{(s-1)/2} \left[\frac{2\pi a}{b} \sqrt{\sum_{j=1}^h n_j^2 \left(\sum_{k=1}^q m_k^2 + M^2 \right)} \right], \end{aligned} \quad (58)$$

where $K_\nu(z)$ is the modified Bessel function of the second kind. Having performed already the analytic continuation, this expression is ready for the substitution $s = -1$, and yields

$$\begin{aligned} \rho_\phi &= -\frac{1}{a^p b^{q+1}} \sum_{h=0}^p \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sqrt{\frac{\sum_{k=1}^q m_k^2 + M^2}{\sum_{j=1}^h n_j^2}} \\ &\quad \times K_1 \left[\frac{2\pi a}{b} \sqrt{\sum_{j=1}^h n_j^2 \left(\sum_{k=1}^q m_k^2 + M^2 \right)} \right]. \end{aligned} \quad (59)$$

Now, from the behaviour of the function $K_\nu(z)$ for small values of its argument,

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) (z/2)^{-\nu}, \quad z \rightarrow 0, \quad (60)$$

we obtain, in the case when M is very small,

$$\begin{aligned} \rho_\phi &= -\frac{1}{a^p b^{q+1}} \left\{ M K_1 \left(\frac{2\pi a}{b} M \right) + \sum_{h=0}^p \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \frac{M}{\sqrt{\sum_{j=1}^h n_j^2}} \right. \\ &\quad \left. \times K_1 \left(\frac{2\pi a}{b} M \sqrt{\sum_{j=1}^h n_j^2} \right) + \mathcal{O} \left[q \sqrt{1+M^2} K_1 \left(\frac{2\pi a}{b} \sqrt{1+M^2} \right) \right] \right\}. \end{aligned} \quad (61)$$

At this stage, the only presence of the mass-dim parameter μ is as M/μ everywhere. This does not conceptually affect the small- M limit, $M/\mu \ll b/a$. Using (60) and inserting now in the expression the \hbar and c factors, we finally get

$$\rho_\phi = -\frac{\hbar c}{2\pi a^{p+1} b^q} \left[1 + \sum_{h=0}^p \binom{p}{h} 2^h \alpha \right] + \mathcal{O} \left[q K_1 \left(\frac{2\pi a}{b} \right) \right], \quad (62)$$

where α is some finite constant, computable and under control, which is obtained as an explicit geometrical sum in the limit $M \rightarrow 0$. It is remarkable that we here obtain such a well defined limit, independent of M^2 , provided that M^2 is small enough. In other words, a physically very nice situation turns out to correspond, precisely, to the mathematically rigorous case. This is moreover, let me repeat, the kind of expression that one gets not just for the model considered, but for *many* other cases, corresponding to different fields, topologies, and boundary conditions —aside from the sign in front of the formula, that may change with the number of compactified dimensions and the nature of the boundary conditions (in particular, for Dirichlet boundary conditions one obtains a value in the same order of magnitude but of opposite sign).

6.4 Numerical results

For the most common variants, the constant α in (62) has been calculated to be of order 10^2 , and the whole factor, in brackets, of the first term in (62) has a value of order 10^7 .

This shows the value of a precise calculation, as the one undertaken here, together with the fact that just a naive consideration of the dependencies of ρ_ϕ on the powers of the compactification radii, a and b , is *not enough* in order to obtain the correct result. Notice, moreover, the non-trivial change in the power dependencies from going from Eq. (61) to Eq. (62).

For the compactification radii at small scales, b , we shall simply take the magnitude of the Planck length, $b \sim l_P$, while the typical value for the large scales, a , will be taken to be the present size of the observable universe, $a \sim R_U$. With this choice, the order of the quotient a/b in the argument of K_1 is as big as: $a/b \sim 10^{60}$. Thus, we see immediately that, in fact, the final expression for the vacuum energy density is completely independent of the mass M of the field, provided this is very small (eventually zero). In fact, since the last term in Eq. (62) is exponentially vanishing, for large arguments of the Bessel function K_1 , this contribution is zero, for all practical purposes, what is already a very nice result. Taken in ordinary units (and after tracing back all the transformations suffered by the mass term M) the actual bound on the mass of the scalar field is $M \leq 1.2 \times 10^{-32}$ eV, that is, physically zero, since it is lower by several orders of magnitude than any bound coming from the more usual SUSY theories –where in fact scalar fields with low masses of the order of that of the lightest neutrino do show up,[96] which may have observable implications.

ρ_ϕ	$p = 0$	$p = 1$	$p = 2$	$p = 3$
$b = l_P$	10^{-13}	10^{-6}	1	10^5
$b = 10 l_P$	10^{-14}	[10^{-8}]	10^{-3}	10
$b = 10^2 l_P$	10^{-15}	[10^{-10}]	10^{-6}	10^{-3}
$b = 10^3 l_P$	10^{-16}	10^{-12}	[10^{-9}]	(10^{-7})
$b = 10^4 l_P$	10^{-17}	10^{-14}	10^{-12}	(10^{-11})
$b = 10^5 l_P$	10^{-18}	10^{-16}	10^{-15}	10^{-15}

Table 1: Orders of magnitude of the vacuum energy density contribution, ρ_ϕ , of a massless scalar field to the cosmological constant, ρ_V , for p large compactified dimensions and $q = p + 1$ small compactified dimensions, $p = 0, \dots, 3$, for different values of the small compactification length, b , proportional to the Planck length l_P . In brackets are the results that more closely match the observational value of the cosmological constant, and in parenthesis the next approximations to that value. A small matching region is clearly determined.

By replacing all these values in Eq. (62), we obtain the results listed in Table 1, for the orders of magnitude of the vacuum energy density corresponding to a sample of different numbers of compactified (large and small) dimensions and for different values of the small compactification length in terms of the Planck length. Notice again that the total number of large space dimensions is three, as corresponds to our observable universe. As we see from Table 1, good coincidence with the observational value for the cosmological

constant is obtained for the contribution of a massless scalar field, ρ_ϕ , for $p = 2$ (or 1) large compactified dimensions and $q = p + 1$ small compactified dimensions, and this for values of the small compactification length, b , of the order of 10 to 1000 times the Planck length l_P (what is actually a very reasonable conclusion, according also to more fundamental approaches coming from string theory).

To be noticed is the fact that full agreement is obtained only for cases where there is exactly one small compactified dimension in excess of the number of large compactified dimensions. We must point out that the p large and q small dimensions are not all that are supposed to exist (in that case p should be at least, and at most, 3 and the other cases would lack any physical meaning). In fact, as we have pointed out before, p and q refer to the compactified dimensions only, but there may be other, non-compactified dimensions (exactly $3 - p$ in the case of the ‘large’ ones), what translates into a slight modification of the formulas above, but does not change the order of magnitude of the final numbers obtained, assuming the most common boundary conditions for the non-compactified dimensions (see e.g. [12] for an explanation of this technical point). In particular, the cases of pure spherical compactification and of mixed toroidal (for small magnitudes) and spherical (for big ones) compactification can be treated in this way and yield results in the same order of magnitude range. Both these cases correspond to (observational) isotropic spatial geometries. Also to be remarked again is the non-triviality of these calculations, when carried out exactly, as done here, to the last expression, what is apparent from the use of the generalized Chowla-Selberg formula. Simple power counting is absolutely unable to provide the correct order of magnitude of the results.

Dimensionally speaking, within the global approach adopted in the present paper everything is dictated, in the end, by the two basic lengths in the problem, which are its Planck value and the radius of the observable Universe. Just by playing with these numbers in the context of this precise calculation of the Casimir effect, we have shown that the observed value of ρ_V may be remarkably well fitted, under general hypothesis, for the most common models of the space-time topology. Notice also that the most precise fits with the observational value of the cosmological constant are obtained for b between $b = 100 l_P$ and $b = 1000 l_P$, with (1,2) and (2,3) compactified dimensions, respectively. The fact that the value obtained for the cosmological constant is so sensitive to the input may be viewed as a drawback but also, on the contrary, as a very *positive* feature of our model. For one, the Table 1 has a sharp discriminating power. In other words, there is in fact no tuning of a ‘free parameter’ in our model and the number of large compactified dimensions could have been fixed beforehand, to respect what we know already of our observable universe.

Also, it proves that the observational value is not easy at all to obtain. Table 1 itself proves that there is only very little chance of getting the right figure (a truly narrow window, since very easily we are off by several orders of magnitude). In fact, if we trust this value with the statistics at hand, we can undoubtedly claim –through use of the model– that the ones so clearly picked up by Table 1 are *the* only two possible configurations of our observable universe (together with a couple more coming from corresponding spherical compactifications). And all them correspond to a marginally closed universe, in full agree-

ment too with other completely independent analysis of the observational data.[91, 87, 88]

Many questions may be posed to the simple models presented here, as concerning the dynamics of the scalar field, its couplings with gravity and other fields, a possible non-symmetrical behaviour with respect to the large and small dimensions, or the relevance of vacuum polarization (see Ref.[107] concerning this last point). Above we have already argued that they can be proven to have little influence on the final numerical result (cf., in particular, the mass obtained for the scalar field in Ref.[105], extremely close to our own result, and the corresponding discussion there). From the very existence and specific properties of the cosmic microwave radiation (CMB) –which mimics somehow the situation described (the ‘mass’ corresponding to the CMB is also in the sub-lightest-neutrino range)– we are led to the conclusion that such a field could be actually present, unnoticed, in our observable universe. In fact, the existence of scalar fields of very low masses is also demanded by other frameworks, as SUSY models, where the scaling behaviour of the cosmological constant has been considered.[96]

Let us finally recall again that the Casimir effect is an ubiquitous phenomena. Its contribution may be small (as it seems to be the case, yet controverted, to sonoluminescence), of some 10-30% (that is, of the right order of magnitude, as in wetting phenomena involving He in condensed matter physics). Here we have seen that it provides a contribution of the right order of magnitude, corresponding to our present epoch in the evolution of the universe. The implication that this calculation bears for the early universe and inflation is not clear from the final result, since it should be adapted to the situation and boundary conditions corresponding to those primeval epochs, what cannot be done straightforwardly.

In a different direction, more elaborate models for the dark energy can be found in the recent contributions [108], which have drawn considerable attention.

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