Singularity Theorems in General Relativity

José M M Senovilla

8th June 2005

eman ta zabal zazu

25 URTE AÑOS

Penrose-Hawking singularity theorems

In relativity, there are several versions of the **Penrose-Hawking singularity theorem**. Most versions state, roughly, that if there is a trapped null surface and the energy density is nonnegative, then there exist geodesics of finite length which can't be extended.

Last updated: 05-13-2005 01:01:13

Be careful!

Gato $\neq$ Liebre !!

What do ST say?

They predict, under certain circumstances, the existence of **singularities** in classical spacetimes.

But, what is this thing they predict? Singularities?

Singularity

The intuitive ideas (such as "infinite concentration" of matter/energy) are **not** what the ST predict!

The main idea: they imply the **sudden** appearance, or disappearance, of physical particles.

In mathematical terms, they imply the existence of **incomplete** endless causal geodesics

Oops!

Singularity

This particle (a rocket, or light,…) has reached the "edge" of the spacetime;
In **finite** time and distance!

Singularity

This particle (a rocket, or light,…) comes from the "edge" of the spacetime;
In **finite** time and distance!

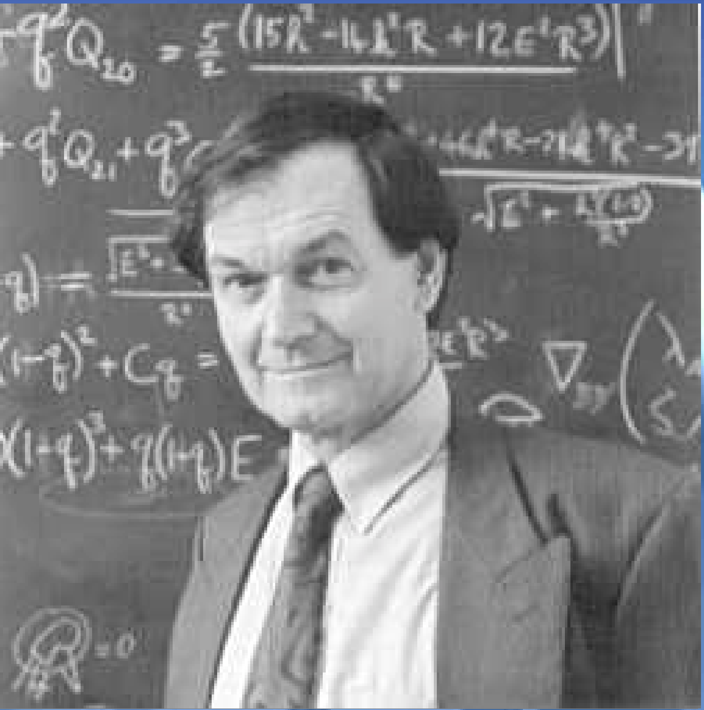
Singularities

This means that one can reach in finite time, something which is at finite distance and where the spacetime

ENDS

BEGINS

People



Pattern singularity theorem

- If the spacetime is <u>smooth</u> and satisfies:

- A condition on the curvature
- A causality condition
- And a Boundary/Initial condition

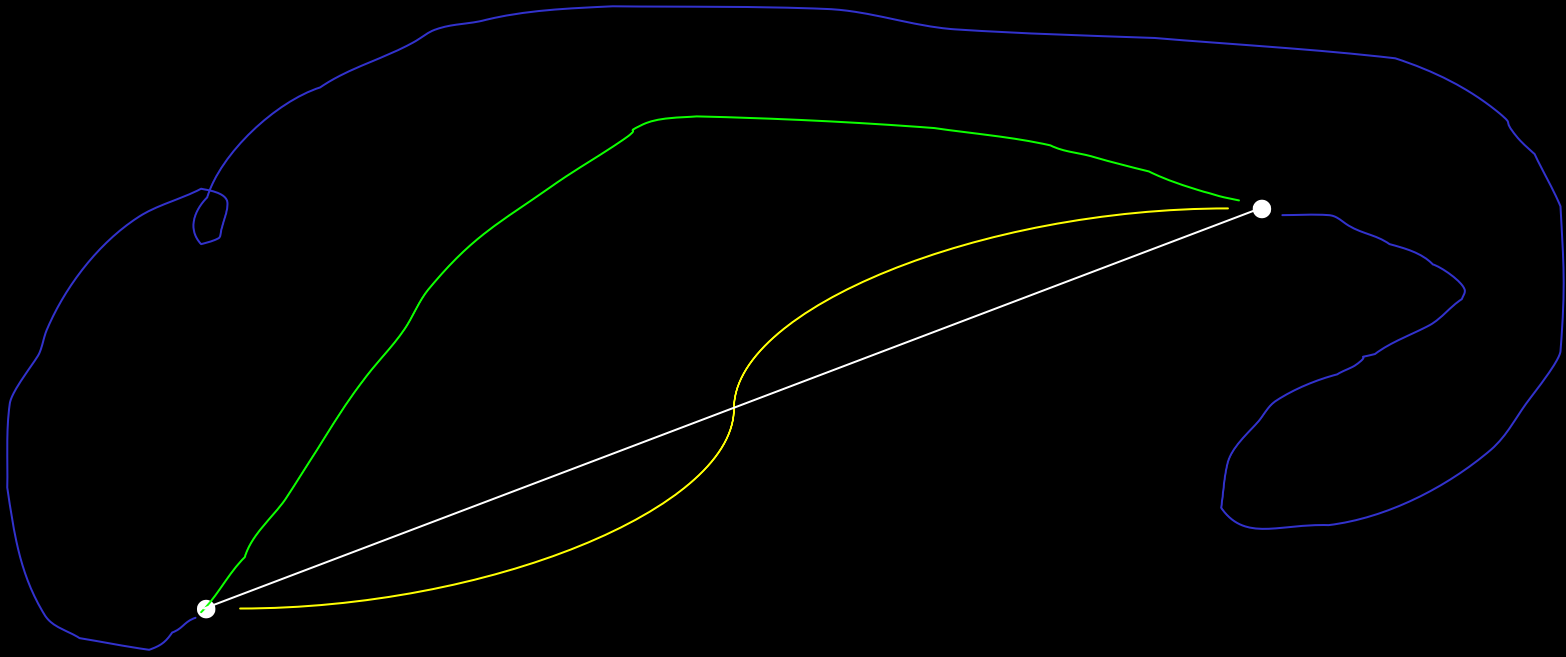
- THEN, it has incomplete causal geodesics (i.e., Singularities).

How do these conditions work?

- What are the effects and consequences of each of the above hypotheses?
- What are they needed for?
- And what do they imply?

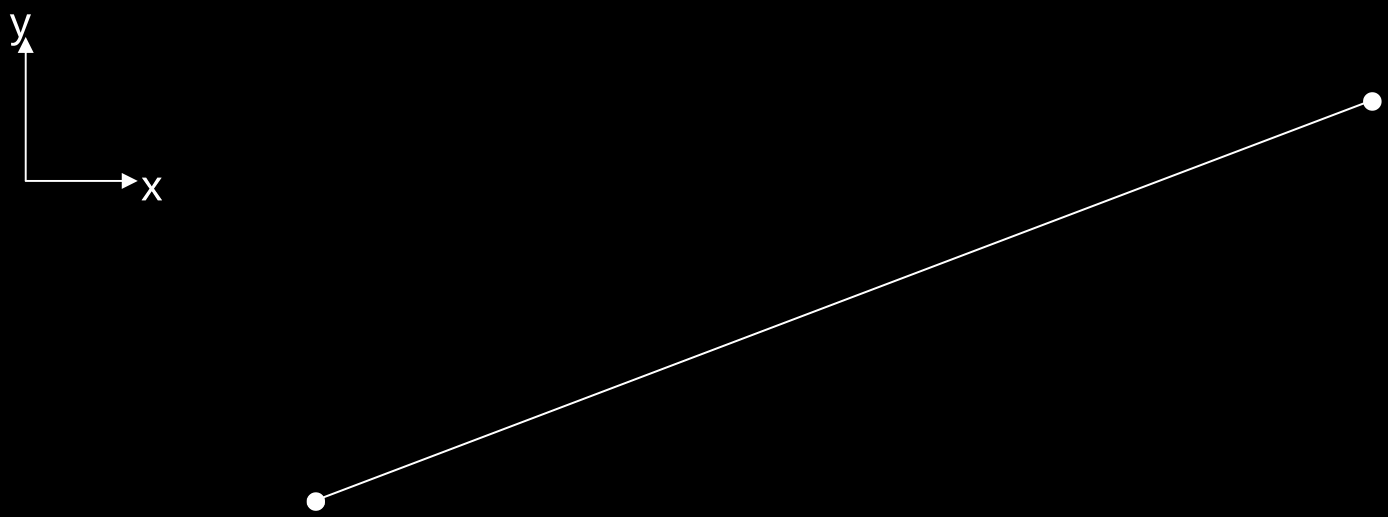
Understandable with elementary geometry !

What is the shortest curve?



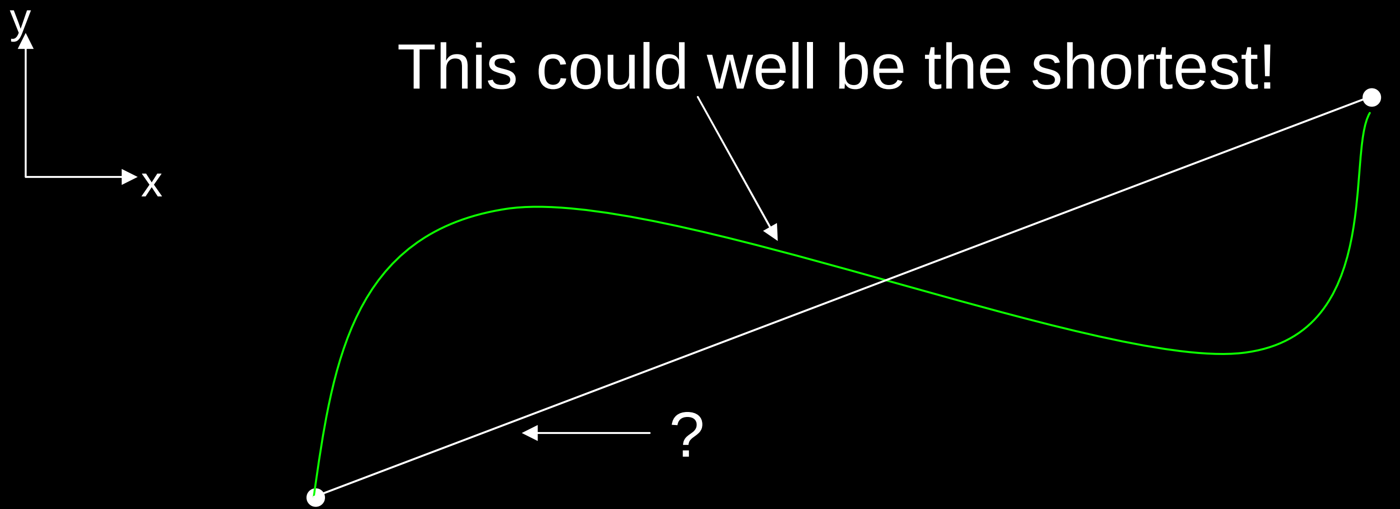
It depends on the “length element”!!

What is the shortest curve?



If the length is “euclidean”, $dl^2=dx^2 + dy^2$.

What is the shortest curve?



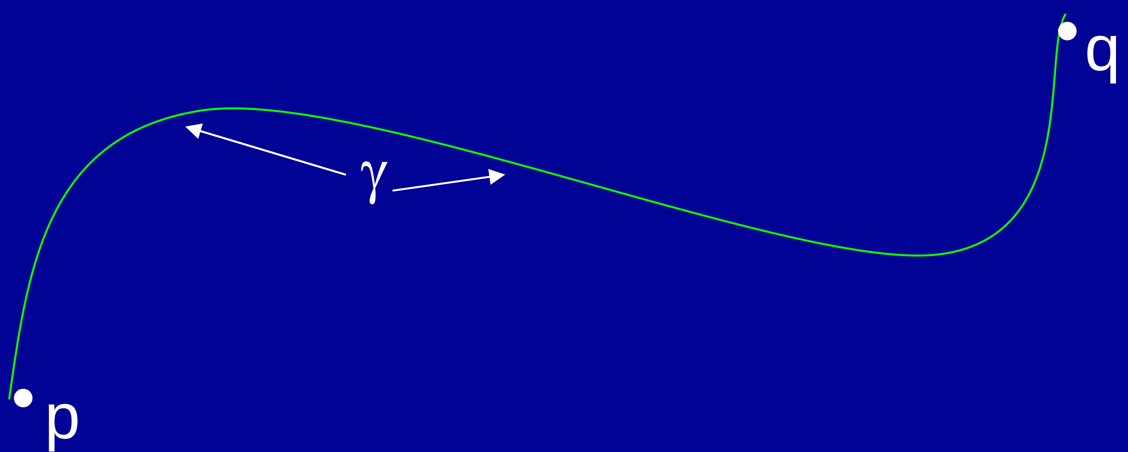
If the length is “riemannian/gaussian”:
e.g. $dl^2 = F^2(x,y)\, dx^2 + G^2(x,y)\, dy^2$.

How can one control this?

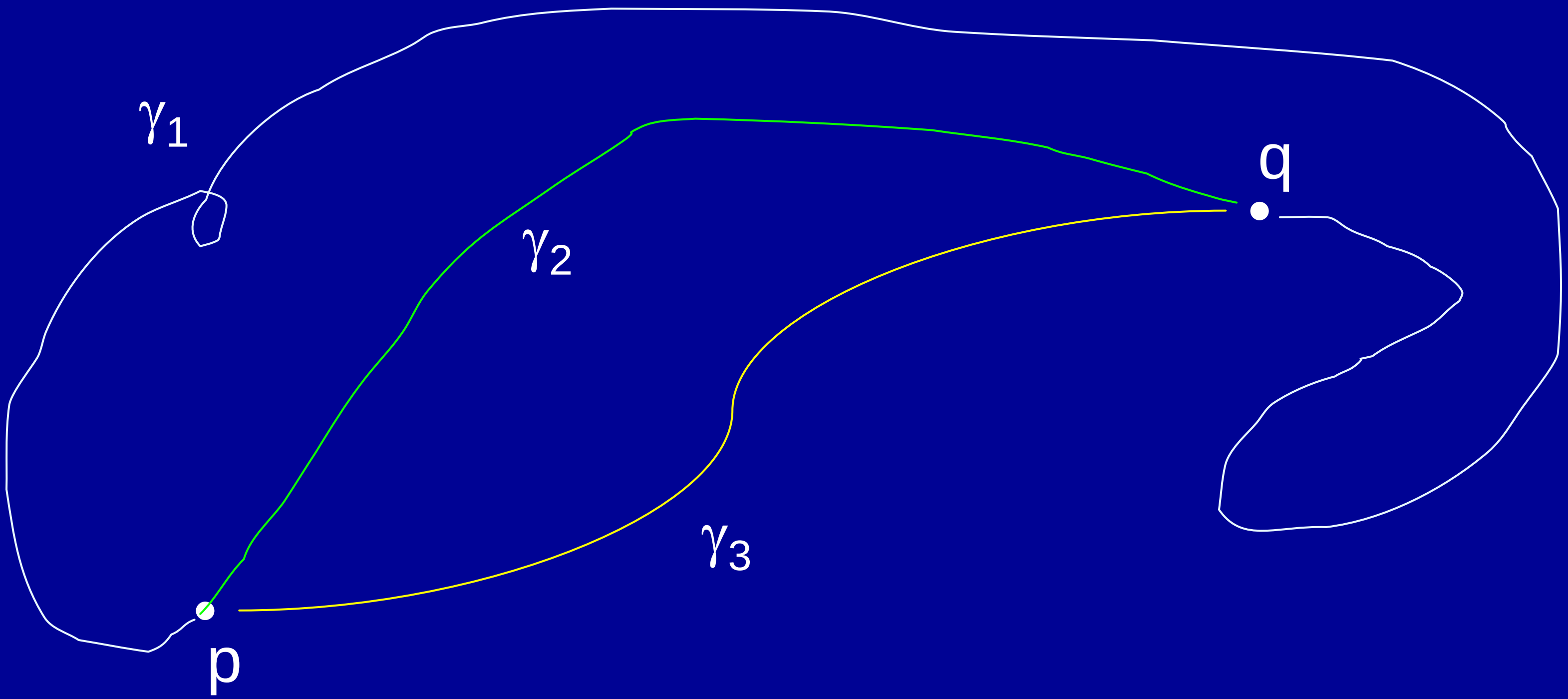
How can we properly define “shortest”?

LENGTH

$$L(p \xrightarrow{\gamma} q) = \int_p^q dl \Big|_\gamma$$



Minimum length?

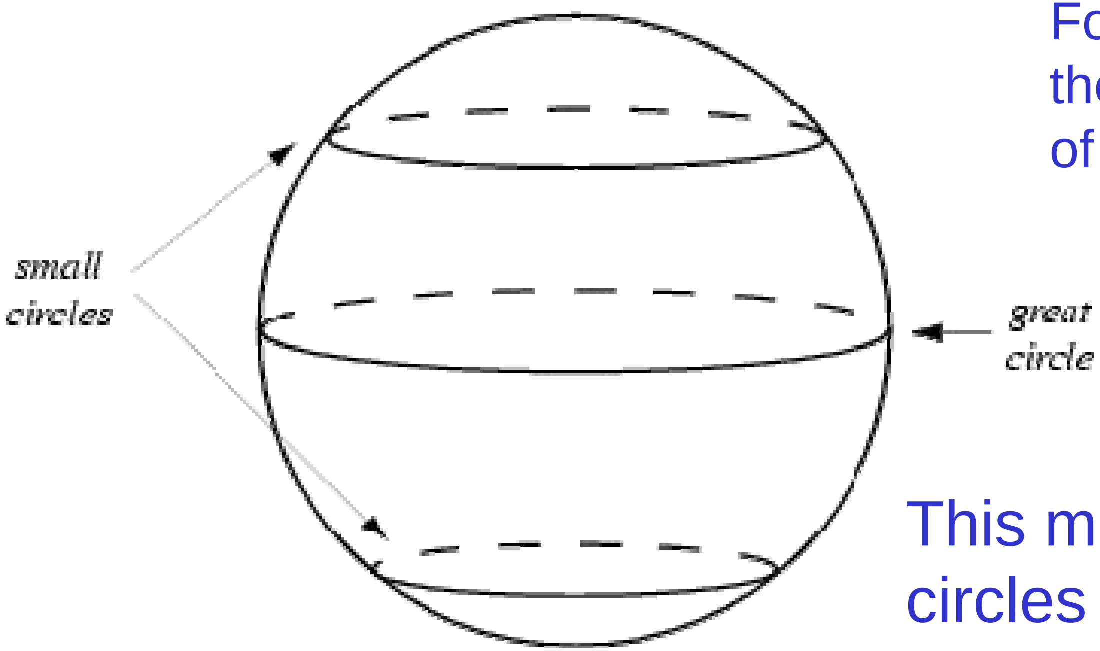


$$\frac{\delta L}{\delta \gamma}\left(p \xrightarrow{\gamma} q\right) = 0$$

Geodesics

Geodesics are equivalently characterized by a vanishing proper curvature vector (or ”acceleration” for physicists). Observe that this is an ***extrinsic*** property!

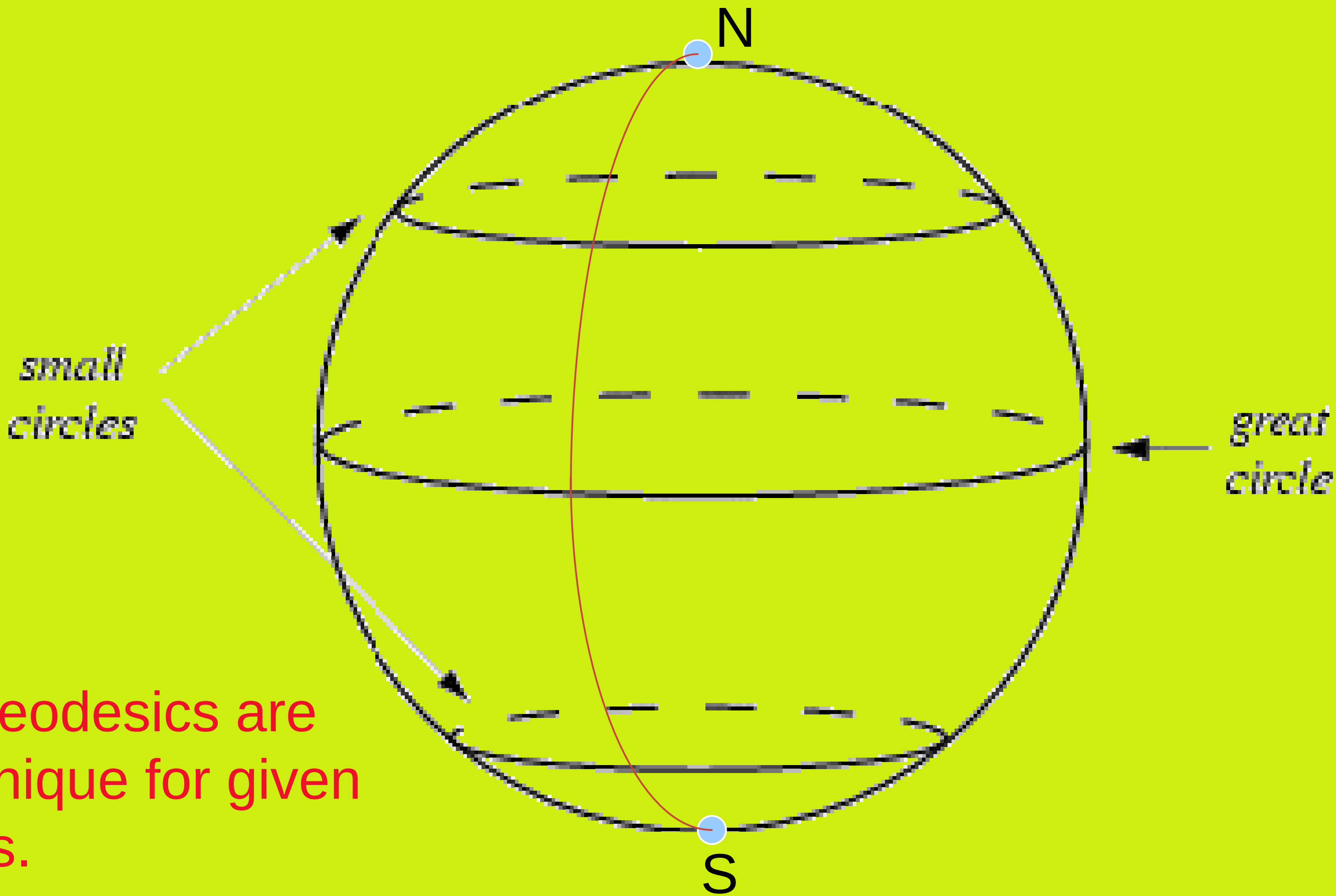
The geodesic condition is **<u>LOCAL</u>**



For instance, in a sphere
the geodesics are segments
of a great ”circle” (ortodrome).

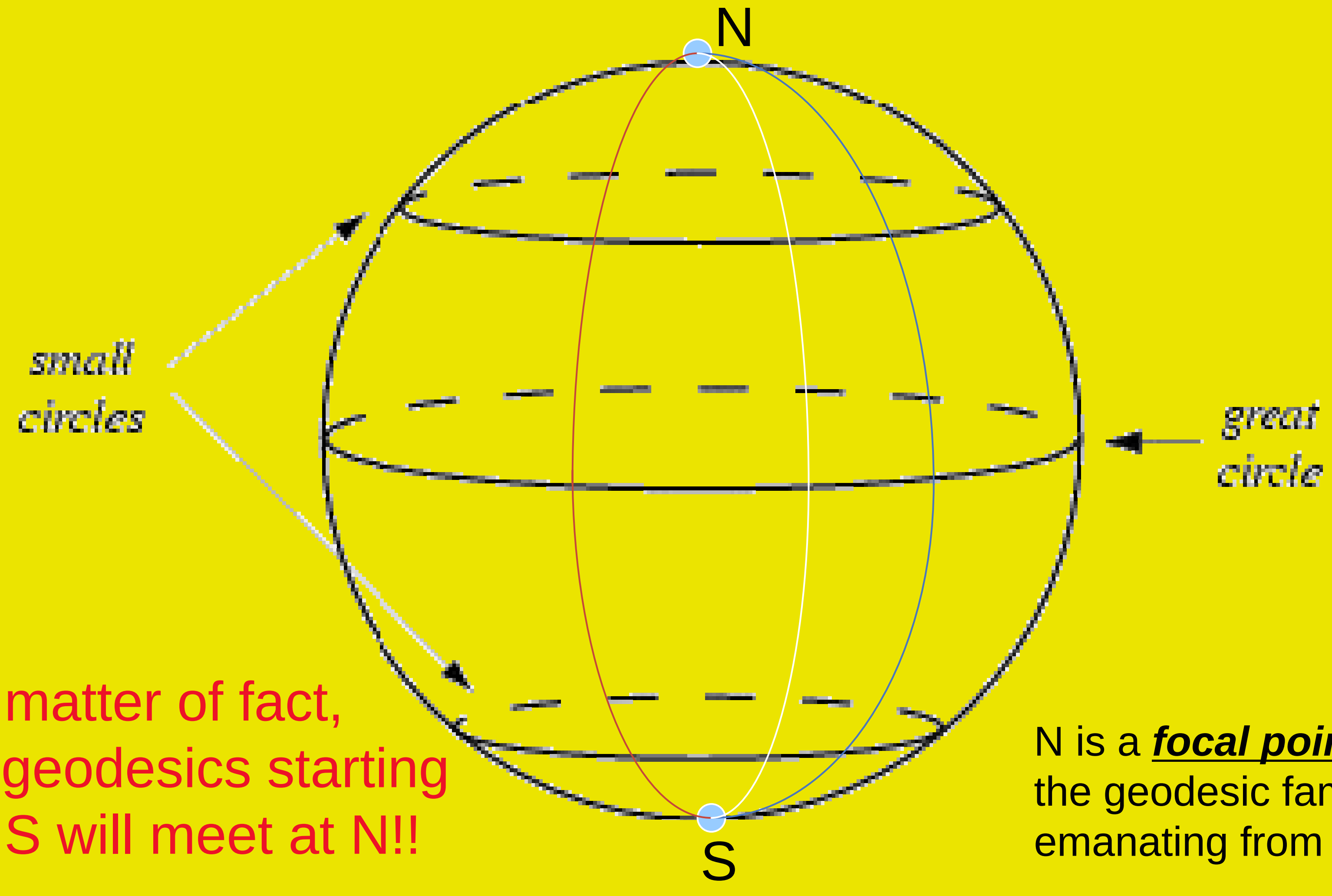
This means that the great circles minimize, **<u>locally</u>**, the distance between any two of their points. But, what happens at large?

CORNERS, CONJUGATE/FOCAL POINTS



So, geodesics are not unique for given points.

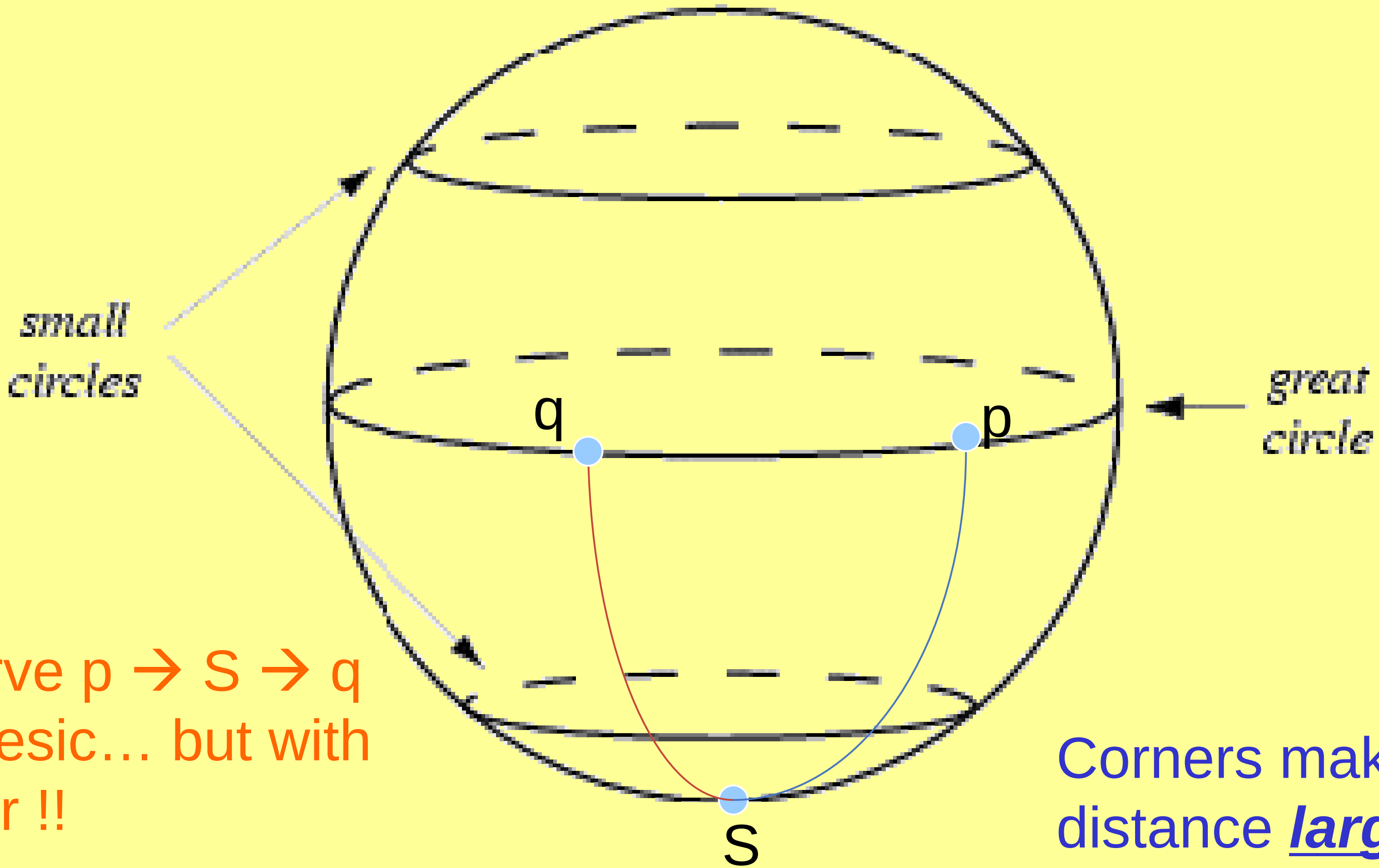
CORNERS, CONJUGATE/FOCAL POINTS



As a matter of fact, **ALL** geodesics starting from S will meet at N!!

N is a ***focal point*** fo
the geodesic family
emanating from S

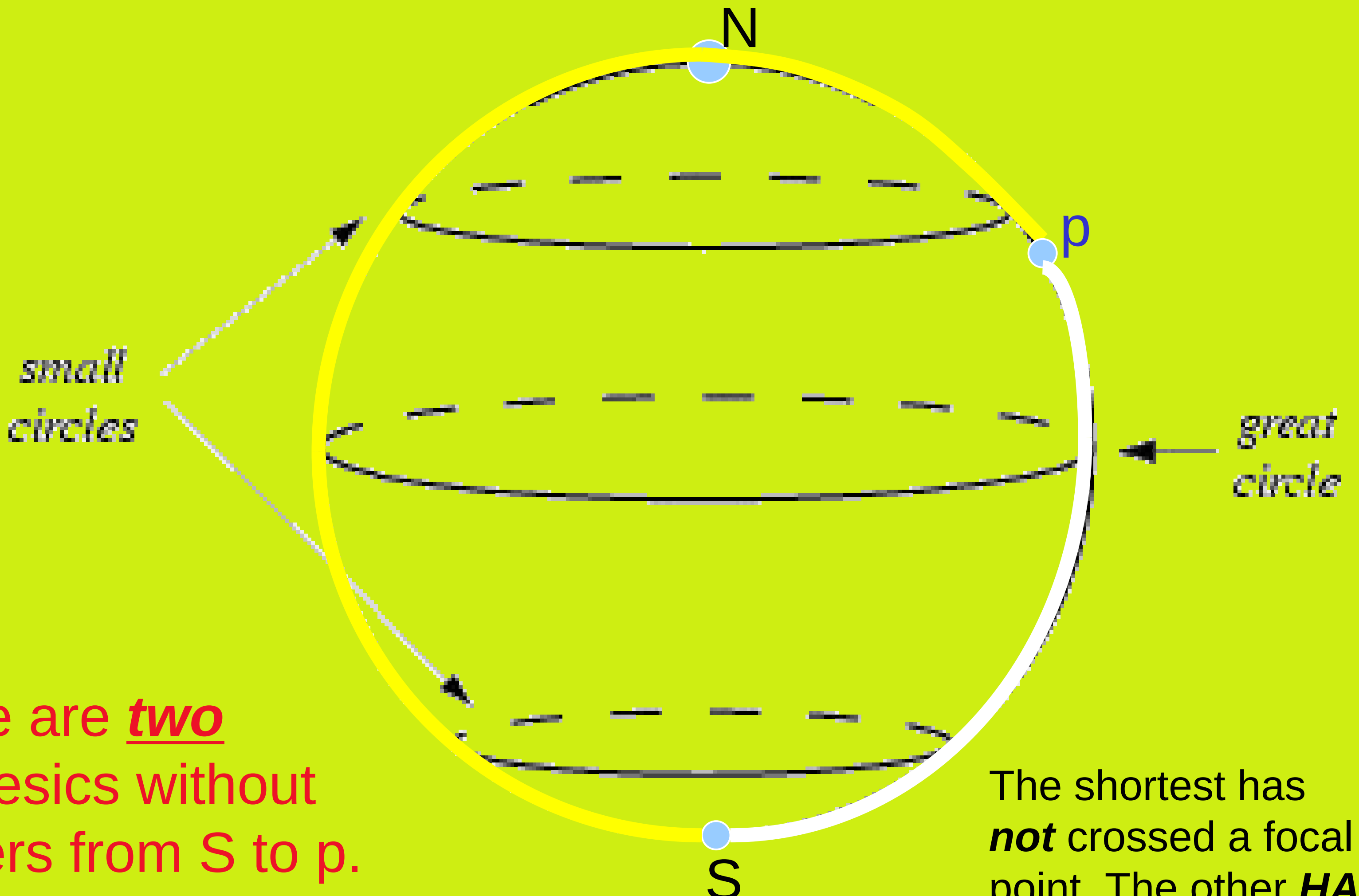
CORNERS, CONJUGATE/FOCAL POINTS



The curve p → S → q is geodesic… but with a corner !!

Corners make distance ***larger***

CORNERS, CONJUGATE/FOCAL POINTS



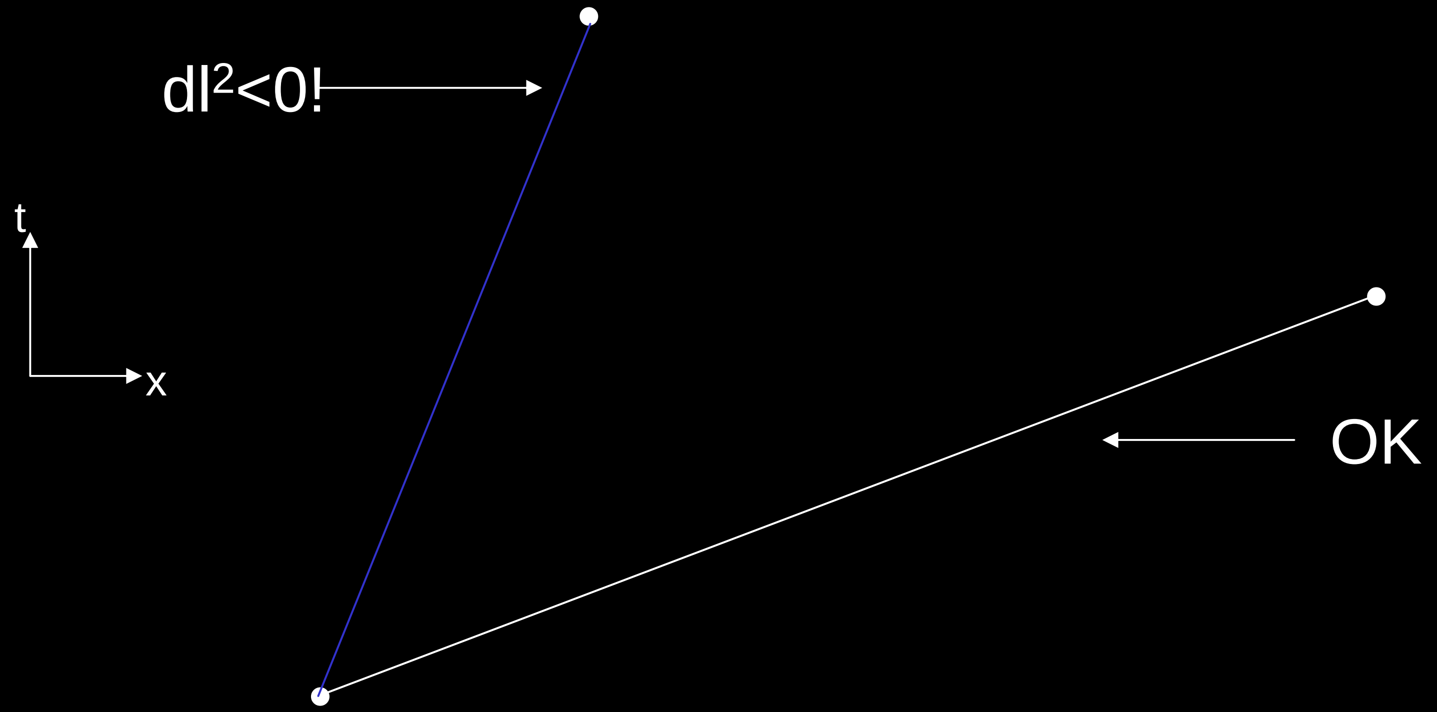
There are ***two*** geodesics without corners from S to p.

The shortest has ***not*** crossed a focal point. The other ***HAS***.

Minimum Length

A curve from p to q has the minimum length between p and q if and only if it is a **geodesic**, with no corners, and *no focal/conjugate* points from p.

Shortest?



If the length is “minkowskian”, $dl^2=dx^2 - dt^2$.

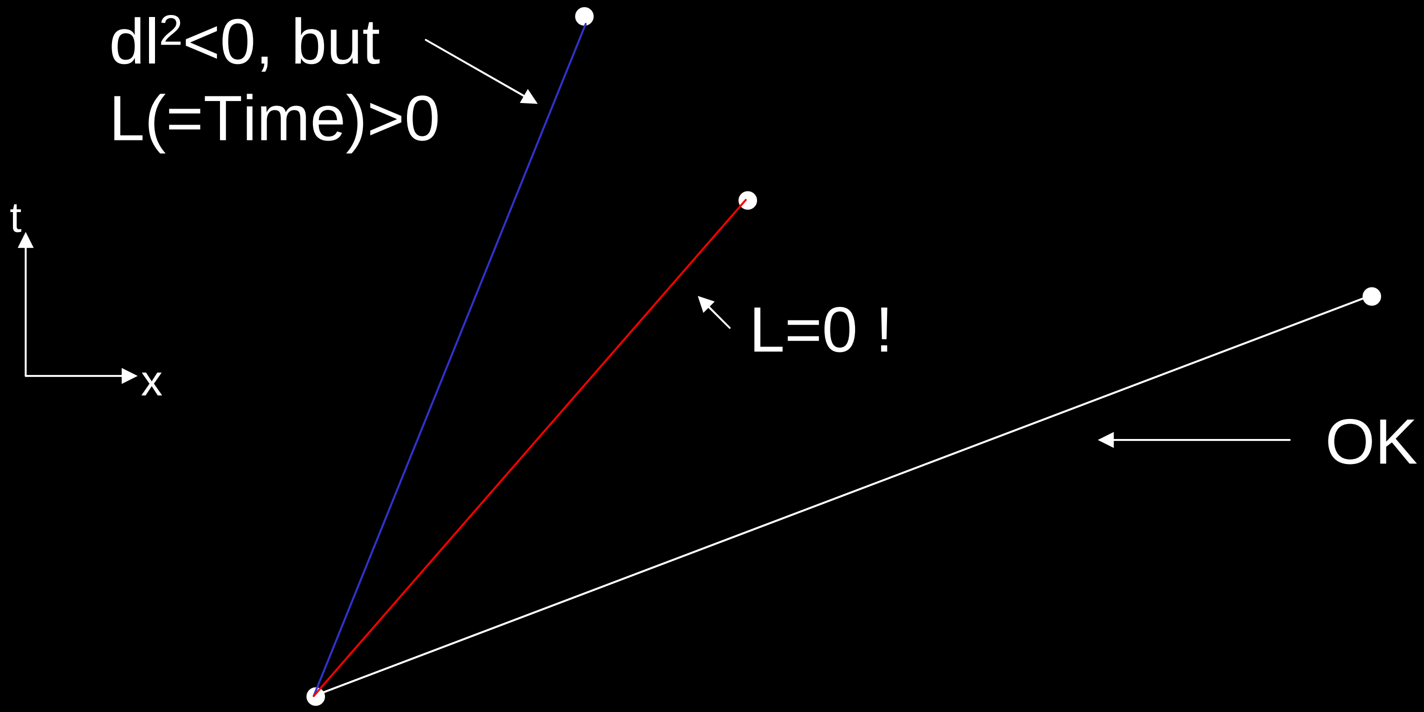
LENGTH/TIME

$$L(p \xrightarrow{\gamma} q) = \int_p^q \sqrt{\left|dl^2\right|}\Big|_\gamma$$

If $dl^2 > 0$ all along γ, L gives **distance**

If $dl^2 < 0$ all along γ, L gives ***time***

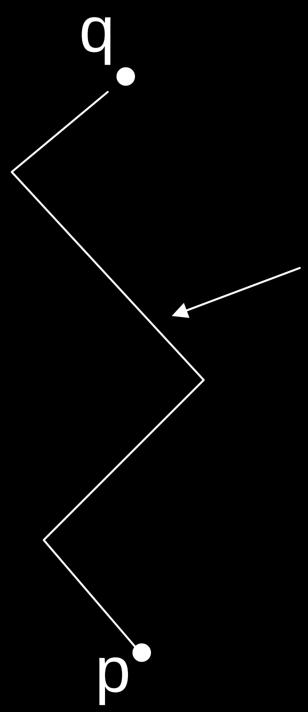
Shortest?



If the length is “minkowskian”, $dl^2=dx^2 - dt^2$.

Shortest?

There cannot be a minimum L when L is time!!!



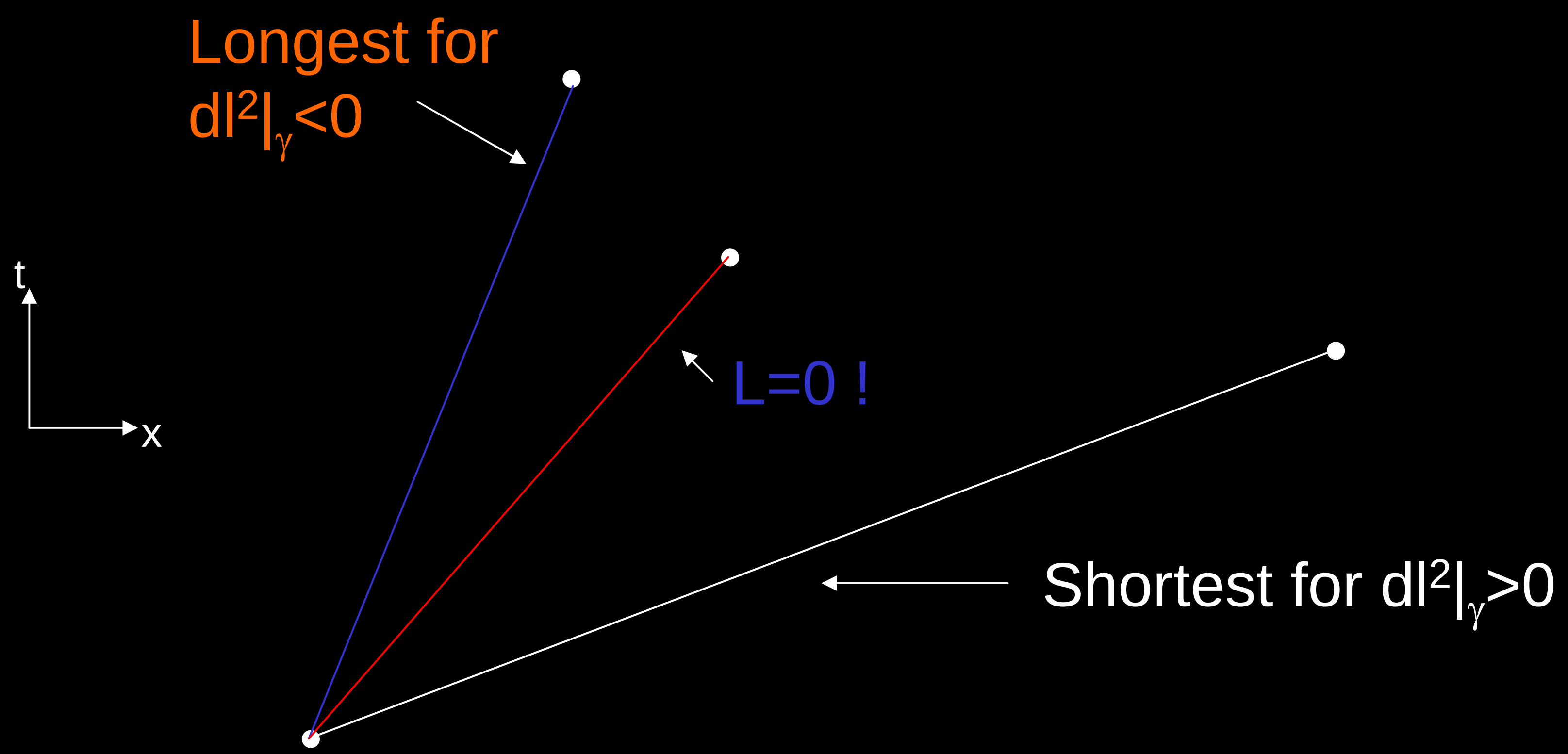
This curve, for instance, has many corners but total L = 0 !

What are the meaning of <u>**geodesics**</u> in these **SPACE-TIMES**?

$$\frac{\delta L}{\delta \gamma}(p \xrightarrow{\gamma} q) = 0$$

Of course, this geodesic condition implies that they are extremal curves with respect to L. But, what about the <u>**second variation**</u>?

Shortest/Longest !

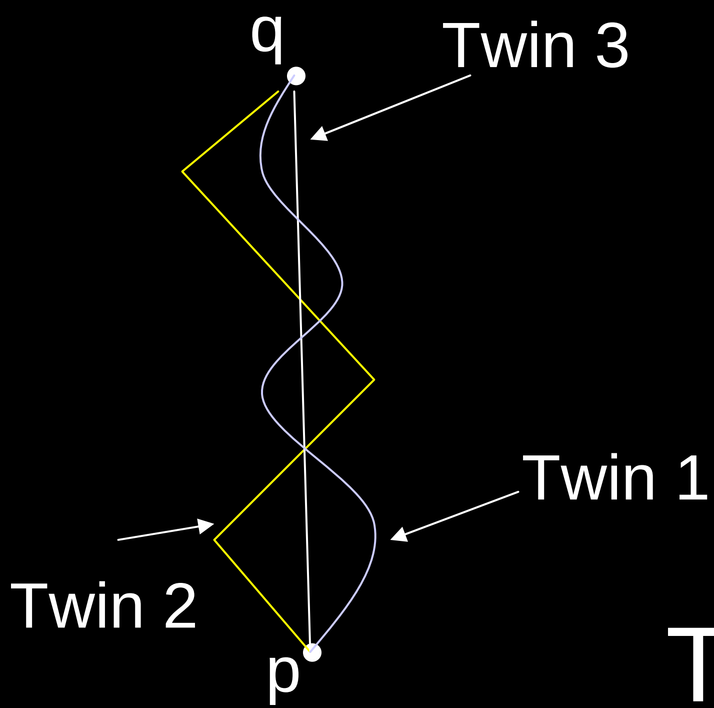


If the length is "minkowskian", $dl^2=dx^2 - dt^2$.

Maximum Time !

A curve from p to q has the maximum time between p and q if and only if it is a (<u>causal</u>) **geodesic**, with <u>no corners</u>, and *no focal/conjugate* points from p.

Twin "paradox"



The oldest twin is always the one in **<u>free fall</u>** (= geodesic)

Pattern singularity theorem

- If the spacetime is <u>smooth</u> and satisfies:

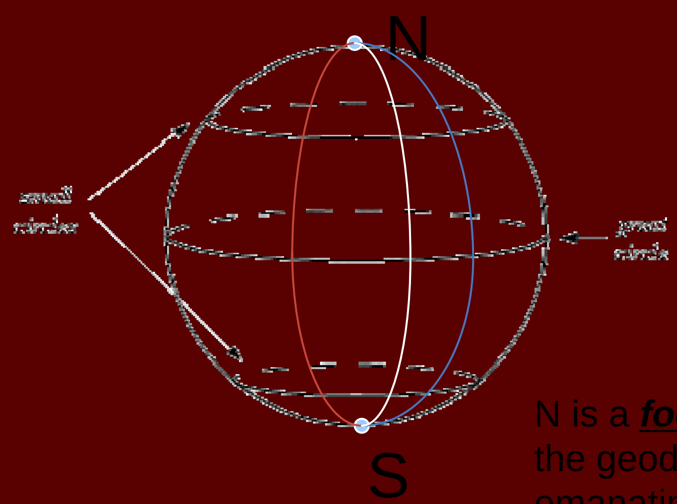
- A condition on the curvature
- A causality condition
- And a Boundary/Initial condition

- THEN, it has incomplete causal geodesics (i.e., Singularities).

The condition on the curvature

It is needed to ensure the focalization of geodesics.

Recall :



N is a ***focal point*** for the geodesic family emanating from S

We can see, intuitively, that the presence of focal points arises due to the curved form of the sphere, i.e., to the presence of **CURVATURE**.

The condition on the curvature

is many times disguised as an “energy condition”

Riemann

=

Weyl

+

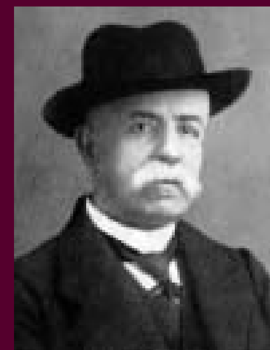
Ricci-Curbastro

The condition on the curvature

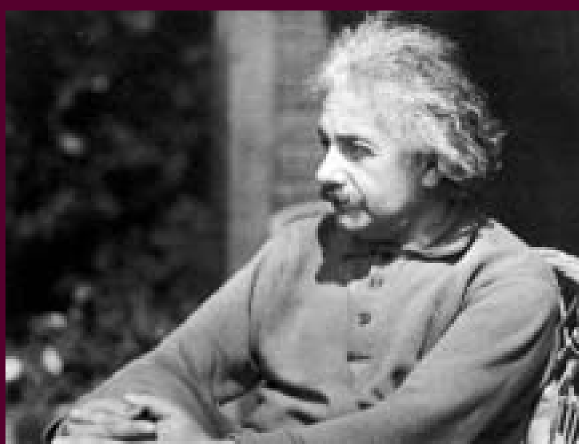
is many times disguised as an “energy condition”

=

+



“genericity”



Einstein’s equations

Energy/mass distribution

Pattern singularity theorem

- If the spacetime is <u>smooth</u> and satisfies:

- A condition on the curvature
- A causality condition
- And a Boundary/Initial condition

- THEN, it has incomplete causal geodesics (i.e., Singularities).

The causality condition

It is necessary in two respects:

(i) To prohibit avoiding the future, the passing of time;

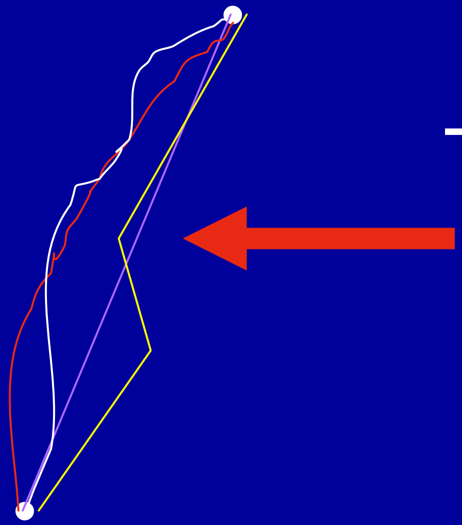
(ii) To ensure the existence of maximal geodesics

The strongest (and more effective for the ST) causality condition is called

<u>**Global Hyperbolicity**</u>

The causality condition

A spacetime is said to be globally hyperbolic if the set of all causal paths between any two points is **compact**



Therefore, there always exists at least one curve with the maximum L (=time).

The causality + curvature conditions

Observe:

(i) Curvature implies the existence of focal points for all endless geodesics

(ii) Global hyperbolicity ensures the existence of maximal curves between any two (causally related) points.

YET, these two conditions by themselves do **NOT provide a contradiction**

Pattern singularity theorem

- If the spacetime is <u>smooth</u> and satisfies:

- A condition on the curvature
- A causality condition
- **A Boundary/Initial condition**

- THEN, it has incomplete causal geodesics (i.e., Singularities).

The boundary/initial condition

It is absolutely essential

- It may adopt several forms:

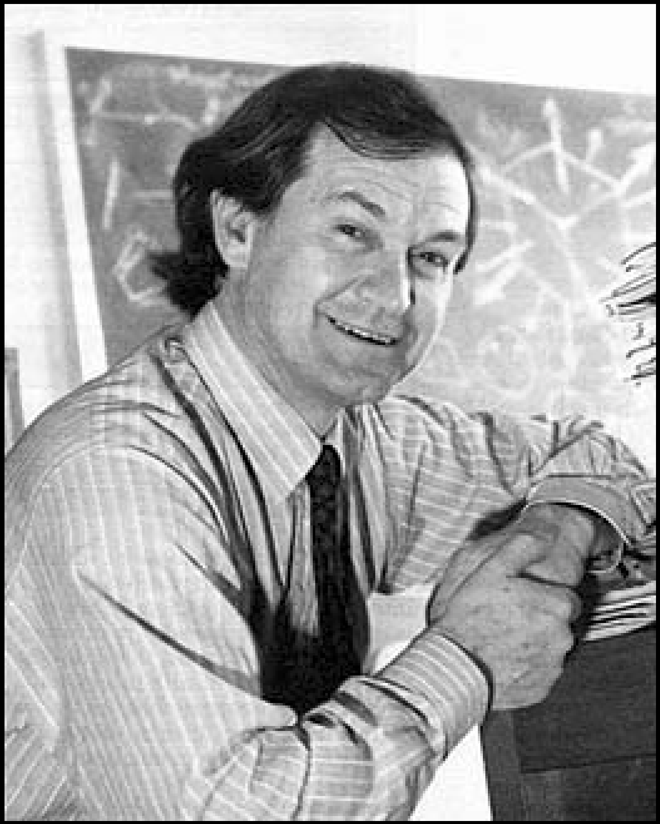
(i) The existence of a ***Trapped Surface***

(ii) An instant of time where ALL the universe is strictly expanding

(iii) Enough curvature (or matter) such that the light cones refocuse completely

(ii) and (iii) are easily understandable, but what about (i) ?

The boundary/initial condition

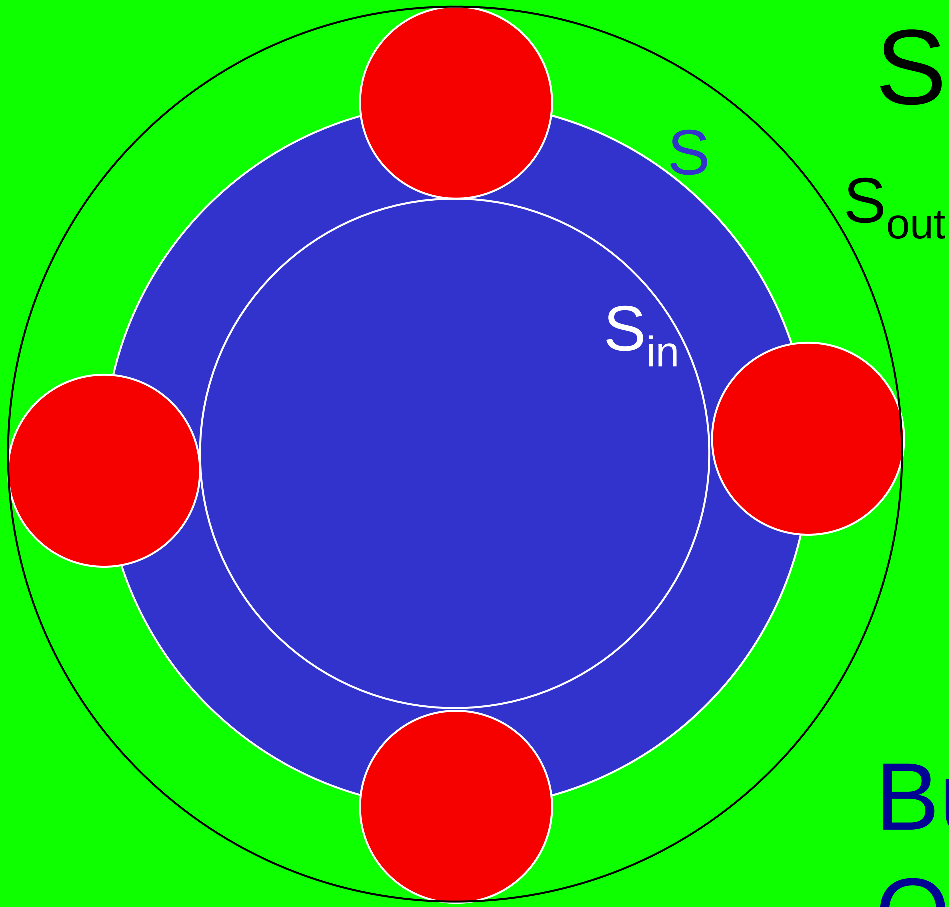


The concept of closed trapped surface is a breakthrough in science, a beautiful and powerful idea, due to Penrose, who had an extraordinary insight and changed relativistic science for ever more.

What is less known, and hardly mentioned, is that the concept of trapped surface is **<u>purely geometric</u>**

PASEN..., Y VEAN!

Evolving surfaces



Of course,

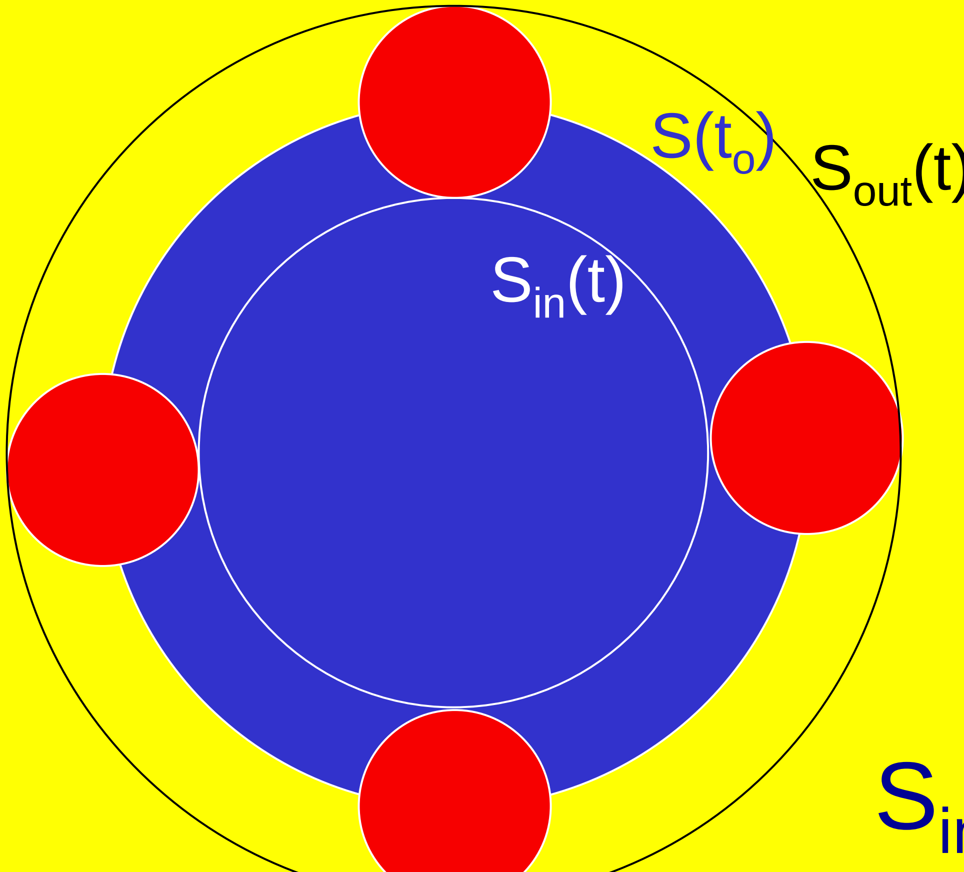
$S_{in} < S_{out}$

But, $S_{in} < S$?

Or $S < S_{out}$?

Trapped surfaces?

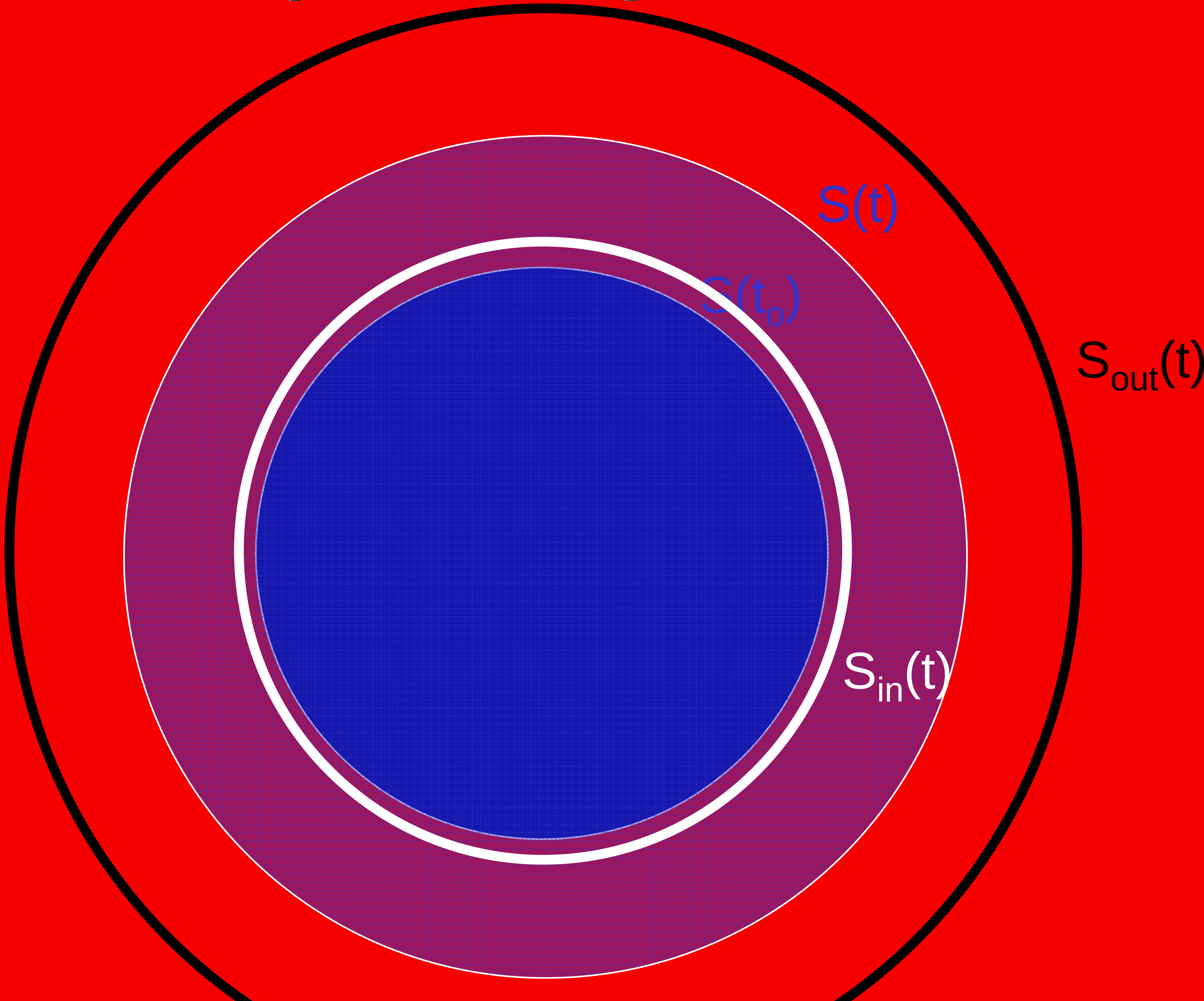
$t > t_o$



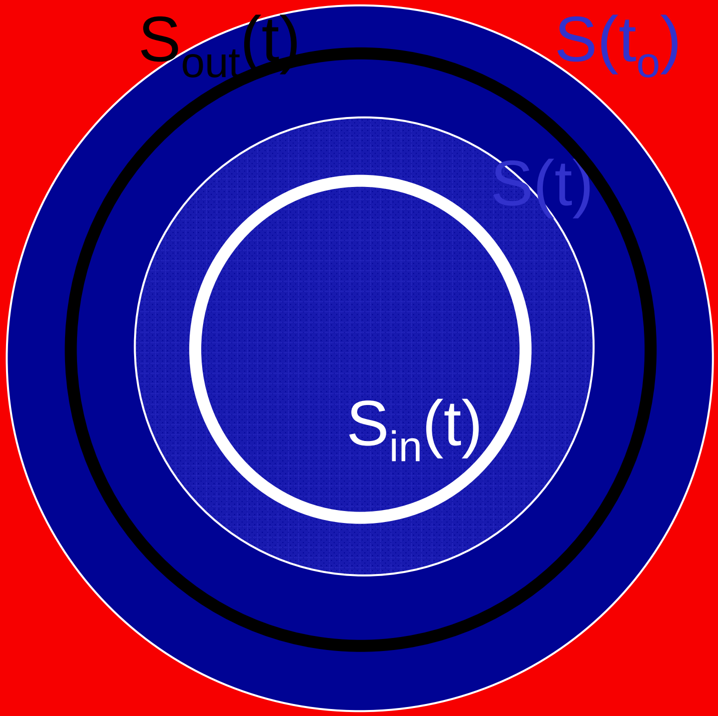
$S_{in}(t) < S(t_o)?$

$S(t_o) < S_{out}(t)?$

Expanding world!



Contracting world!



Trapped surfaces!

Expanding:	$S_{in}(t) > S(t_o)$ is feasible	"Past trapped surface"
Contracting:	$S_{out}(t) < S(t_o)$ is possible	"Future trapped surface"

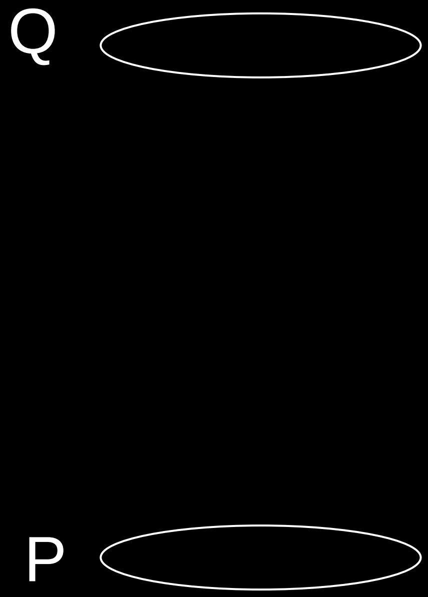
$(t>t_o)$

Of course, this (expansion/contraction) may change with time t!

Trapped surfaces (mathematically revisited)

Less known is the fact that the concept of trapped surface is purely geometric, and closely related to that of ***minimal surfaces*** (and geodesics).

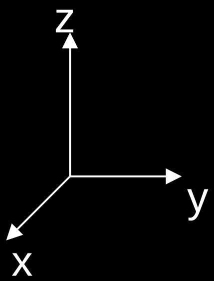
What surface has the minimum area?

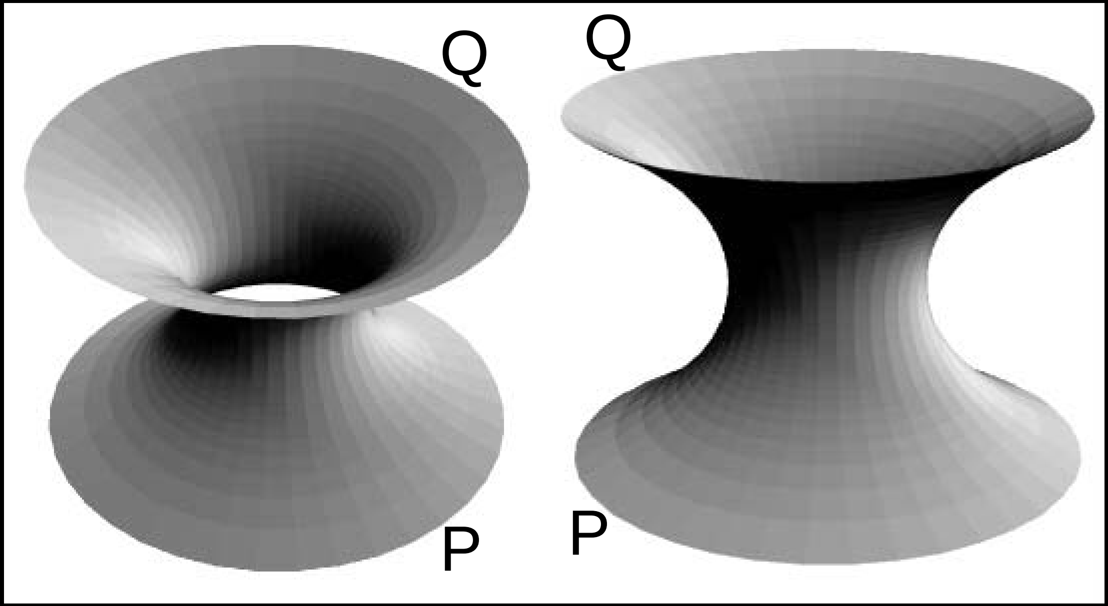


In analogy with the shortest path discussed before, one can ask, for given lines such as P and Q on the left, which surface with boundary on P and Q has the minimum area?

Of course, again the answer depends on the “length element”!!

Minimum area?

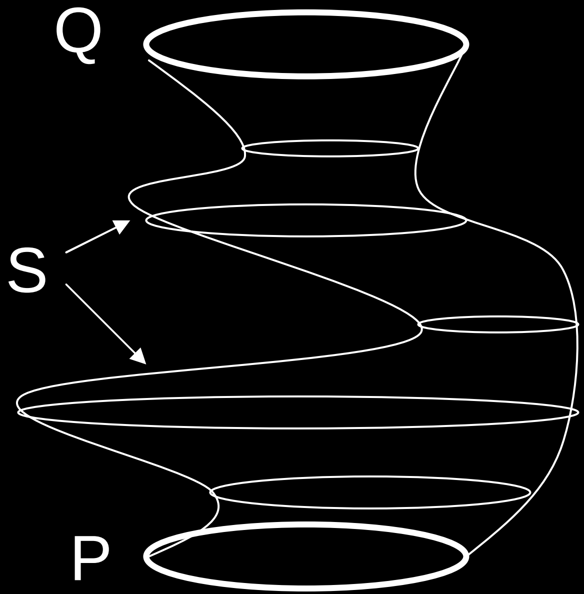




CATENOID (discovered by Meusnier in 1776)

If the line element is "euclidean", $dl^2=dx^2+dy^2+dz^2$.

AREA



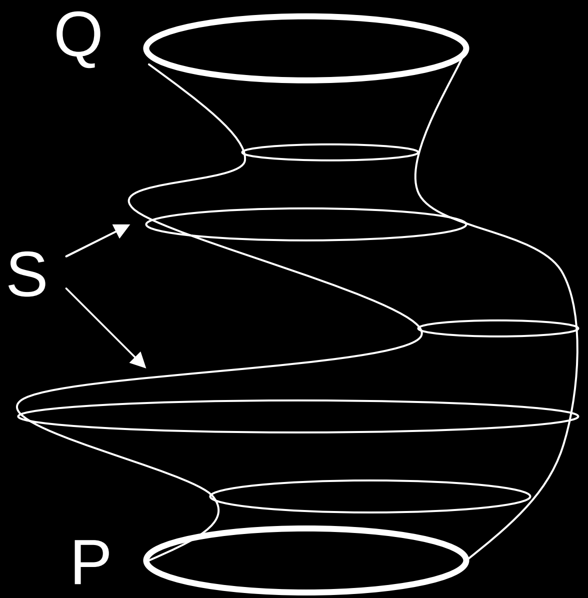
$$A(P \xrightarrow{S} Q) = \int_P^Q dl \Big|_S$$

If the length is riemannian, e.g.:

$dl^2 = F^2(x,y,z)\, dx^2 + G^2(x,y,z)\, dy^2 + H^2(x,y,z)\, dz^2$

(Observe that this is a double integral)

Minimum area?

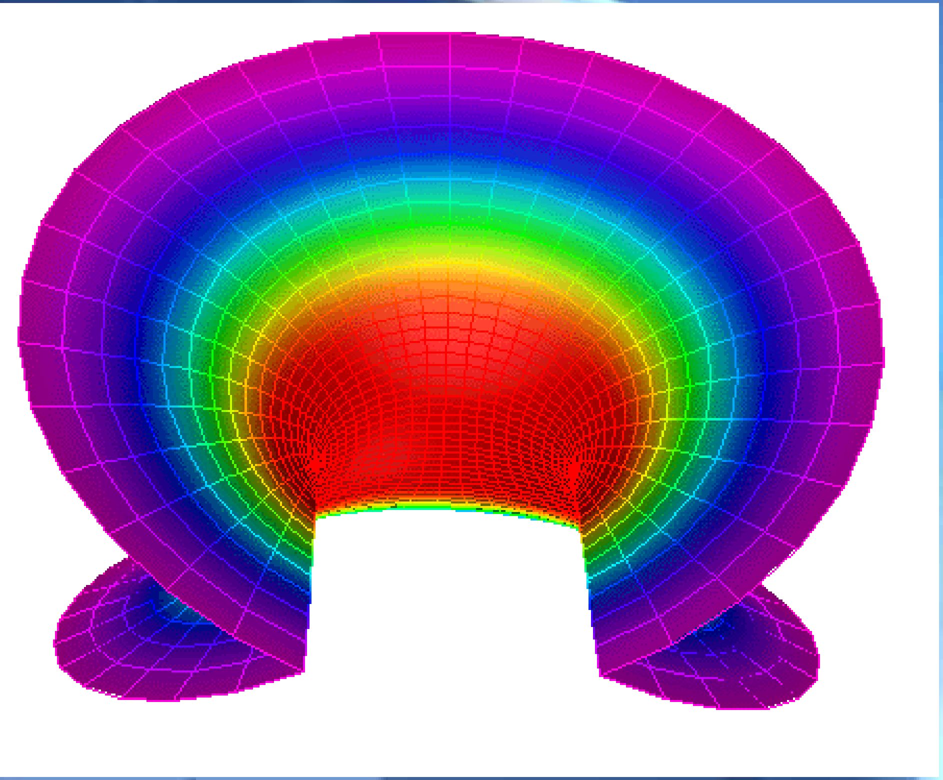


$$\frac{\delta L}{\delta S}\left(P \xrightarrow{S} Q\right) = 0$$

Minimal surfaces

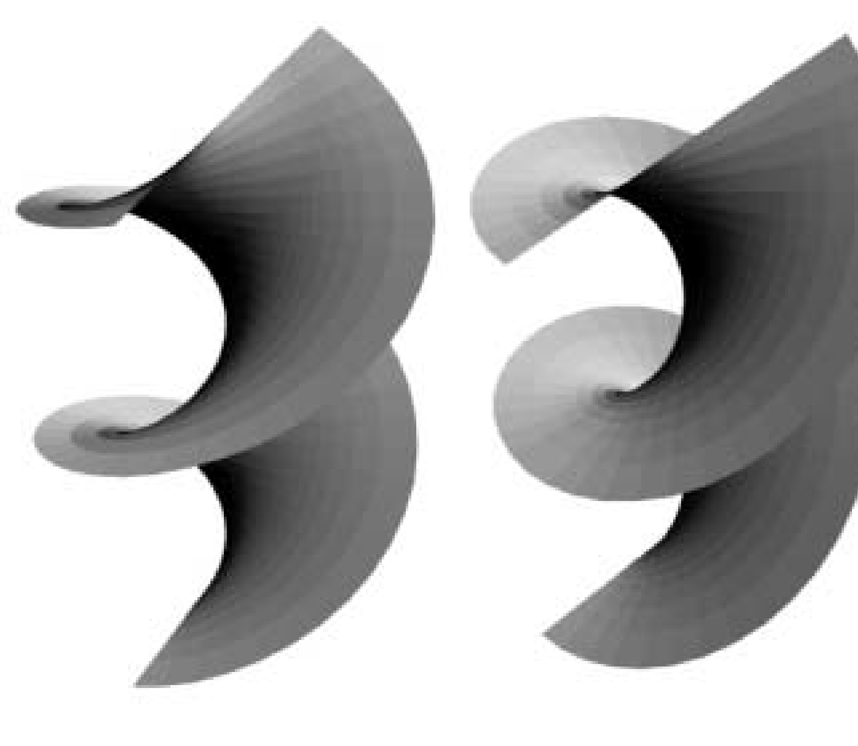
Minimal surfaces are equivalently characterized by the vanishing of the ***mean curvature*** (vector). Observe that again, this is an ***extrinsic*** property!

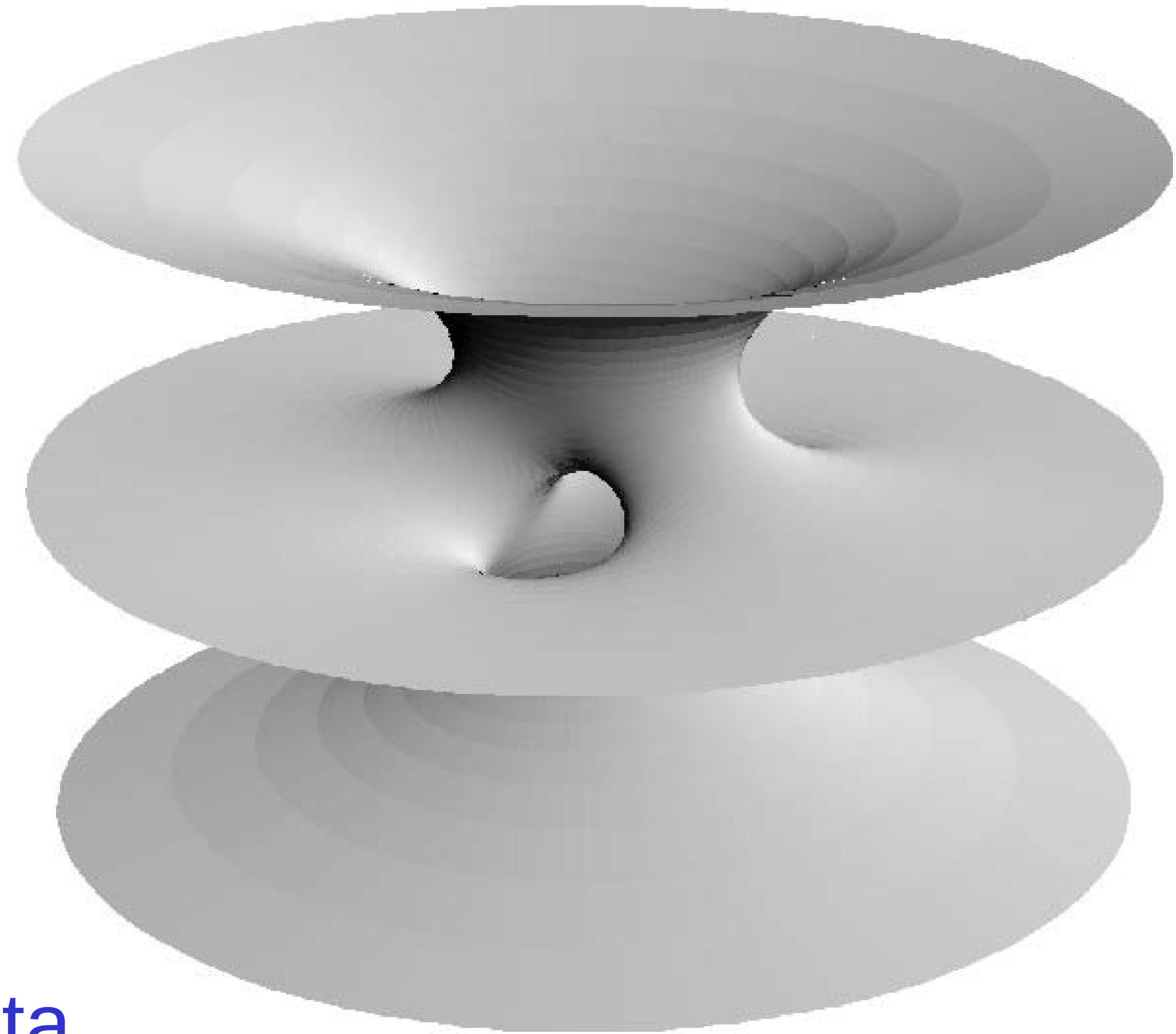
Examples of minimal surfaces in flat space



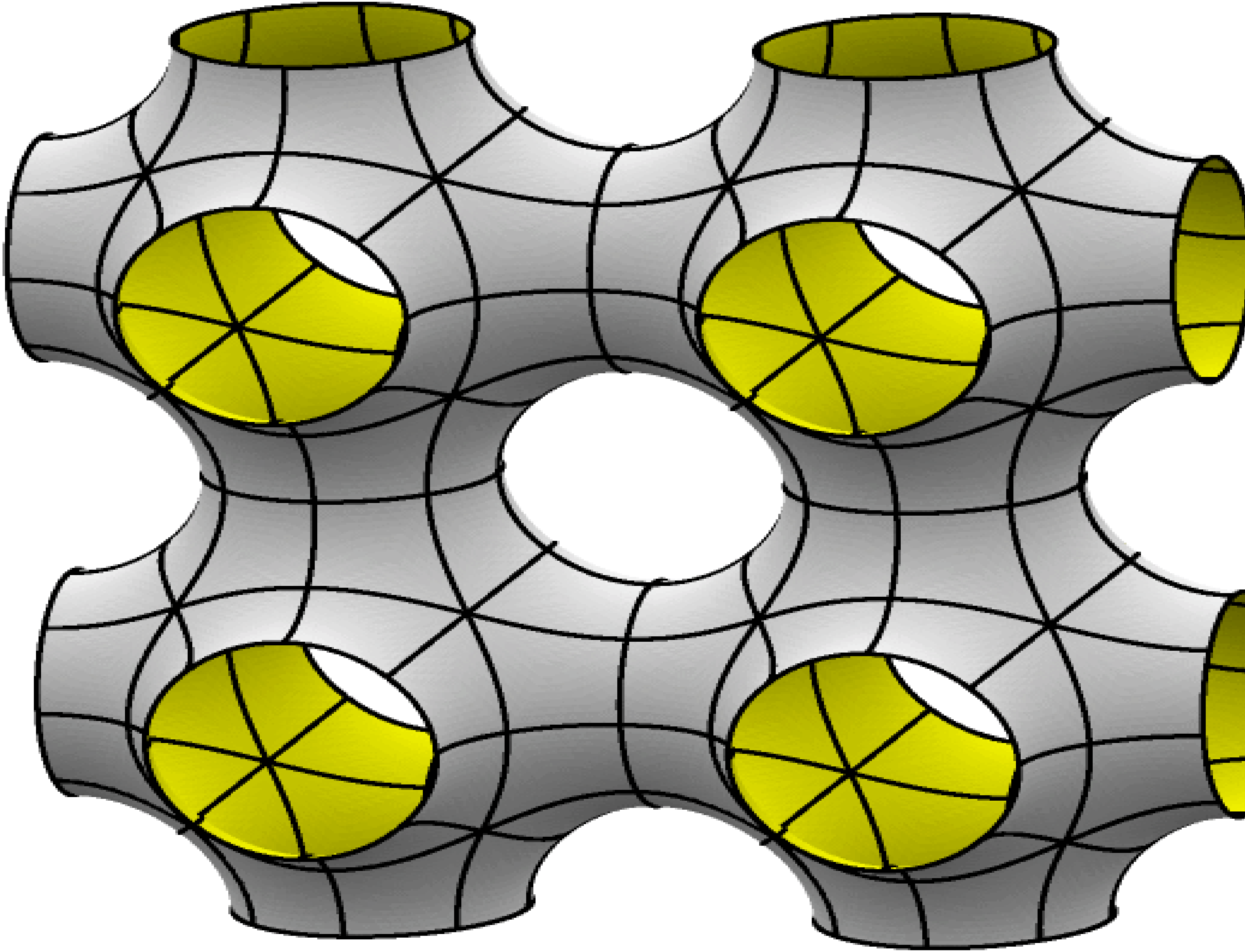
Catalan

Helicoid

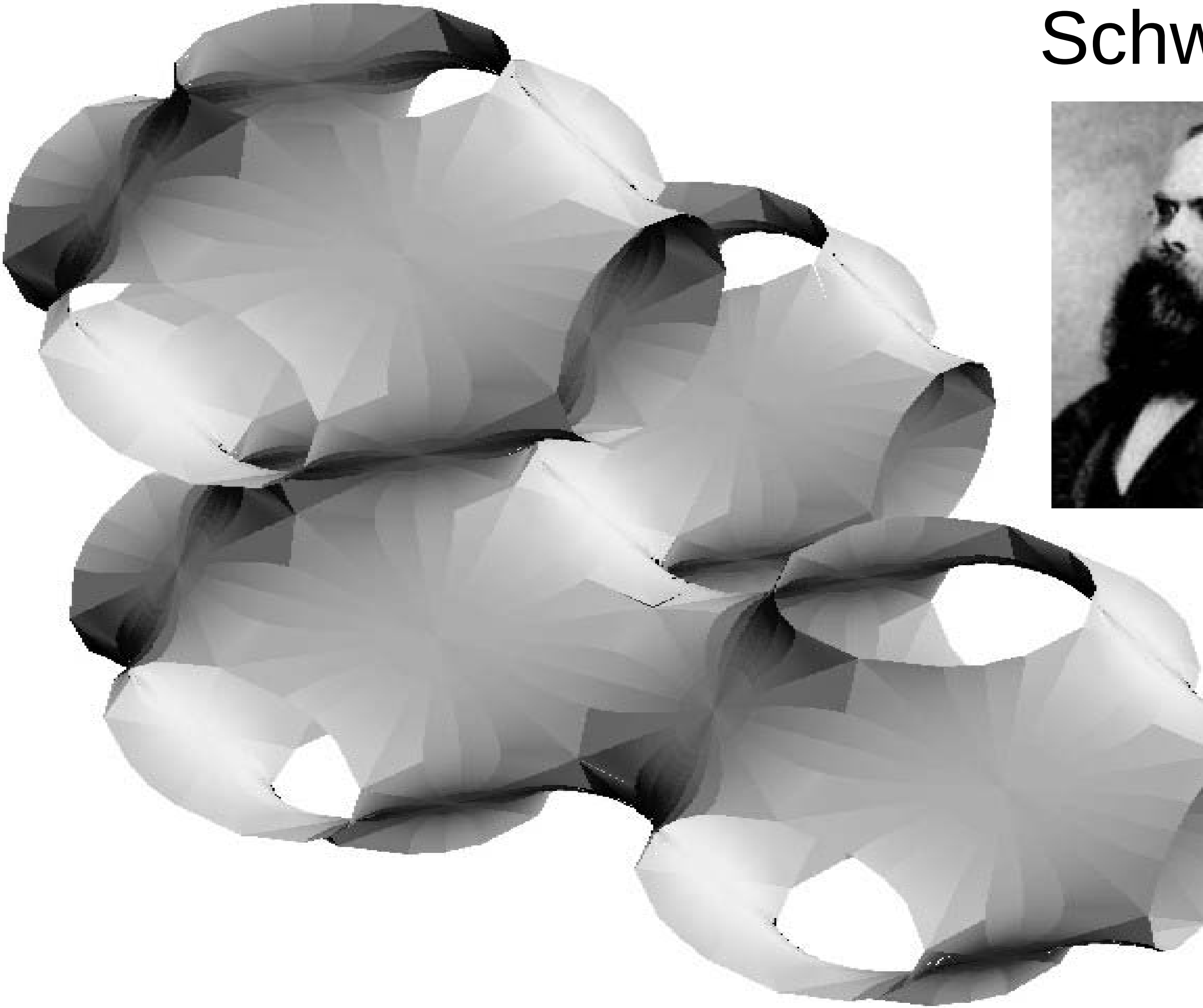




Costa



Schwarz



What about higher dimensions?

The mean curvature looks like a scalar (the mean of the principal curvatures) due to the existence of only **ONE** extra dimension. If the ***co-dimension*** is > 1, then it is very simple to see that the mean curvature is a (normal) *vector*: it may have components along all directions orthogonal to the surface.

This is exactly as with geodesics, which in 3-dimensional space, as is well known, have a curvature vector (the acceleration) which is orthogonal to the curve.

The number of independent components of these vectors is in general the co-dimension.

Four (or higher) dimensions!

Imagine the space has dimension > 3. If the length is positive definite, e.g.

$$dl^2 = dx^2 + dy^2 + dz^2 + \ldots + dw^2$$

(so, no time!).

Then, the mean curvature vector $\vec{H}$, <u>as any other vector</u>, has a non-negative norm, and the norm vanishes if and only if the mean curvature vector vanishes:

$\vec{H} \bullet \vec{H} \geq 0$; furthermore, $\vec{H} \bullet \vec{H} = 0 \Longleftrightarrow \vec{H} = \vec{0}$

<u>The only distinguished case is that of minimal surfaces!!</u>

Four (or higher) dimensions!

However, if there is time

$dl^2 = - dt^2 + dx^2 + dy^2 + dz^2 + \ldots + dw^2$

Then, vectors may have negative and vanishing norm!

There appear TWO new possibilities:

- ☒ $\vec{H} \cdot \vec{H} < 0$; These are **TRAPPED SURFACES**!!
- ☒ $\vec{H} \cdot \vec{H} = 0$, *BUT* $\vec{H} \neq \vec{0}$; These are "marginally trapped": they form the **HORIZONS** !

Do ST apply to actual, physical, situations?

Who cares?

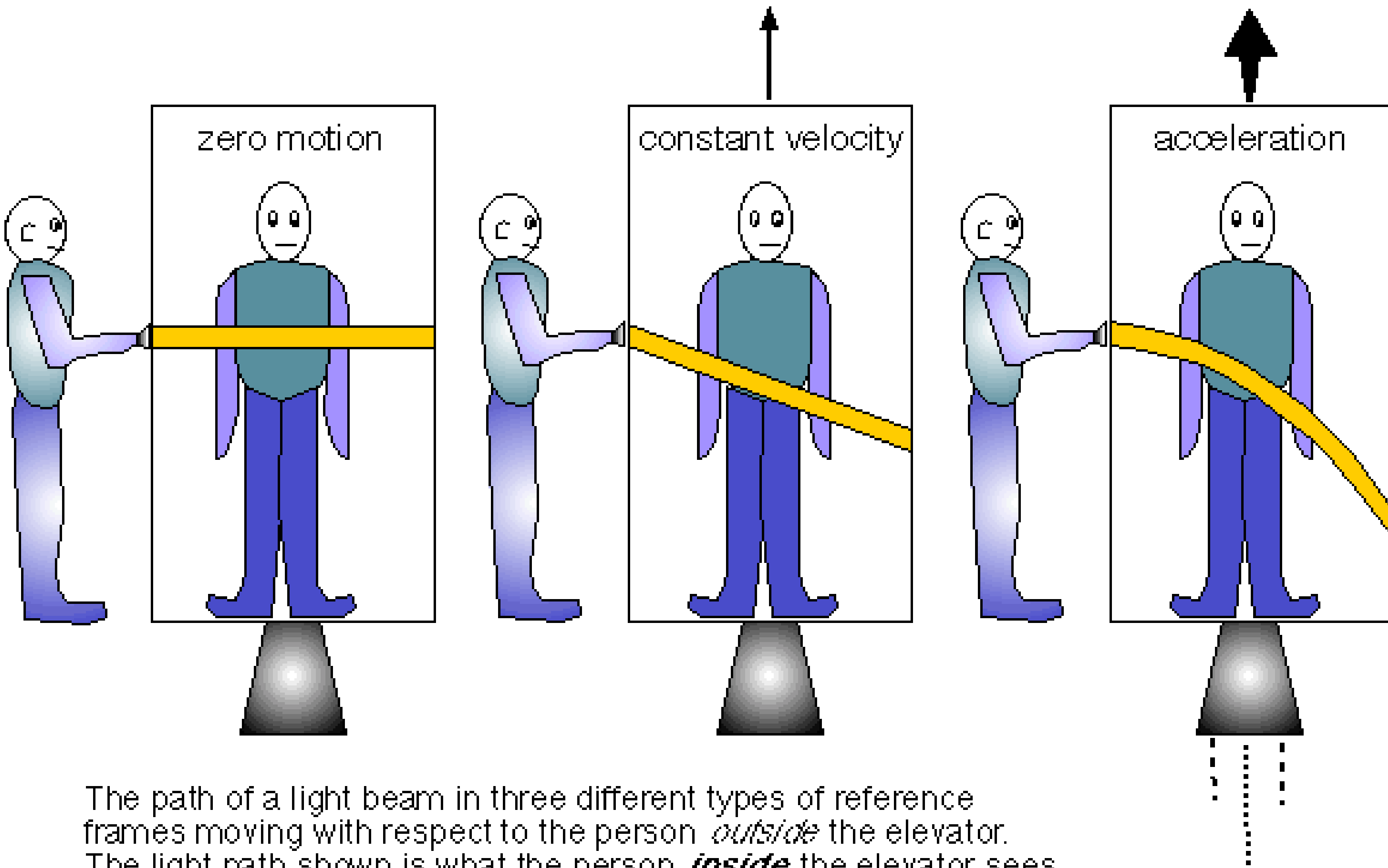
Yes, mainly to two real physical situations:

(i) Gravitational collapse (and formation of “black holes”.)
(ii) Cosmology (and the classical origin or end of everything.)

Why? How?

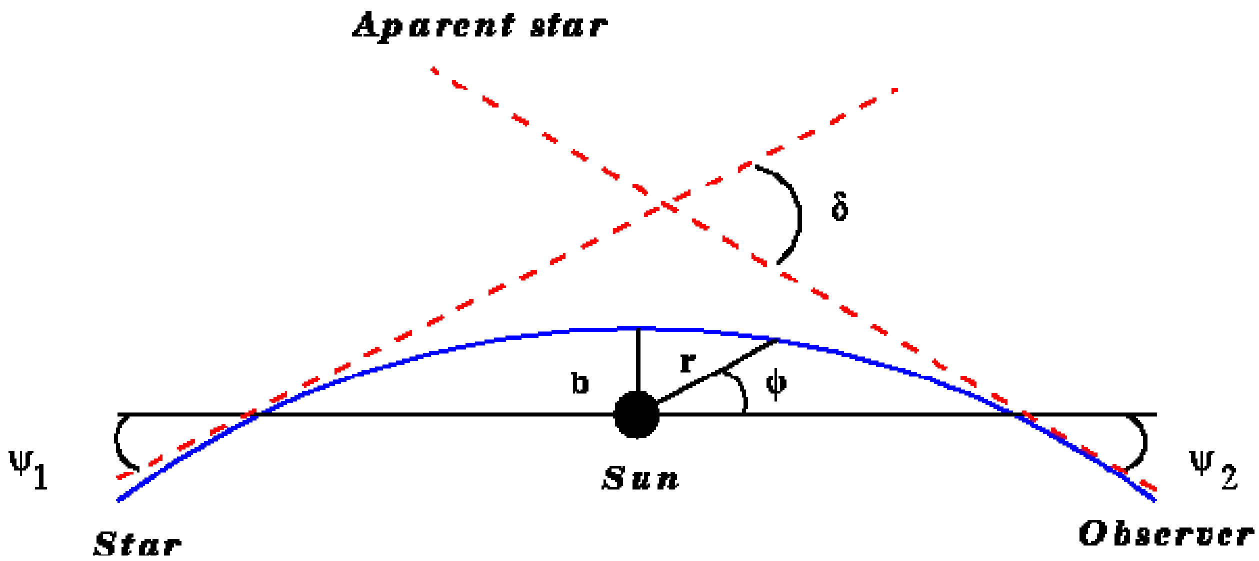
Gravitational collapse

The gravitational field affects light!

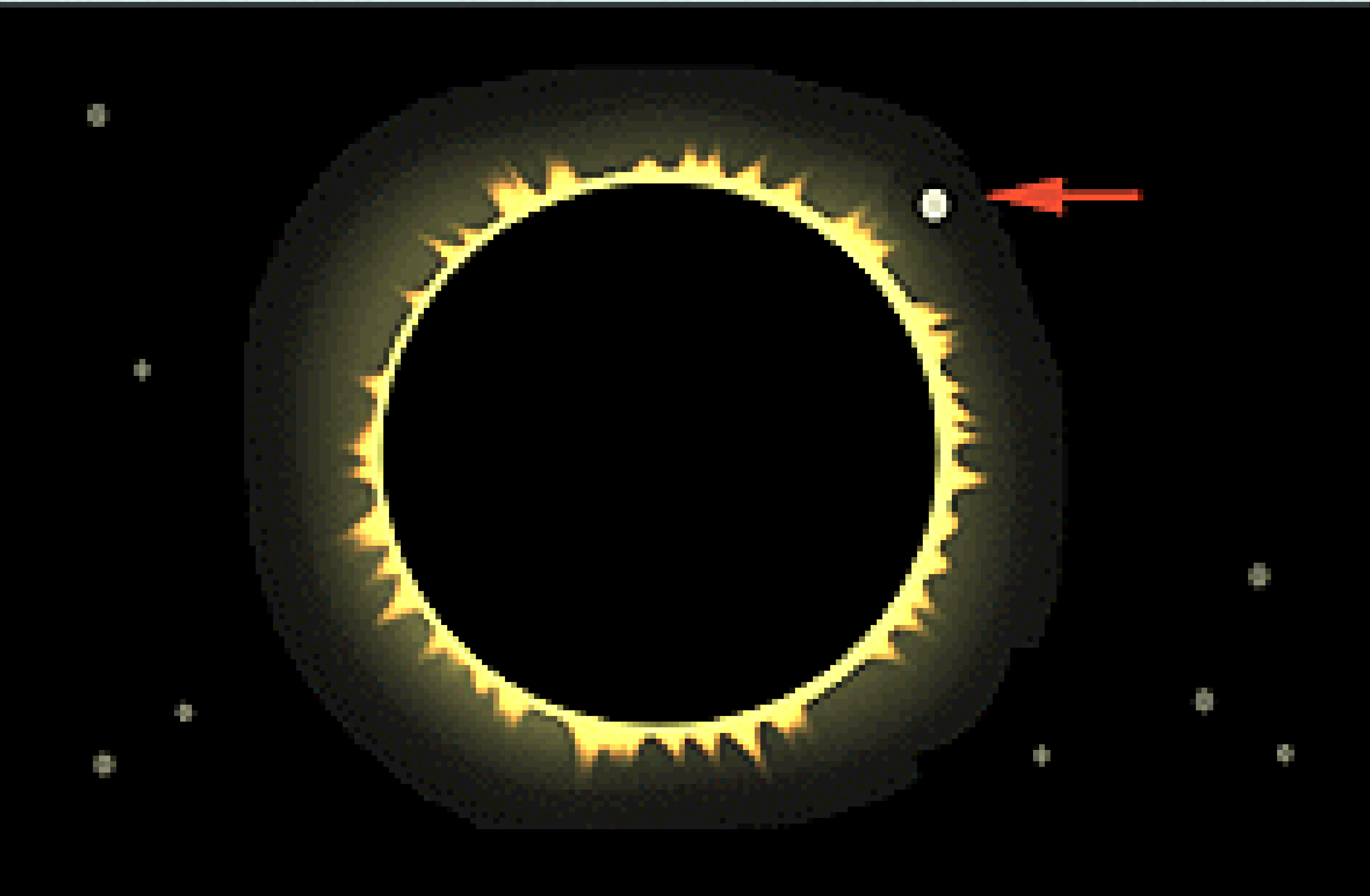


The path of a light beam in three different types of reference frames moving with respect to the person *outside* the elevator. The light path shown is what the person ***inside*** the elevator sees. Under large acceleration, the beam of light will curve downward. It should also do that in a region of strong gravity.

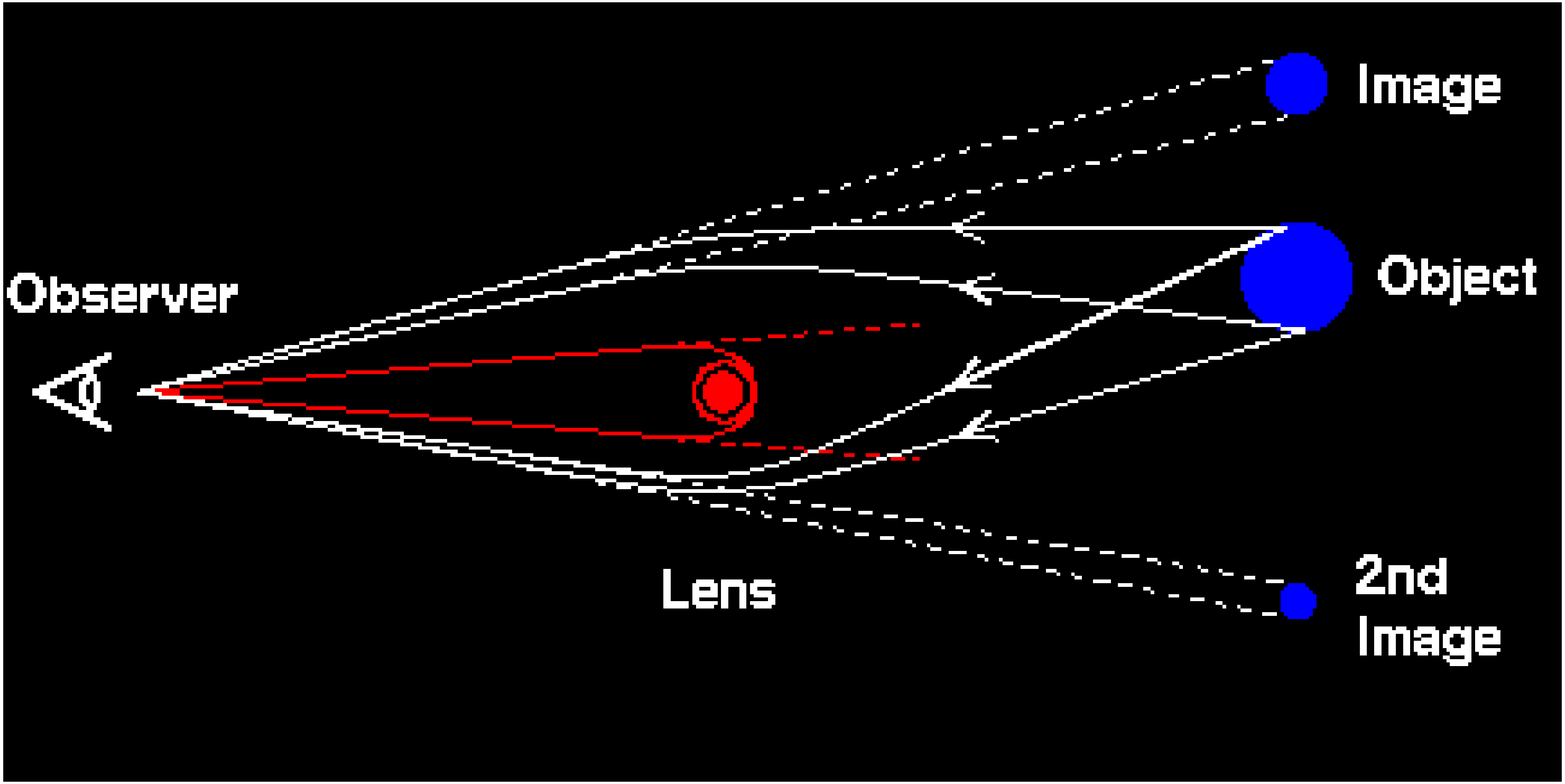
Bending of light by a large mass

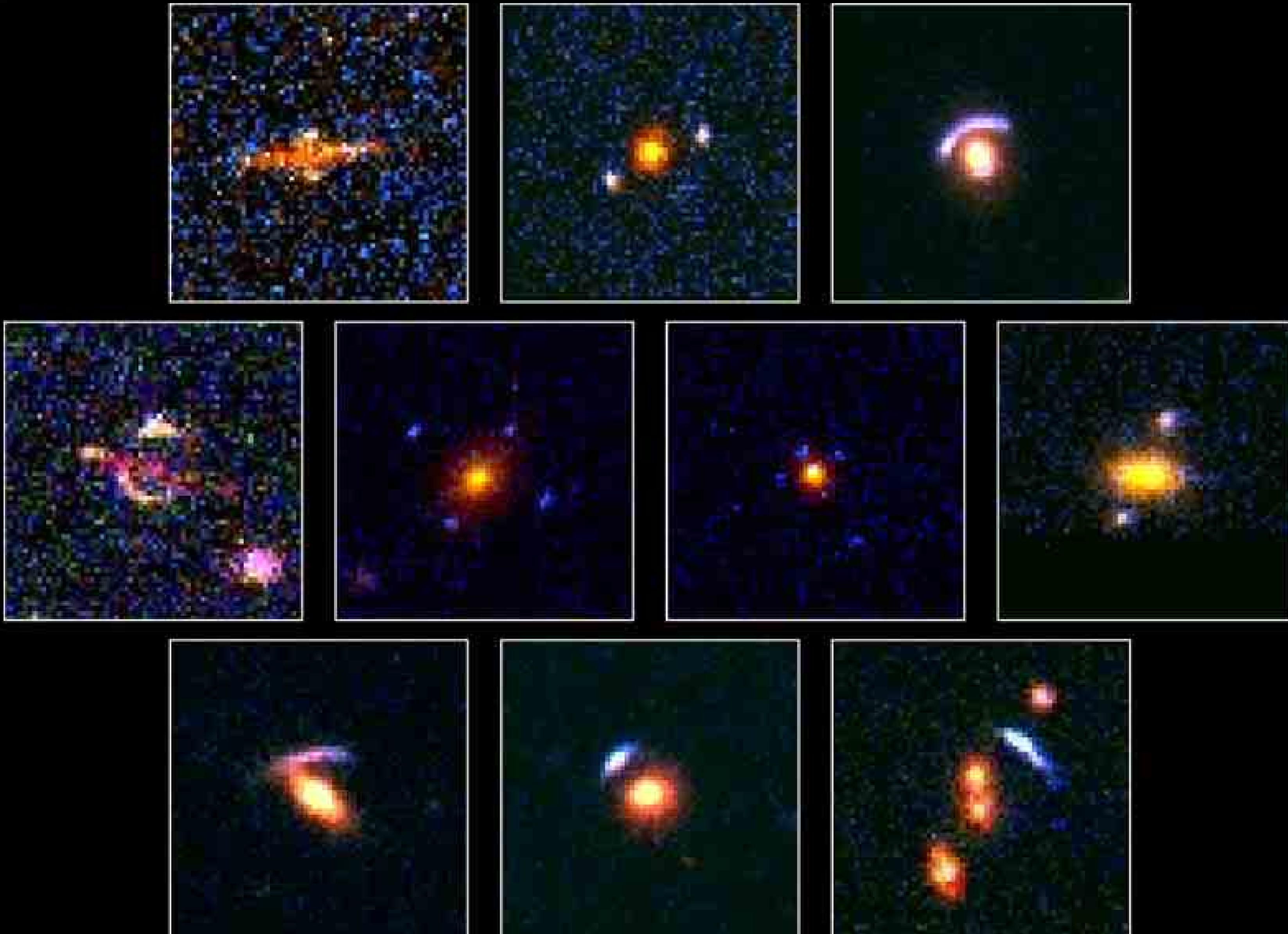


BENDING LIGHT



Gravitational lenses!

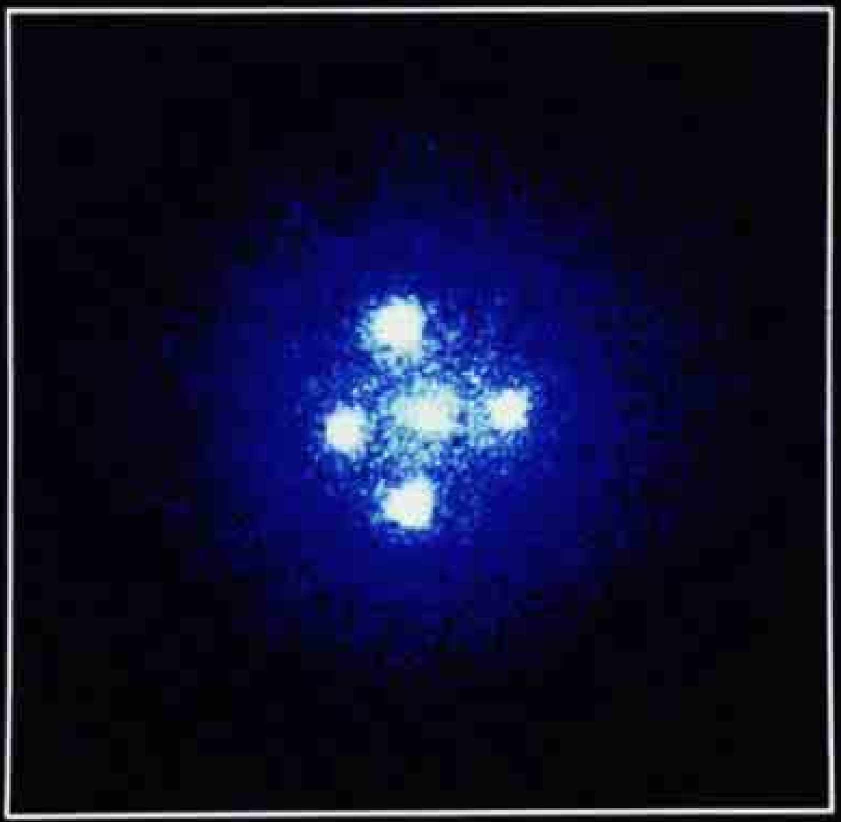




Gallery of Gravitational Lenses **HST • WFPC2**

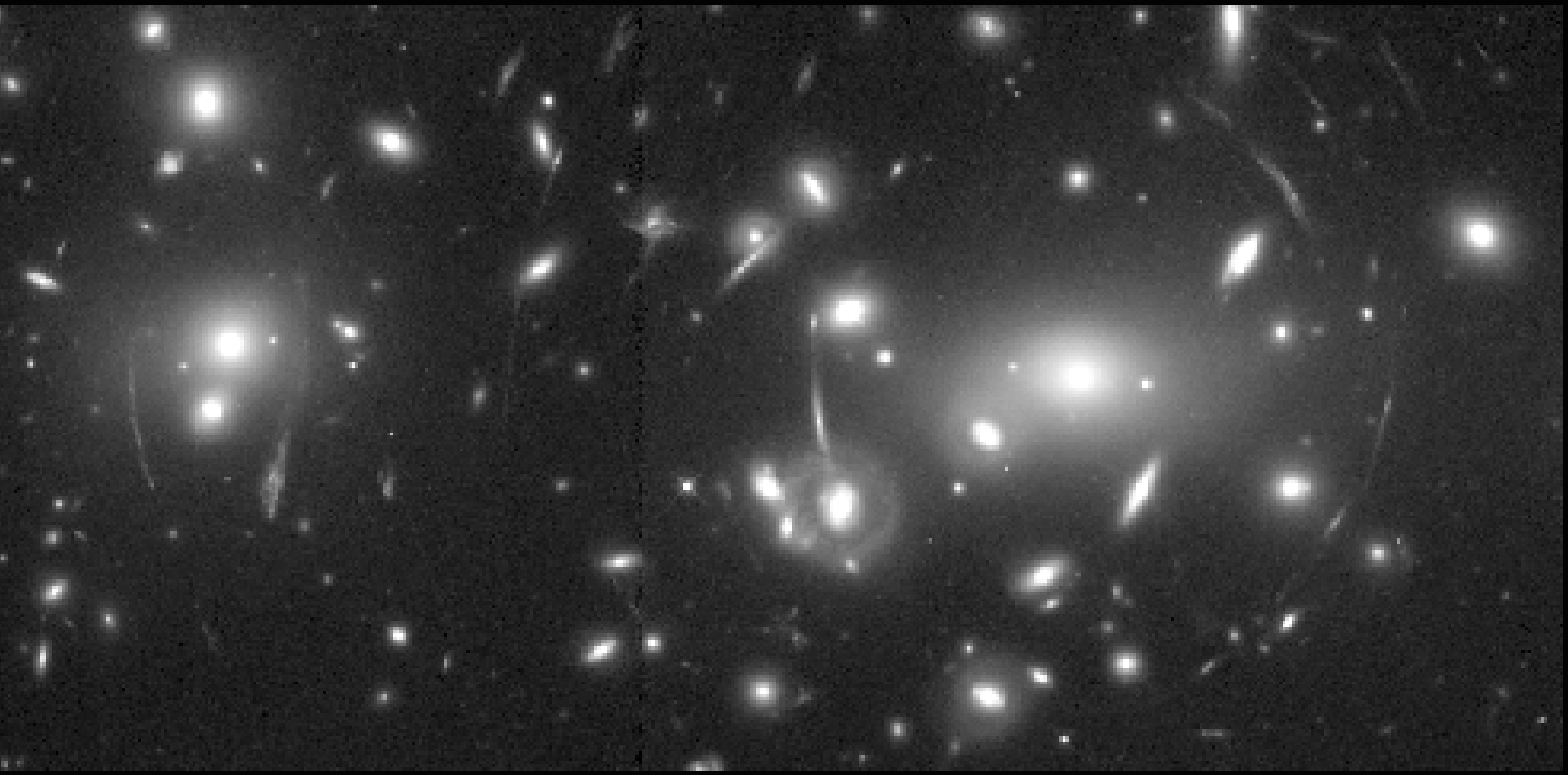
PRC99-18 • STScI OPO • K. Ratnatunga (Carnegie Mellon University) and NASA

The Einstein Cross



Gravitational Lens G2237+0305

Abell 2218: A Galaxy Cluster Lens



Gravitational collapse

The gravitational field affects light!

→ **Trapped surface?**

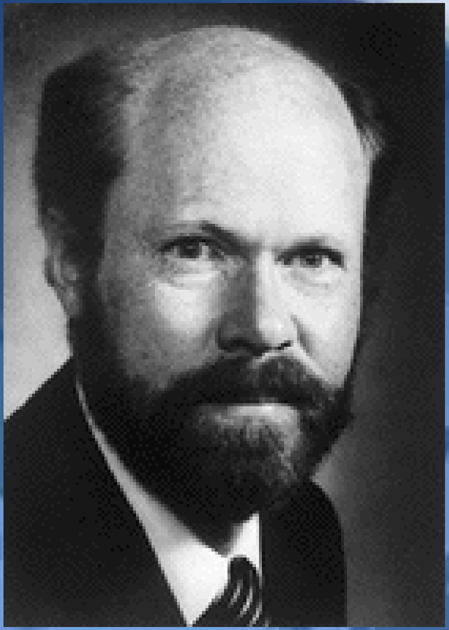
The formation of trapped surfaces in realistic processes of gravitational collapse is one the main areas of research and interest in **numerical relativity**.

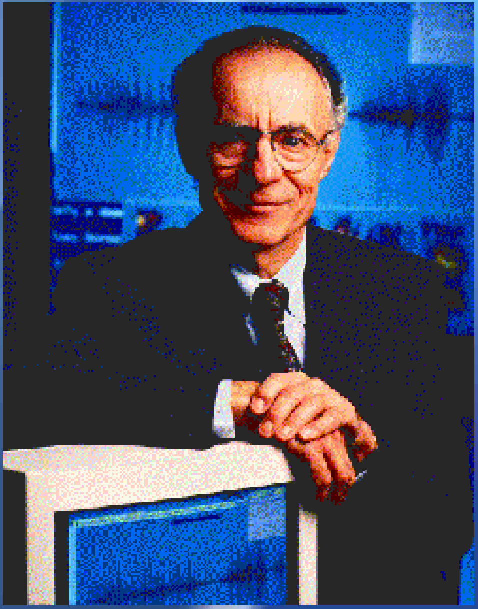
Cosmology

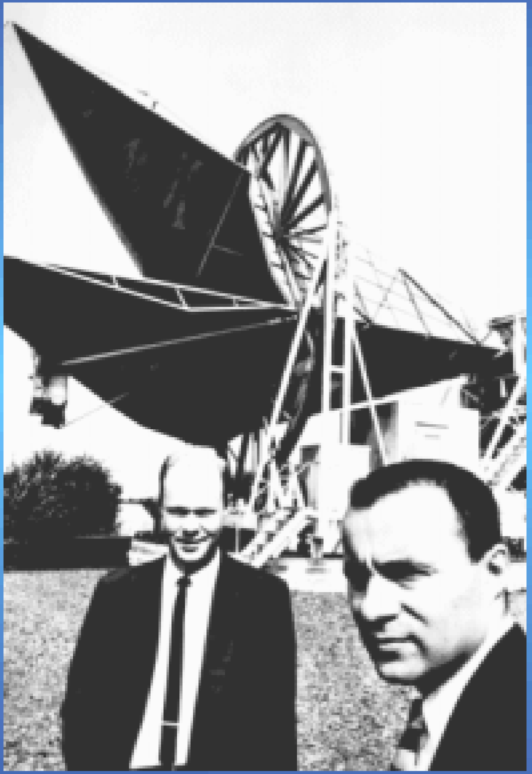
The ST may apply to the Universe as a whole, and imply the (classical) singular origin --- and possibly end--- of the entire matter and energy content of the Universe due to THREE ***observed facts***

1) Expansion

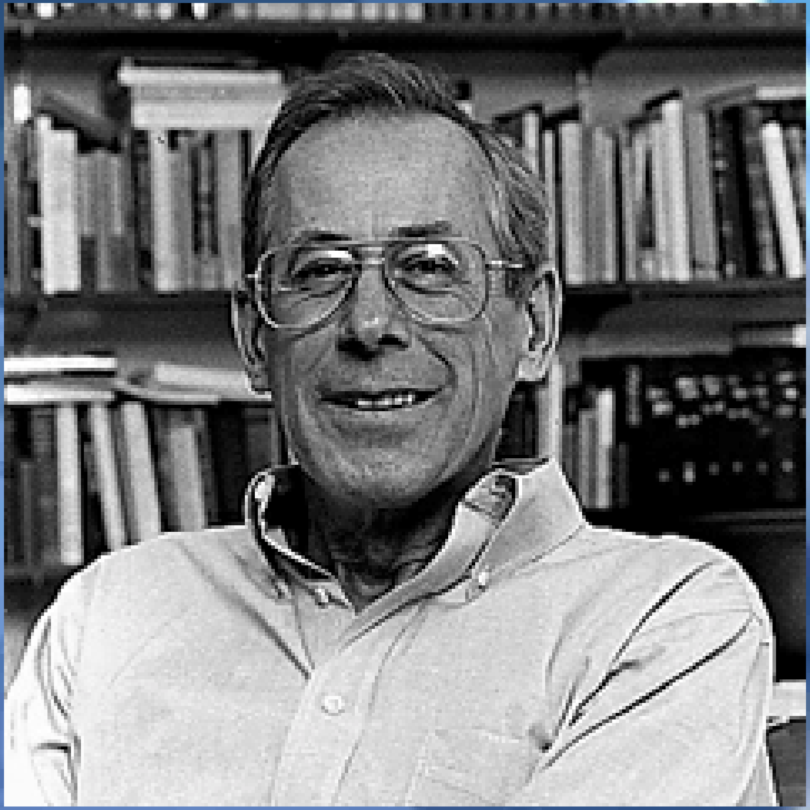
2) Isotropy of MWBR







3) Homogeneity + origin of light elements



The Friedman-LRW geometries

Curvature index

$$dl^2 = -dt^2 + a^2(t)\,[dr^2 + S^2(r,k)\,d\Omega^2]$$

Scale factor

k = 0, ±1

S = sinh r, r, sin r
for k= -1 , 0, 1

Take any 2-sphere defined by
t=T , r=R
with T and R constants.

A trivial calculation provides: $\vec{H} = \frac{1}{a^2(T)}\left(-\acute{a}(T)\,a(T)\frac{\partial}{\partial t} + \frac{S'(R)}{S(R)}\frac{\partial}{\partial r}\right.$

The Friedman-LRW geometries

$dl^2 = -dt^2 + a^2(t)\,[dr^2 + S^2(r,k)\,d\Omega^2]$

$$\vec{H} = \frac{1}{a^2(T)}\left(-\dot{a}(T)a(T)\frac{\partial}{\partial t} + \frac{S'(R)}{S(R)}\frac{\partial}{\partial r}\right)$$

Matter density

$$\Rightarrow \vec{H}\bullet\vec{H} = \frac{1}{a^2}\left(\frac{S'^2}{S^2} - \dot{a}^2\right) = \frac{1}{a^2 S^2} - \frac{\rho}{3}$$

Using Einstein's equations

The Friedman-LRW geometries

$$\vec{H} \bullet \vec{H} \leq 0 \Leftrightarrow a(T)S(R) \geq \sqrt{\frac{3}{\rho(T)}}$$

This is, of course, always possible for big enough R (provided the matter density is positive!!)

ALL FRWL models have trapped spheres !

The Universe?

CONCLUSIONS

There is a LOT of nice, intriguing, beautiful, appealing, interesting, (differential) geometry behind GRAVITATION. ??

And many unexplored zones!!