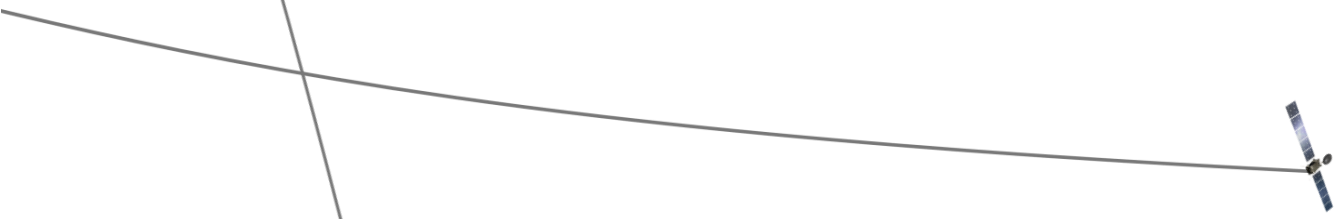

A Mission to L_4

Calculation of the valid regions in the R3BP to send a satellite that goes directly through the equilibrium point L_4 of the Earth-Moon system.

Author: Dedekind

"The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living. I am not speaking, of course, of the beauty which strikes the senses, of the beauty of qualities and appearances. I am far from despising this, but it has nothing to do with science. What I mean is that more intimate beauty which comes from the harmonious order of its parts, and which a pure intelligence can grasp."

Henri Poincaré

Abstract

In this mathematical essay we investigate and calculate the valid regions in the *R3BP* where a satellite sent from different points on the surface of Earth can be found if we want it to go through the equilibrium point L_4 of the Earth-Moon system. A Runge-Kutta 45 Method called Dormand&Prince has been programmed in order find a numerical approximation solution to the differential system of equations for the movement of the third body. We have determined the coordinates of the launching points, the possible angles, direct angles, and the possible velocities of firing off for our satellite among other parameters. Furthermore, we have concluded the article by creating some graphs where the results obtained with the code in order to answer the research question are clearly stated and commented.

Keywords: Three-body problem, ordinary differential equations, equilibrium points, numerical integration.

MSC 2010: 34K28, 70F07, 70M20

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Finally, I would like to thank my family for being so comprehensive when it has not been possible for me to be with them as much as I would have wanted to and for having helped me that much.

Contents

Introduction

Immensity. That's all that I felt when looking at the sky last summer. I observed the Moon and how it changed its position from day to day; I was sure that there was a lot of mathematics involving this beautiful topic. I can say firmly that curiosity and the attitude of wanting to know more about planets and orbits, combined with my passion for mathematics were the principal causes that made me want to develop this work.

Like almost in every assignment or task, my first idea of what I would do was very different from the final result. My intention was to study the trajectory equation and focus on the eccentricity of orbits in order to finally explain why does this parameter change as time passes (talking in a large scale, of course), with Earth as the principal example. While I was reading books, articles and thesis about the two and three body problems, orbits and their eccentricity; I discovered the Lagrangian points, particularly L_4, which appealed my attention from the very first moment. In fact, in one of my first researches I found the system of differential equations for the trajectory of a third body whose solution permits us to know where it is at a given time, which cannot be solved analytically.

My main objectives in this paper are the following:

1. To explain accurately the mathematical (and physical) background, and to proof a good number of the formulated propositions.
2. To combine different areas of mathematics and establish bridges between them in order to give an answer to my research question.
3. To show how that the concepts studied are required and applied on my way to finding the mentioned solution.
4. The most important and principal objective, to find the solution of my research question, which can be stated as follows: which are the regions where a satellite launched from different points on the surface of our planet Earth can be found if we want it to go through the equilibrium point L_4?

Other general objectives are to learn how mathematics can be found in a topic about which I knew nothing before starting to work with it; and finally to challenge myself programming in C++ and doing some elaborated illustration with GeoGebra [a] and GNUplot [b].

In order to accomplish my principal aim, I have divided my work in five chapters. It is important to remark that I needed a solid physical background that would define the problem in which I would be working during all the exploration and would give me the equations that I would be analysing and working with. This is the reason why I have dedicated two whole chapters to talk about the two-body problem and the three-body problem, which discuss the dynamics of two and three bodies in space respectively. Added to that, I felt that previous chapter was necessary in order to explain some concepts about derivation, integration, a few important laws of physics and some important definitions so that when explaining the two and three body problem, everything would be clear and revised previously. As we will see further on, the three-body problem gives us a system of differential equations that describes the motion of the third body. As I wanted it to go from Earth to L_4, I had to solve this system of differential equation using numerical integrators, which would give a numerical approximation of the solution. Therefore, I decided to assign an entire chapter to the explanation of numerical methods to solve differential equations, and I would chose the one with less error in order to solve the problem myself.

At this point, I must admit that the first four chapters are only the introduction and the background of my real work. My objective is accomplished in Chapter 5, where I launch my satellite from Earth and find the solution of the 3BP during a period of time until it reaches L_4 from three points on the surface of Earth. It is important to remark that the units that I work with are normalised, and that all the illustrations and programs I have done them myself, including the cover page and the back cover.

Chapter 1

Previous Concepts

1.1 Basic Definitions

The aim of this section is to clarify some physic elements whose definitions I have considered necessary to introduce. These and other complementary definitions can be found at [15] [17].

Definition 1.1.1. ***Reference frame*** *A coordinate system is always introduced in order to describe physical events that occur in space and time, such as the motion of a body. In particular, the position of a moving body can be described by space-time events specified by its space-time coordinates. An observer can be placed at the origin of the coordinate system, and both the observer and the coordinate system act as a reference frame for describing the position, velocity, and acceleration of bodies. The choice of the origin, which as we said is the location of the observer, will determine the position vector of the body; but the displacement, velocity, and acceleration vectors are independent of the location of the observer.*

Definition 1.1.2. ***Inertial reference frames*** *If no forces act on an object, any reference frame for which the acceleration of the object remains zero is an inertial reference frame.*

Definition 1.1.3. ***Mechanical Energy*** *We define the sum of the kinetic and potential energies of a system as the mechanical energy of the system:*

$$E_{mech} := K + U, \tag{1.1}$$

where K includes the kinetic energy, i.e. all the energy due to motion, of all moving members of the system; and U includes all types of potential energy in the system, i.e. the energy that an object has due to its position in a force field.

Definition 1.1.4. ***Linear momentum*** *The linear momentum of a particle or an object that can be modeled as a particle of mass m moving with a velocity $\vec{v}$ is defined to be the product of the mass and velocity of the particle:*

$$\vec{p} = m\vec{v}. \tag{1.2}$$

Definition 1.1.5. ***Angular Momentum*** *The instantaneous angular momentum $\vec{L}$ of a particle relative to an axis through the origin O is defined by the cross product of the particle's instantaneous position vector $\vec{r}$ and its instantaneous linear momentum $\vec{p}$:*

$$\vec{L} := \vec{r} \times \vec{p}. \tag{1.3}$$

Definition 1.1.6. ***Periapsis*** *It is the closest point a second body comes to the first one during an orbit. If the body of greater mass is the Sun, the point where the orbiting body is nearest to it is called the perihelion. The contrary case, i.e. the point where an orbiting body is furthest from the body with greater mass, is called apoapsis.*

Definition 1.1.7. ***Centre of mass*** *We consider a system consisting of two particles with masses m_1 and m_2. The position of the center of mass of the system can be described as being the average position of the system's mass. It is located somewhere on the line that goes through the two particles, and is obviously closer to the particle having larger mass.*

The x-coordinate of the centre of mass of the pair of particles is given by:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{1.4}$$

Extending this concept to a system of n particles with masses m_i in three dimensions, the x coordinate of the centre of mass of n particles is defined to be:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M} = \frac{1}{M} \sum_i m_i x_i. \tag{1.5}$$

1.2 Newton's Laws

Isaac Newton stated that *"An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line"*. Galileo Galilei recognised the idea that force produces motion many centuries before Newton's birth, but this last scientist extended the concept of force to any circumstance that produces

acceleration. All this knowledge can be summed up in his three laws of motion. Extended information can be found at [15, Ch.5,].

Newton's First Law of Motion: The Law of Inertia

In the absence of external forces and when viewed from an *inertial reference frame*, an object at rest remains at rest and an object in motion continues in motion with a constant velocity, that is, with a constant speed in a straight line (linear motion).

Newton's first law makes no distinction between an object at rest and an object moving with constant (nonzero) velocity. Whether an object remains at rest or remains moving with constant velocity depends on the *reference frame* in which the object is observed. Suppose you are on a train that is going along a straight path and at constant altitude. You carefully place a tennis ball on your seat tray (which is horizontal). Relative to the plane, the tennis ball will remain at rest as long as the train continues to move at constant velocity relative to the ground. Relative to the ground, the tennis ball remains moving on a straight line with the same velocity as the plane.

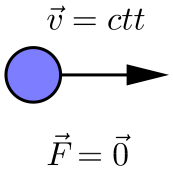

Figure 1.1: Newton's First Law.

Now, suppose that the train suddenly accelerates forward (relative to the ground). You will then observe that the tennis ball on your tray starts to roll toward the rear of the train, accelerating (relative to the plane) even though there is no horizontal force acting on it. In this accelerating reference frame of the plane, Newton's first-law statement does not apply. Newton's fist-law statement applies only in reference frames known as *inertial reference frames*.

Newton's Second Law of Motion

When viewed from an inertial reference frame, the acceleration of an object is directly proportional to the net force acting on it and the reciprocal of the mass of the object is the constant of proportionality. Thus,

$$\vec{a} = \frac{\vec{F}_{net}}{m} \quad \text{where} \quad \vec{F}_{net} = \sum_i \vec{F}_i. \tag{1.6}$$

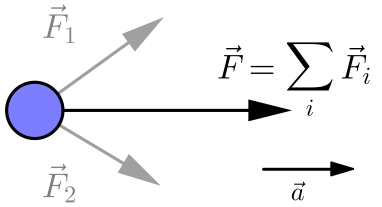

Figure 1.2: Newton's Second Law.

Isolating $\vec{F}_{net}$ we obtain the following mathematical statement:

$$\vec{F}_{net} = m\vec{a}. \tag{1.7}$$

In both the textual and mathematical statements of Newton's second law, we have indicated that the acceleration is due to the net force $\vec{F}_{net}$ acting on an object. The *net force* on an object is the vector sum of all forces acting on the object. Equation (1.7) is a vector expression and hence is equivalent to three component equations:

$$F_{net,x} = ma_x, \qquad F_{net,y} = ma_y, \qquad F_{net,z} = ma_z. \tag{1.8}$$

Newton's Third Law of Motion

When two bodies interact, the force $\vec{F}_{BA}$ exerted by object B on object A is equal in magnitude and opposite in direction to the force $\vec{F}_{AB}$ exerted by object A on object B. Thus,

$$\vec{F}_{BA} = -\vec{F}_{AB}. \tag{1.9}$$

Newton's third law describes an important property of forces: forces always occur in pairs. If a force is exerted on some object A, there must be another object B exerting the force. This law states that the forces are equal in magnitude and opposite in direction. That is, if object A exerts a force on object B, then B exerts an equally strong but oppositely directed force on A.

Figure 1.3: Newton's Third Law.

Each pair of forces is called a Newton's third-law *(N3L) pair*. It is common to refer to one force in the pair as an action and the other as a reaction. This terminology is unfortunate because it sounds like one force "reacts" to the other, which is not the case. The two forces occur simultaneously. Either can be called the action and the other the reaction. If we refer to a force acting on a particular object as an action force, then the corresponding reaction force must act on a different object.

Newton's Law of Universal Gravitation

It is a well-known legend that Isaac Newton was having a nap under a tree when suddenly an apple fell from the tree and smacked on his head. From this observation, he imagined that perhaps all objects in the Universe are attracted to each other as the apple was attracted to the Earth. Analysing astronomical data on the motion of the Moon around the Earth, Newton stated that the force law governing the motion of planets was the same as the force law that attracted a falling apple to the Earth.

Newton's Law of Universal Gravitation asserts that

> *every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

Putting into a formula this information, he stated that the magnitude of the gravitational force F_g is:

$$F_g = G\frac{m_1 m_2}{r^2}, \tag{1.10}$$

where

- m_1 and m_2 the masses of the two particles,
- r is the distance between both particles,
- G is the universal gravitational constant, whose value in SI units, measured by Henry Cavendish in 1798, ($G = 6.674 \cdot 10^{-11}$ N $\cdot$ m^2 / kg^2).

1.3 Conic Sections

In this section we will study basic properties of conic sections, which are the curves that can be described as graphs of second-degree equations in two variables, obtained by intersecting a plane and a right circular cone. A circle can easily be obtained by cutting the cone with a plane perpendicular to the cone's axis. A plane parallel to a side of the cone produces a parabola; and a plane at an arbitrary angle to the axis of the cone forms an ellipse. Finally, an hyperbola can be obtained by cutting the cone with a plane parallel to the cone's axis. If we extend the cone through its vertex and form a second cone, we will find the second branch of the hyperbola.

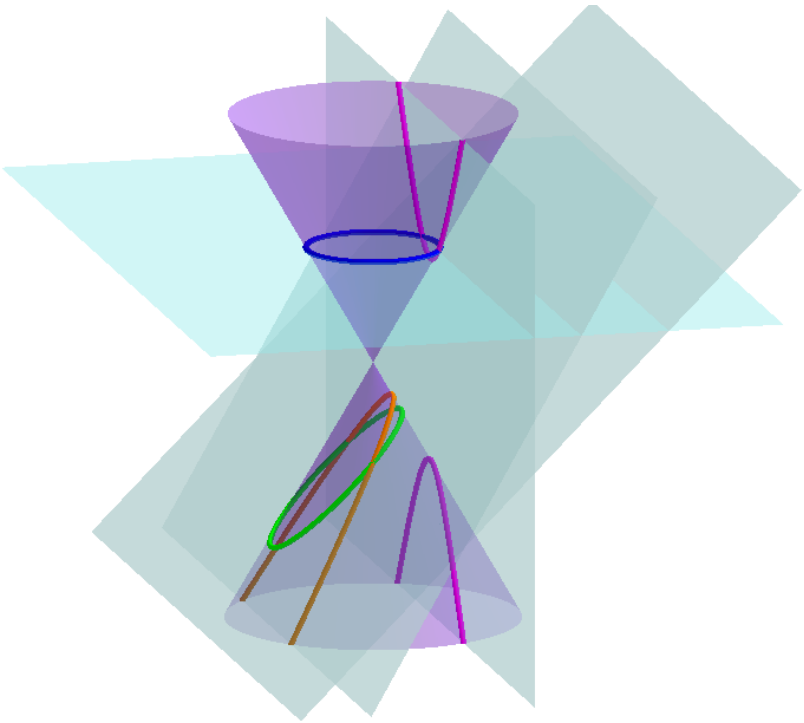

Figure 1.4: Conic sections obtained by cutting a cone with a plane having an arbitrary inclination.

The four named *conic sections* are shown in Figure 1.4 (circle, ellipsis, parabola and hyperbola). As it has been said, we can obtain them by cutting the cone with a plane and an angle α to the cone axis. These and additional information to amplify the concepts explained can be found in [1].

1.3.1 Circles

A circle is the closed curve formed by a set of points in a plane that are equidistant from a given point, called the *centre*. The distance between any of this points to the centre is called the *radius*:

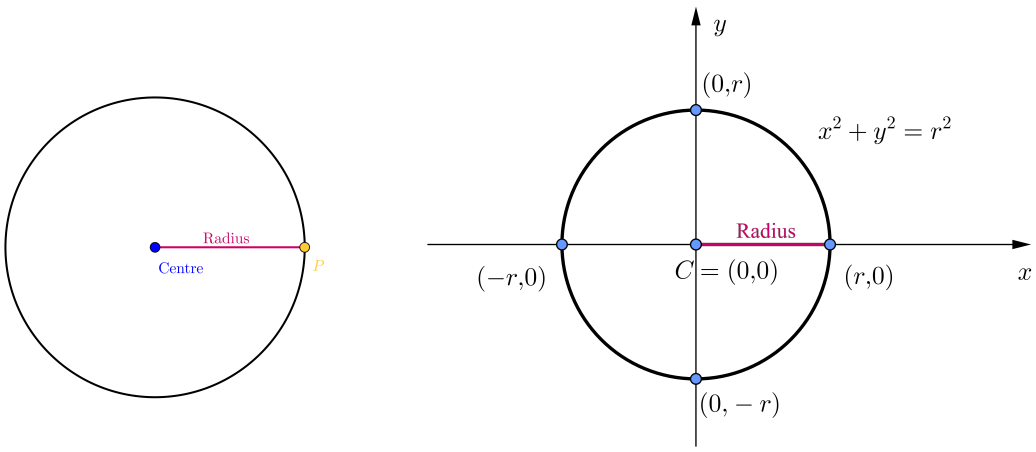

Figure 1.5: Definition of a circle: major elements.

The centre-radius form of the equation of a circle with its centre at the origin and radius r is:

$$x^2 + y^2 = r^2, \tag{1.11}$$

where

- r is the radius of the circle.
- The origin is located at the centre of the circle.
- The two x-intercepts are at the points $(r, 0)$ and $(-r, 0)$.
- The two y-intercepts are at the points $(0, r)$ and $(0, -r)$.

Now, considering the circle of radius R centred at the point $(0, 0)$, the parametric equations for this circle are the following:

$$x = r\cos(t); \qquad y = r\sin(t); \qquad t \in [0, 2\pi]. \tag{1.12}$$

1.3.2 Ellipses

An ellipse is the closed curve in a plane such that the sum of the distances (d_1 and d_2) from each point to two fixed points is constant:

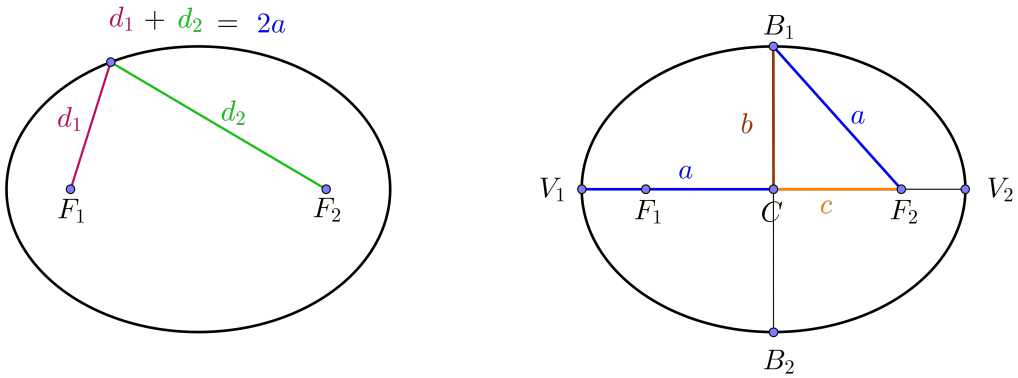

Figure 1.6: Some characteristics of an ellipse.

Where

- F_1 and F_2 are each of the two fixed points called *foci* (singular, *focus*).
- V_1 and V_2 are the points where the line containing both foci intersects the ellipse.
- The *semi-major axis* (a) is half of the distance between the vertices.
- The *centre* C of the ellipse is the midpoint of the major axis.
- B_1 and B_2 are the points where the line perpendicular to the major axis that goes through the centre intersects the ellipse.
- The *semi-minor axis* (b) is half of the distance between the co-vertices.
- The *focal distance* (c) is the distance between the centre and one of the two foci.
- The *eccentricity* (e) is the quotient of the distance between the foci and the semi-major axis ($e = \frac{c}{a}$).

The equation of an ellipse with its centre at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1.13}$$

where a is the semi-major axis, b is the semi-minor axis; the x-intercepts are at the points $(a, 0)$ and $(-a, 0)$; and the y-intercepts are at the points $(0, b)$ and $(0, -b)$.

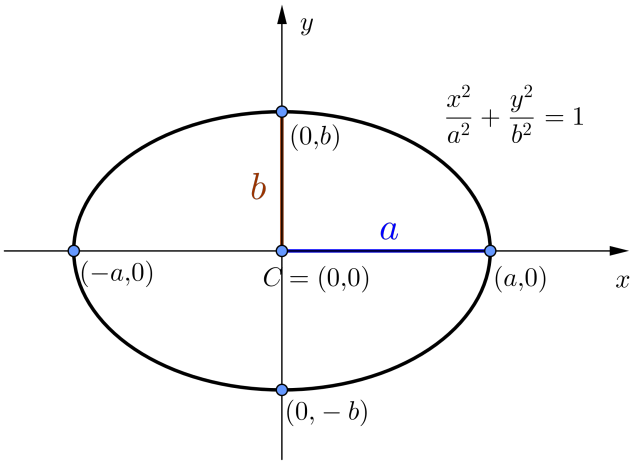

Figure 1.7: Important points of an ellipse: x, y-intercepts

The parametric equations of an ellipse are the following (considering equation (1.13)):

$$x = a\cos(t); \qquad y = b\sin(t); \qquad t \in [0, 2\pi]. \tag{1.14}$$

1.3.3 Parabolas

The set of all points in a plane that are equidistant from a line and a point not on the line is called a parabola:

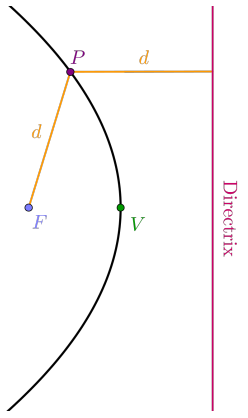

Figure 1.8: Main elements of a parabola.

At figure 1.8 we can see the different elements of a parabola:

- F is called the *focus* of the parabola, and it is the point mentioned above at the definition.
- *directrix* is the line from which any point of the parabola is equidistant with the focus.
- V is the *vertex* of the parabola: its highest or lowest point, also known as the maximum or the minimum.

Parabolas are quadratic functions, and the standard form to describe them is given by the equation

$$x = a(y-h)^2 + k, \qquad (1.15)$$

where a, h and k are real numbers. The vertex has coordinates (k, h) and the focus $(k + \frac{1}{4a}, h)$. Talking about the parametric equations of a parabola, they are as follows:

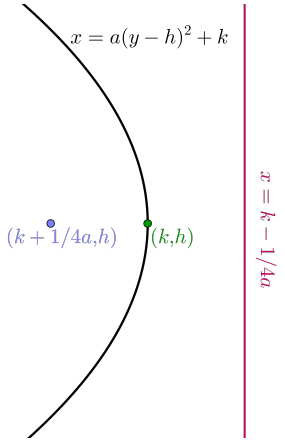

Figure 1.9: Coordinates and equations of the major elements of parabolas.

$$x = \tan^{-1}(t); \qquad y = [\tan^{-1}(t)]^2; \qquad t \in \mathbb{R}. \qquad (1.16)$$

1.3.4 Hyperbolas

The definition of a hyperbola is the set of points in a plane such that the absolute value of the difference of the distance of each point from two fixed points ($foci$, singular $focus$) is constant. Both foci are united by means of the transverse axis. Two branches form the graph of a hyperbola, and each of these branches intersects the transverse axis at a point called vertex:

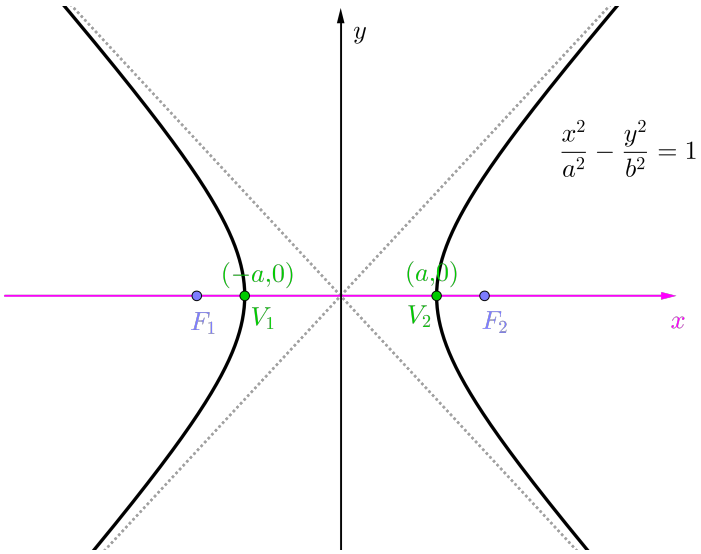

Figure 1.10: Important parts of a hyperbola with their coordinates and formulae.

Explaining into detail Figure 1.10,

- F_1 and F_2 are the foci of the hyperbola.
- T is the transverse axis: the line that goes through both foci.
- V_1 and V_2 are the vertices that cross the x axis at points $(-a, 0)$ and $(a, 0)$ respectively.

The equation of a horizontal parabola with centre at the origin and x-intercepts at $(a, 0)$ and $(-a, 0)$ (figure 1.10) is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1.17}$$

where $a > 0$ and $b > 0$.

Considering a hyperbola following this equation: $x^2 - y^2 = r^2$; the parametric equations for the right branch are:

$$x = r\cosh(t); \qquad y = r\sinh(t); \qquad t \in \mathbb{R}, \tag{1.18}$$

where: $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$ and $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$.

1.4 Integral Calculus

1.4.1 Multiple integrals

A multiple integral is a generalisation of a definite integral, i.e. an integral with start and end values, which graphically represents the signed area of the region in the xy-plane that is bounded by the graph of the function f; to functions of more than one real variable, for instance $f(a, b)$, or $f(a, b, c)$. Particularly, integrals of a function over a region in $\mathbb{R}^2$, that is a function depending on two variables, are called double integrals, and integrals of a function of three variables over a region of $\mathbb{R}^3$ are called triple integrals. Mathematically, a multiple integral is defined as follows:

$$\iint \dots \int_{\mathbf{D}} f(x_1, x_2, ..., x_n)\, dx_1 \, ... \, dx_n, \tag{1.19}$$

where $f(x_1, x_2, \dots, x_n)$ is a function in n variables over D, which is the domain of integration. Normally, D is represented by nested limits of integration in the reverse order

of execution, that is, the leftmost integral is computed the last and so on. The difficulty added in multiple integrals falls on the fact that sometimes it is not trivial to find the limits of integration in the domain D.

1.4.2 Fubini's Theorem

Providing conditions for interchanging the order of integration in a multiple integral, Fubini's theorem is a very powerful and useful tool. Given that sums are essentially special cases of integrals (with respect to discrete measures), it also gives conditions for interchanging the order of summations, or the order of a summation and an integration. The theorem's statement is the following:

Theorem 1.4.1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Then, there exist the integrals*

$$\begin{aligned} &a) \int_a^b \left(\int_c^d f(x, y)\ dy \right) dx, \\ &b) \int_c^d \left(\int_a^b f(x, y)\ dx \right) dy, \end{aligned} \tag{1.20}$$

which coincide.

Observations

1. If we have a more complex integral such that domain of integration D is not given by intervals (it is not rectangular) but it is defined by $\phi_1(x)$ and $\phi_1(x)$ we can generalise the expression above by stating that

$$\begin{aligned} &a) \int_{x_1}^{x_2} \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\ dy\ dx, \\ &b) \int_{y_1}^{y_2} \int_{\gamma_1(y)}^{\gamma_2(y)} f(x, y)\ dx\ dy \end{aligned} \tag{1.21}$$

   are equal if the conditions stated at the theorem apply.

2. If $f : [a, b] \times [c, d] \to \mathbb{R}$ can be expressed as $g(x)h(y)$, equations (1.20) are equivalent to

$$\int_a^b g(x)\ dx \int_c^d h(y)\ dy. \tag{1.22}$$

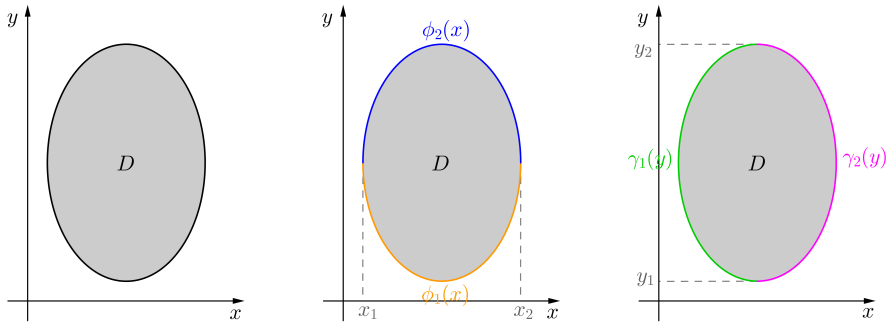

Figure 1.11: Example of different limits of integration of (1.21).

1.4.3 Change of variables for multiple integrals

Sometimes it is useful to describe the points of the plane using other coordinates than the Cartesian ones (x, y), simply because there are limits of integration that can be expressed easier in other systems of coordinates. The change of one variable for another is in a geometric sense a transformation from a determined space to a different one, and this transformation requires a change in the integration region and the addition of a co-relation factor to the differential. Given a transformation that follows the relation:

$$f(y_1, \ldots, y_n) \longrightarrow f(y_1(x_1, x_2, \ldots, x_n), \ldots, y_n(x_1, x_2, \ldots, x_n)). \tag{1.23}$$

Expressing the domain in the new coordinates x_i and multiplying by the co-relation factor, we obtain a multiple integral equal to the original one:

$$\iint \ldots \int_{D(\vec{y})} f(y_1, \ldots, y_n) dy_1 \ \ldots \ dy_n = \iint \ldots \int_{D(\vec{x})} f(x_1, \ldots, x_n) \ |J| \ dx_1 \ \ldots \ dx_n, \tag{1.24}$$

where:

$$J = \frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}, \tag{1.25}$$

see [12].

In the following lines, we will study how to change from Cartesian coordinates to Spherical ones.

Change to Spherical Coordinates

Supposing that we want to change our function $f(x, y, z)$ to a spherical one following that $D = \{x^2 + y^2 + z^2 \leq R^2\}$.

Expressing x, y and z with spherical coordinates ($r \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$):

$$\begin{cases} x(r, \theta, \phi) = r \cos\theta \sin\phi \\ y(r, \theta, \phi) = r \sin\theta \sin\phi \\ z(r, \phi) = r \cos\phi \end{cases} \tag{1.26}$$

Now, if we calculate $|J|$ from equation (1.25), we get that:

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta \sin\phi & -r \sin\theta \sin\phi & r \cos\theta \cos\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\phi & 0 & -r \sin\phi \end{vmatrix} \\ &= \cos\phi(-r^2 \sin^2\theta \sin\phi \cos\phi - r^2 \cos^2\theta \sin\phi \cos\phi) \\ &\quad + (-r \sin\phi)(r \cos^2\theta \sin^2\phi + r \sin^2\theta \sin^2\phi) \\ &= -r^2 \cos^2\phi \sin\phi(\sin^2\theta + \cos^2\theta) - r^2 \sin^3\phi(\cos^2\theta + \sin^2\theta) \\ &= -r^2 \cos^2\phi \sin\phi - r^2 \sin^3\phi \\ &= -r^2 \sin\phi(\cos^2\phi + \sin^2\phi) \\ &= -r^2 \sin\phi. \end{aligned}$$

Taking the absolute value of this last result, we get that $|J| = r^2 \sin\phi$. Finally, we can express equation (1.25) as:

$$\int_{-R}^{-R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}} f(x, y, z)\, dx\, dy\, dz = \int_0^R \int_0^{2\pi} \int_0^{\pi} f(r, \theta, \phi) r^2 \sin\phi\, d\phi\, d\theta\, dr. \tag{1.27}$$

1.5 Ordinary differential equations

This section is intended to give a general idea and some basic definitions and theorems about ordinary differential equations (*ODE's* in abbreviated form). We will contextualise historically differential equations in general and will define them before coming into ordinary differential equations. For further information see [9].

1.5.1 An outline of the historical context

Everything started at 1202 with the observations that Leonardo de Pisa made with respect to the rise in a population of rabbits and, supposing that only the rabbits of the two previous generations take part in each reproductive period and that each couple generated a new one; de Pisa expressed his observations in the following mathematical equations:

$$r_{n+1} = r_n + r_{n-1},$$

where

- r_n is the number of male-female couples of rabbits.
- n is the particular generation of rabbits, and hence $n \geq 0$.
- r_0 and r_1 equal to one, and the two rabbits that form these two generations are called the *initial couple*.

This is the well-known Fibonacci sequence, which has the general solution $c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. The number $\frac{1+\sqrt{5}}{2}$ is labelled as φ, and it is named the golden ratio. Strange as it may, seem the golden ratio occurs in the solution of differential equations (see [10]).

Intuitively, a *differential equation* relates a function f with its derivatives $f', ..., f^n$. Differential equations are studied from a large amount of different perspectives. It is important to remark that generally, these kind of equations don't have a solution, only the simplest differential equations are solvable by explicit formulae. Nevertheless, mathematicians have developed several methods in order to find some properties of their solutions without finding their exact form.

Differential equations can be classified into some groups, for instance *ordinary differential equations (ODE's)*, which are the ones with one or more unknown functions, depending only on one independent variable; and *partial differential equations (PDE's)*, which are differential equations where the unknown function or functions depend on more than one independent variable, and are pretty more complex . In this chapter we will only study the first group. Both ODE's and PDE's are separated as *linear* (if the unknown function and its derivatives appear to the power of one) and *nonlinear* otherwise.

Newton listed three kinds of differential equations in his 1671 work *Methodus fluxionum et Serierum Infinitarum*:

$$\begin{aligned} &\frac{dy}{dx} = f(x) \\ &\frac{dy}{dx} = f(x,y) \\ &x_1\frac{\partial y}{\partial x_1} + x_2\frac{\partial y}{\partial x_2} = y. \end{aligned} \tag{1.28}$$

In 1676, Leibnitz referred to these equations as *equatio differentialis*. Both Newton and Leibnitz developed some methods in order to solve physical and geometric problems.

Euler had a very important role in this area of mathematics in the 18th century. He was the first mathematician who understood and defined clearly what the concept of *function* meant, and this fact made easier the resolution of ODE's.

Peano, Lipschitz, Cauchy and Picard analized mathematically differential equations, and the first results that proved the solutions of ODE's under some general conditions were estabished.

Henry Poincaré made the most important contribution at the beginning of the 20th century: the named *qualitative theory of ordinary differential equations*.

As all concepts and elements, differential equations has been a subject whose meaning and way of understanding has evolved throughout history. In the past, *finding a solution* of the differential equation meant *to empress it in terms of elemental functions*. However, as we have said before, this was hardly impossible in the large majority of the cases, so mathematicians would be satisfied when they found implicit relations in the solutions, without finding the *closed formulae*. In the 17th century Newton stated that *all ODE's can be solved using power series with indeterminate coefficients*. Nowadays, numerical calculus has been strongly developed during this last century at the same time with informatics, computers and the qualitative theory, which tries to find rigorously properties of these solutions without knowing them explicitly.

1.5.2 Differential equations

Formally, a differential equation is any equation in which a function or more than one are the unknowns, and depend on one or more independent variables, which appear together with their derivatives or partial derivatives in the equation. The simpler differential equation requires the calculation of the primitive function: given the function $f(x,y)$, we look for the unknown function $y = y(x)$ such that

$$y'(x) = f(x,y). \tag{1.29}$$

The *order* or *degree* of a differential equation is the number of the highest derivative in the equation.

1.5.3 ODE's

Introduction

An *ordinary differential equation* of order m has the form:

$$f(t, x, x', x'', ..., x^{(m)}) = 0, \tag{1.30}$$

where

$$F : S \subset \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}.$$

being S is an open subset, i.e. a subset which doesn't contain any of its boundary points. Each variable $(t, x, ..., x^{(m)})$ goes to $F(t, x, x', ..., x^{(m)})$.

A function is a solution of an ODE of m order if its m first derivatives have a certain relation at every point.

Let $x = \xi(t)$, where $\xi : (a, b) \subset \mathbb{R} \to \mathbb{R}$. $\xi(t)$ is a solution of (1.30) if

$$F(t, \xi(t), \xi'(t), ..., \xi^{(m)}(t)) = 0, \quad \forall t \in (a, b). \tag{1.31}$$

We will illustrate the concept of an ordinary differential equation with an example. We have to find the solution of the first order ODE equation $y' = y$: $F(x, y, y') = y' - y$. Integrating both sides of the first equation we get that:

$$\begin{aligned} \int \frac{dy}{y} &= \int dx \\ \ln y &= x + c \\ y &= \tilde{c}\, e^x. \end{aligned} \tag{1.32}$$

We will say that a n-dimensional ODE is *autonomous* if the function f does not depend on the independent variable, i.e. $y' = f(y)$, and *non autonomous* otherwise.

An ODE of order m in *normal form*, i.e. in a way such that the derivative of greatest order appears isolated in function of all the other ingredients of the equation, can be re-written as a first-order system with equivalent m dimension (where equivalent means that if we solve one of the equations we can look for the solutions of the other).

Consider the equation of the pendulum system

$$y''(t) + \frac{g}{l}\sin y(t) = 0, \tag{1.33}$$

where y is the angle with respect to the vertical axis, g is the acceleration due to gravity and l is the length of the pendulum. This is an autonomous ODE of order two, and we want to express it in normal form, i.e. $y^{(m)} = f(t, y, y', ..., y^{(m-1)})$ and then re-write this n-ordered scalar equation to a vector equation (i.e. of first order) and dimension n:

$$\begin{aligned} \vec{x}' &= \vec{F}(t, \vec{x}) \\ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}' &= \begin{pmatrix} f_1(t, x_1, x_2, ..., x_m) \\ \vdots \\ f_n(t, x_1, x_2, ..., x_m) \end{pmatrix}. \end{aligned} \tag{1.34}$$

As mentioned, the first step is to express the pendulum equation in such a way: $y''(t) = -a^2 \sin y$, being $a^2 = \frac{g}{l}$. Now, we define $\vec{x} = (x_1, x_2)$ as $x_1 = y$ and $x_2 = y'$. Hence,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= y'' = -a^2 \sin x_1. \end{aligned}$$

Now, $\vec{x}' = \vec{f}(\vec{x})$, where $\vec{f}(\vec{x}) = (x_2, -a^2 \sin x_1)$, as we wanted.

Existence and uniqueness of solutions

Given the n-dimensional system of ODE's $y' = f(t, y)$ and the values $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}n$, known as *initial conditions*, the associated *initial value problem (IVP)* or *Cauchy Problem* consists on finding a solution function $y(t)$ of the ODE verifying $y(t_0) = y_0$.

For instance, consider the 1-dimensional IVP

$$\begin{cases} y' = ay \\ y(t_0) = y_0, \end{cases} \tag{1.35}$$

being $a \in \mathbb{R}$ a fixed value. As calculated in example (1.32), the general solution of the problem equation is $y = ce^{at}$, where $c \in \mathbb{R}$ is a constant. From the initial condition $y(t_0) = y_0$, we get that $c = y_0$, so the unique solution of the Cauchy problem is $y(t) = y_0 e^{a(t-t_0)}$.

Picard's Theorem is a basic and important result about existence and uniqueness of ODE's solutions. Before stating it, we need some previous definitions.

Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, we will say that F is *Lipschitz* or *L-Lipschitz* with Lipschitz constant $L > 0$ if and only if

$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|, \forall x_1, x_2 \in U, \tag{1.36}$$

where $\|\cdot\|$ indicate any norm in $\mathbb{R}^n$ and $\mathbb{R}^m$.

We will say that a function F is locally Lipschitz if around every point we can define a Lipschitz constant, i.e. given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, U open, we will say that the function is locally Lipschitz if $\forall x_0 \in U$ there exist $L_{x_0} > 0$ and $V_{x_0} \subset U$ around x_0 such that $F_{\|V_{x_0}} : V_{x_0} \subset \mathbb{R}^n \to \mathbb{R}^m$ is L_{x_0}-Lipschitz.

After these definitions, we can state *Picard's Theorem*:

Theorem 1.5.1. *Given that $f = f(t, x)$ is a function defined in $\Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, Ω an open subset, f continuous and locally Lipschitz with respect to x. Given $(t_0, x_0) \in \Omega$, being $a, b > 0$ such that the "rectangle"*

$$R_{a,b}(t_0, x_0) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \|t - t_0\| \leq a, \|x - x_0\| \leq b\}$$

verifies $R_{a,b}(t_0, x_0) \subset \Omega$. We define

$$M := \sup_{(t,x)\in R_{a,b}(t_0,x_0)} \|f(t, x)\|,$$
$$\alpha := min\{a, b/M\}.$$

Then there exists a unique solution x(t) for the IVP

$$x' = f(t, x); \qquad x(t_0) = x_0;$$

defined for $t \in I_\alpha(t_0) := [t_0 - \alpha, t_0 + \alpha]$.

Further on, we will say that as our ODE is $\mathcal{C}^1$, i.e. it is continuously differentiable; therefore it is locally Lipschitz, and hence it has a unique solution for out IVP. Nevertheless, we have to prove that if a function is $\mathcal{C}^1$ it is locally Lipschitz (see lemma 1.5.1).

Lemma 1.5.1. *Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ that is continuously differentiable (i.e. $\mathcal{C}^1$), it is also locally Lipschitz.*

Proof. Let x, ξ be respectively the centre and the radius of some ball $B(\tilde{x}, \xi)$. As the closure is compact, $\frac{\partial f}{\partial x}$ is bounded by some ball. Given the points $x, y \in B(\tilde{x}, \xi)$, we have:

$$f(x) - f(y) = \int_0^1 \frac{\partial f(y + t(x - y))}{\partial x}(x - y)\, dt.$$

Hence we can get the bound:

$$\|f(x) - f(y)\| \leq \int_0^1 \|\frac{\partial f(y + t(x - y))}{\partial x}\| \, \|(x - y)\| \, dt \leq L\|x - y\|.$$

□

Chapter 2

The Two-Body Problem

2.1 First Approach

Going back in time, despite the fact that Newton found a geometrical solution of the Two-Body Problem about 1685, curiously, the analytic solution of the problem for spheres of finite size was not accomplished until many years later. In Europe, the methods of the calculus were developed at the beginning of the eighteenth century, but Newton's system of mechanics did not find immediate acceptance. It wasn't until Voltaire vigorously supported the Newtonian theory, after his visit to London in 1727, that the French accepted Newton's explanations. Before that, they preferred the vortex theory of René Descartes. In a parallel way, in England, mathematicians continued to employ the geometrical methods of the *Principia*, and this fact also delayed the analytical solution of the problem. Daniel Bernoulli was probably the mathematician who first gave an analytical solution for the problem, but the 2BP was certainly solved in detail by Euler in 1744 in his *Theoria motuum planetarum et cometarum.*

In this chapter we consider the problem of two isolated bodies of masses m_1 and m_2 respectively, with r_1 and r_2 denoting the position vectors of the two bodies relative to a fixed origin O. Moreover, Newton's laws can be applied in our problem, since we are in an inertial reference frame and no force is acting on the bodies except for the force of mutual gravitational attraction. Specifically, it asks

> *"Given at any time the positions and velocities of two massive particles moving under their mutual gravitational force, the masses also being known, calculate their position and velocities for any other time".*

The two-body problem is the easiest specific case of the n-body problem. It describes how two unconstrained rigid bodies in close proximity having arbitrary spatial distribution of mass, charge or similar field quantity, orbit around each other; and has a wide range of areas of applications, for instance molecular dynamics or satellite formation flying. Its

importance relies on the fact that, contrarily to the Three-Body Problem (and N-Body Problem), a general solution can be found. This is due to the fact that the 2BP can be reduced to a central-force problem, as it will be shown later on in the chapter.

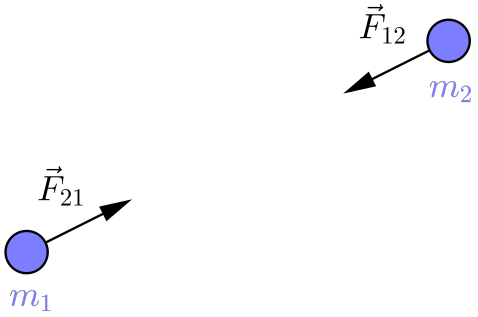

Figure 2.1: Simple approach to the two-body problem: two punctual points with their respective attractive forces.

2.2 Formal Approach

A good way to define formally the 2BP is that it specifies and analyses the dynamics of two otherwise unconstrained rigid bodies in close proximity that only interact with each other, which we will consider as punctual points.

It is important to mention the fact that we have made two **hypothesis** in this problem. **(1)** On the one hand, we assume that both bodies are symmetrically spherical, and hence we can reduce them to punctual masses (see 2.2.1). **(2)** On the other hand, our second hypothesis is that the two bodies are isolated; thus, the only force that acts is the attractive gravitational force between them. As we stated that Newton's Laws can be applied, we can say that the attractive gravitational force that m_1 exerts on m_2, $\vec{F}_{12}$, is equal in magnitude and opposite in direction to the force that m_2 exerts on m_1 (by Newton's Third Law of Motion).

Lemma 2.2.1. *Given that a body has a spherical symmetry, i.e. its density only depends of its distance to the centre, it can be reduced to a punctual mass.*

Proof. Defining $V = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq R^2\}$ we can express the total mass of the body M as:

$$M = \iiint_V \rho(\sqrt{x^2 + y^2 + z^2})\ dx\ dy\ dz. \tag{2.1}$$

Applying $\lim_{n\to\infty}$ at equation (1.5) for the Centre of Mass:

$$\vec{r}_{CM} = \frac{\iiint_V \rho(\sqrt{x^2+y^2+z^2})\ (x,y,z)\ dx\ dy\ dz}{M}. \tag{2.2}$$

We can change the coordinates of the numerator of equation (**??**) from Cartesian coordinates to the Spherical ones following the procedure shown at 1.4.3.

For each of the coordinates x, y and z:

$$\begin{aligned}
x &: \int_0^R\int_0^{2\pi}\int_0^{\pi} \rho(r)r^3\cos\theta\sin^2\phi\ dr\ d\theta\ d\phi = \int_0^R \rho(r)r^3dr \int_0^{2\pi}\cos\theta\ d\theta \int_0^{\pi}\sin^2\phi\ d\phi = 0;\\
y &: \int_0^R\int_0^{2\pi}\int_0^{\pi} \rho(r)r^3\sin\theta\sin^2\phi\ dr\ d\theta\ d\phi = \int_0^R \rho(r)r^3dr \int_0^{2\pi}\sin\theta\ d\theta \int_0^{\pi}\sin^2\phi\ d\phi = 0;\\
z &: \int_0^R\int_0^{2\pi}\int_0^{\pi} \rho(r)r^3\cos\phi\sin\phi\ dr\ d\theta\ d\phi = \int_0^R \rho(r)r^3dr \int_0^{2\pi}d\theta \int_0^{\pi}\cos\phi\sin\phi\ d\phi = 0 \qquad .
\end{aligned} \tag{2.3}$$

As the numerator of equation (2.2) is zero, we have shown that $\vec{r}_{CM} = (0,0,0)$. □

In this section we will explain how, applying Newton's second law of motion and the Gravitational law, after a mathematical procedure, we end up reducing our initial equation to a Central-Force Problem.

Firstly, it is important to remember that the Two-Body Problem is set in an inertial reference frame, and therefore we can apply equations (1.7) and (1.10). Our first hypothesis stated that the only force that interacted between the two bodies was the gravitational force, so for the body with mass m_1, since $F_{net} = F_g$ we can equalise the previous equations obtaining a new one:

$$m_1a_1 = G\frac{m_1m_2}{r^2}. \tag{2.4}$$

Since we know that the acceleration is the second derivative of the position, we can re-write equation 2.4 in vectorial form as:

$$m_1\ddot{\vec{r}}_1 = \frac{Gm_1m_2}{\|\vec{r}_2-\vec{r}_1\|^3}(\vec{r}_2-\vec{r}_1). \tag{2.5}$$

Applying Newton's third law of motion, we get the following equation for the body with mass m_2:

$$m_2\ddot{\vec{r}}_2 = -\frac{Gm_1m_2}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1). \tag{2.6}$$

Once we have the system of equation formed by equations (2.5) and (2.6), we can detect some observations:

1. It's an **autonomous system**: the position, velocity and acceleration vectors are time-dependent, but the system does not depend explicitly on the independent variable (the time).
2. We have 2 equations, each of them three-dimensional, and each of them of second order. Hence, we have **12 unknowns** $(\vec{r_1}, \vec{r_2}, \dot{\vec{r}}_1, \dot{\vec{r}}_2)(t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$:

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{v}_1\\ \vec{r}_1\\ \vec{v}_2\\ \vec{r}_2\end{pmatrix} = \begin{pmatrix}\frac{Gm_2}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1)\\ \vec{v}_1\\ -\frac{Gm_1}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1)\\ \vec{v}_2\end{pmatrix}. \tag{2.7}$$

   In equation (2.7) we have introduced the notation $\dot{\vec{r}}_1 = \vec{v}_1$ and $\dot{\vec{r}}_2 = \vec{v}_2$.

2.3 Reduction to a Central-Force Problem

Once we have simplified the masses m_1 and m_2 from equations (2.5) and (2.6) respectively, we can subtract the first simplified equation to the second one as follows:

$$\begin{aligned}\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 &= -\frac{Gm_1}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1) - \frac{Gm_2}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1)\\ &= -\frac{G(m_1 + m_2)}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1).\end{aligned} \tag{2.8}$$

If we call $\vec{r} := \vec{r}_2 - \vec{r}_1$, we obtain a new equation with only **6 unknowns**: $(\vec{r}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$. We can also introduce a mass parameter $\mu = G(m_1 + m_2)$, and we will have a problem in which the acceleration $(\ddot{\vec{r}})$ depends on a function which depends only on the radius $(f(r))$; therefore, we will have the reduction to a central-force problem:

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0. \tag{2.9}$$

Equation (2.9) is the 2BP equation of motion. The results obtained from this equation will be only as accurate as the assumptions (1) and (2).

We say that the **Mechanical Energy** (1.1) and the **Angular Momentum** (equation (1.3)); in particular, the quantities $\vec{h} = \vec{r} \times \vec{v}$, called the $Specific$ angular momentum, and $\mathcal{E} = v^2/2 - \mu/r$ are conserved as we are in an isolated system with no nonconservative forces acting and there are no dissipative mechanisms.

Given that the angular momentum is different than zero, the solution of the problem given by equation (2.9) is given by the following equation (see [4]):

$$r = \frac{h^2/\mu}{1 + e\cos\theta}. \tag{2.10}$$

The equation (2.10) is called the Trajectory Equation, and it will be studied accurately in the next section. Nevertheless, it is important to state that:

- h is the $specific$ angular momentum ($\vec{h} = \vec{r} \times \vec{v}$).
- μ is the mass parameter ($\mu = G(m_1 + m_2)$).
- e is the eccentricity of the curve.
- θ is the angle between $\vec{r}$ and the perihelion.

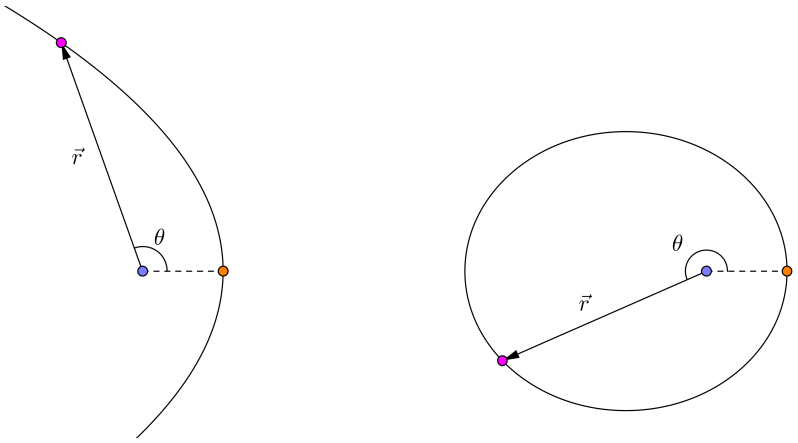

Figure 2.2: Illustration of the trajectory equation (2.10) with a body describing a parabolic (*left*) and an elliptical (*right*) trajectory.

It can be observed from Figure 2.2 that the pink dot is the body orbiting around the origin (blue dot) describing both parabolic and elliptical paths. The position vector of the pink-dotted body with respect to the origin is denoted by $\vec{r}$, and the angle between this position vector and the orange-dotted perihelion is labelled as θ.

If the angular momentum is zero, the position and velocity vectors of the bodies are parallel and go in the same direction (but not necessarily in the same sense!). Hence, the bodies will move in a straight line, and either they will collapse or they will separate until they escape.

2.4 A Study of the Trajectory Equation

Now let's study the case where $h \neq 0$. From the Equation of the Trajectory (2.10), we can classify the different types of trajectories depending on their eccentricity. As it has been said, the eccentricity indicates the deviation of the orbit from a perfect circle. Four trajectories are distinguished:

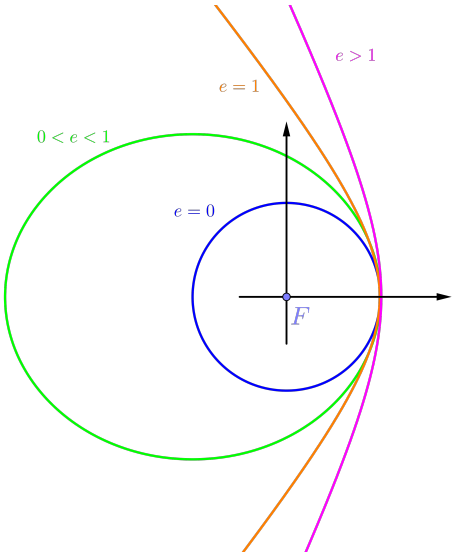

Figure 2.3: Different trajectories represented by their eccentricity.

Eccentricity	Type of Orbit	Colour
$e = 0$	Circular	
$0 < e < 1$	Elliptical	
$e = 1$	Parabolic	
$e > 1$	Hyperbolic	

Table 2.1: Brief summary of graph (2.3).

Furthermore, (see [4, page 29] [5]) we can express eccentricity as:

$$e = \sqrt{1 + \frac{2}{\mu^2}\mathcal{E}h^2}. \tag{2.11}$$

From equation (2.11) we can see that orbits can also be classified by their specific mechanical energy and their specific angular momentum.

We observe that the domain of a square root encompasses all non-negative values. If $\mathcal{E} < 0$, then $e \in [0, 1]$ and so,

$$-\frac{\mu}{\sqrt{2|\mathcal{E}|}} \leq h \leq \frac{\mu}{\sqrt{2|\mathcal{E}|}}. \tag{2.12}$$

For the extreme values of h, $e = 0$, and therefore we will have a circular trajectory. If $h \neq 0$, $e \in (0, 1)$, and the trajectory will be elliptical. The semi-major axis and the semi-minor axis are given by:

$$\begin{aligned} a &= \frac{h^2/\mu}{1-e^2} = \frac{\mu}{2|\mathcal{E}|}; \\ b &= \frac{h^2/\mu}{\sqrt{1-e^2}} = a\sqrt{1-e^2}. \end{aligned} \tag{2.13}$$

We must also consider the case when $\mathcal{E} < 0$ and $h = 0$: we will have a collision line.

If $\mathcal{E} = 0$, taking into account that $h \neq 0$, $e = 1$, and so the trajectory will be parabolic. When h is also zero, there will be a collision line.

The last case to consider is when $\mathcal{E} > 0$. Taking into account that $h \neq 0$ (because if not we would have a collision line), $e > 0$, and so the trajectory will be hyperbolic.

Equation (2.11) is illustrated in Figure 2.4:

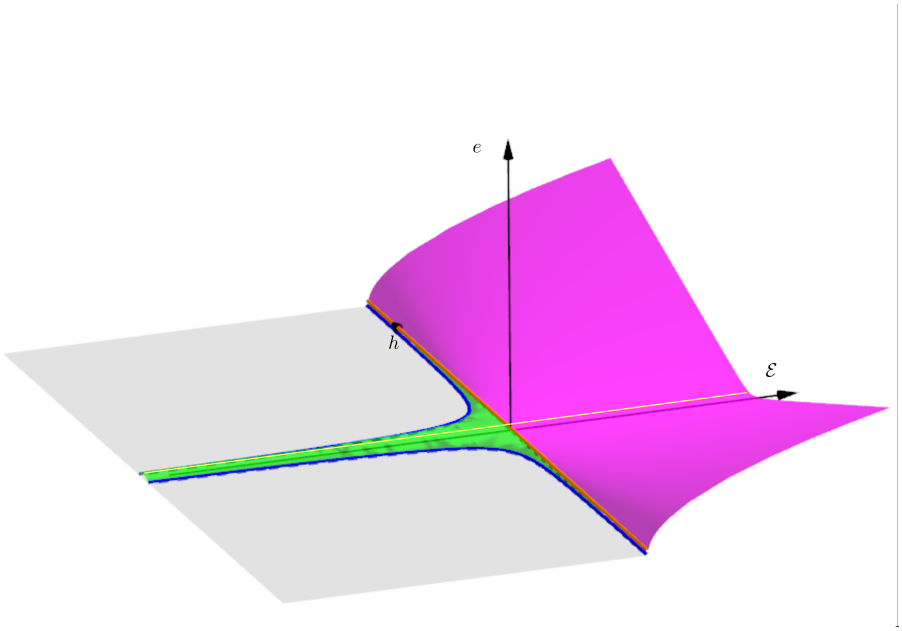

Figure 2.4: Eccentricity in terms of the specific energy and the specific angular momentum.

It can be said from Figure 2.4 that e is the eccentricity of the trajectories of the body represented at Table 2.1, $\mathcal{E}$ its specific energy and h its specific angular momentum

It is important to comment that we can make a distinction between trajectories with $\mathcal{E} < 0$ and the ones with $\mathcal{E} \geq 0$. Trajectories with $\mathcal{E} < 0$, i.e. circular and elliptical paths, have an enclosed movement, whereas parabolic and hyperbolic trajectories have a non-enclosed movement. This observation has a direct implication related to the radius, which we will discuss some lines below. The following graph represents the potential energy ($\mathcal{E}_p = \frac{-\mu}{r}$ see [12]) with respect to the radius:

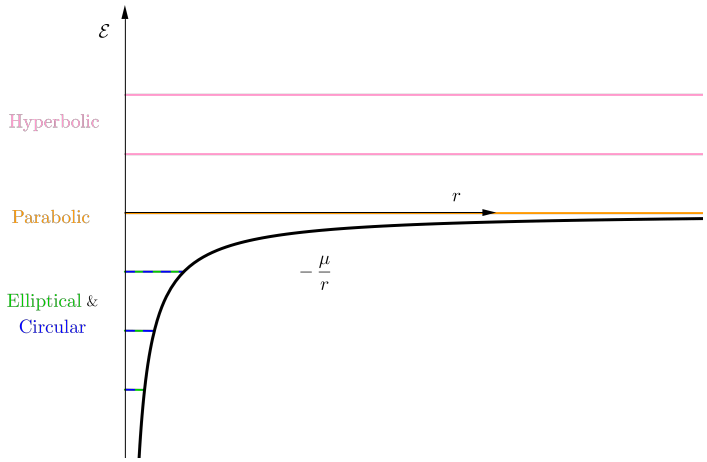

Figure 2.5: Trajectories classified by their potential energy with respect to the radius.

The different trajectories are separated in Figure 2.5 according to their potential energy. Some observations can be made, for instance that elliptical and circular movements ($\mathcal{E}_p < 0$) have an enclosed range for its radius. Particularly, in circular trajectories, we can state that the radius is not only enclosed but it is constant. On the contrary, the radius the parabolic and hyperbolic cases is not enclosed, it can take the values $(0, +\infty)$.

Finally, I have created a summary (see Table 2.2) exposing the type of orbit and their respective properties concerning the eccentricity, the energy, the radius and the movement of the body:

	Circular	**Elliptic**	**Parabolic**	**Hyperbolic**
Eccentricity (e)	$e = 0$	$0 < e < 1$	$e = 1$	$e > 1$
Energy $(\mathcal{E})$	$\mathcal{E} < 0$		$\mathcal{E} = 0$	$\mathcal{E} > 0$
Radius value	constant	not constant		
Type of movement	bounded		not bounded	

Table 2.2: Summary of the different types of orbits and their properties.

2.5 Getting back the two bodies

In the last sections we have shown the solution of the central-force problem and analysed it, but we must remember that this last problem was a reduction of our initial problem, which was the Two-Body Problem. As its own name says, in this problem we have two bodies, so we have to remember our statement ($r = r_2 - r_1$) and undo this change. From equations (2.10) and (1.5), we can state that:

Lemma 2.5.1. *In an inertial reference frame where the origin is set at the centre of mass, the dynamics of two bodies which are at positions r_1 and r_2 is given by:*

$$\begin{aligned} a)\ r_1 &= -\frac{m_2}{m_1+m_2}r \qquad ; \\ b)\ r_2 &= \frac{m_1}{m_1+m_2}r, \end{aligned} \tag{2.14}$$

where r is defined by (2.10).

Proof. By the formula of the centre of mass ((1.4)) we can deduce that:

$$r_1 = \frac{(m_1+m_2)r_{CM} - m_2 r_2}{m_1}.$$

Setting our reference frame at $r_{CM} = 0$, we obtain that:

$$r_1 = -\frac{m_2}{m_1}r_2.$$

Applying that ($r_2 = r + r_1$):

$$r_1 = -\frac{m_2}{m_1}r_2 = -\frac{m_2}{m_1}(r + r_1) \Rightarrow r_1(1 + \frac{m_2}{m_1}) = -\frac{m_2}{m_1}r \Rightarrow r_1 = \frac{-m_2}{m_1+m_2}r.$$

Analogously we could follow the same procedure and find r_2. □

Corollary 2.5.1. *Knowing the type of trajectory described by the position vector r, i.e. if it is circular, elliptical, parabolic or hyperbolic; the type of trajectories depicted by the position vectors of the two bodies, defined as r_1 and r_2, are also known, as they are the same.*

Proof. From equation (2.14), it is known that $r_1 = c_1 r$ and $r_2 = c_2 r$ are multiples of r, if the position vector r follows a particular trajectory, r_1 and r_2 will follow the same one with a variation given by the constants c_1 and c_2, which will make the trajectory more or less wide. □

The Corollary 2.5.1 is illustrated in Figure 2.6, showing the two bodies of masses m_1 and m_2 following the same type of trajectory in different cases. The line that unifies both bodies represents that we can express the position of one body with respect of the other one, and so, it is as if they were "attached":

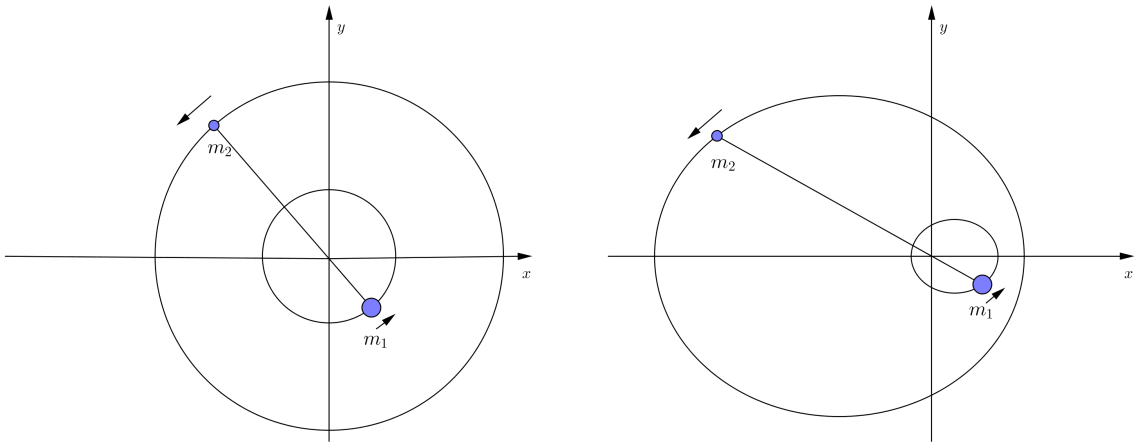

Figure 2.6: Back with the two bodies orbiting with circular (*left*) and elliptical (*right*) trajectories.

Chapter 3

The Restricted Three-Body Problem

3.1 Historical Background

Whittaker described the three body problem as *"most celebrated of all dynamical problems"*, and for Hilbert it fulfilled the necessary criteria for a good mathematical problem. It can simply be stated:

> *"three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion"*.

It seems quite a simple statement, but it belies the complexity of its solution. At the beginning of the 17th century, after having observed carefully what was going on in space, Kepler proposed his three laws for planetary motion, describing the elliptical orbits of the planets around the sun. Since Johannes Kepler first formulated his laws, scientists endeavoured to solve for the equation of motion of these bodies. In 1687, Newton published his *Philosophae Naturalis Principia Mathematica*, one of the most important books in the history of science, where he formalised this ideas and announced his laws of motion and gravitation, stated at section 1.2.

Newton proved Kepler's laws, and then he turned his attention to other systems than a Sun-Planet system, with higher degrees of complexity. He began to study systems containing more than two bodies. One of his main considerations was the Sun-Earth-Mon system. When he tried to analyse these kind of situations, we came up with a lot of difficulties, and he remarked *"...[his] head never ached but with the studies of the moon"*.

After Newton's death, it was not until 1747 that Alexis Clairaut announced that he had successfully constructed a series approximation for the motion of the three masses. Where Newton had aimed to approximate for the perigee of the moon, Clairaut actually succeeded in doing it. He won the St. Petersburg Academy prize for his work on the problem in 1752, and in 1759 the value of his approximations was amply demonstrated

with Halley's comet passed Earth within a month of what his equations had predicted, the margin of error he himself had prescribed.

Also in the eighteenth century, Leonhard Euler had focused on the three body problem. He proposed a simplification of the general problem where the mass of the third body m_3 is taken to be negligible; and also used variation of parameters to study perturbations of the planetary motion. In fact, the *Euler three body problem* is known to be a special case of the mentioned problem, with the remaining two masses orbiting with circular trajectories.

Simultaneously, the mathematician Joseph Lagrange made a big step in the progress of the general three body problem. His major contributions included the reduction of the problem from a 18-th order differential equation system to a system of order 7, and the description of two types of particular solutions to the general problem, which we will state later. It is important to remark his development of *Lagrangian Mechanics*, which has been a crucial tool not just to the three body problem in particular but to the general theory of dynamic systems.

Without knowing Lagrange's advancements, Carl Jacobi reduced the problem containing differential equations of order 18 to a sixth order system and Euler's restricted problem sixth-ordered system to a fourth-ordered one. A constant of motion was found, known as *Jacobi's integral*, which is the only known conserved quantity of the restricted problem. George Hill, in 1878, developed a very useful application of this constant of motion, describing the regions of possible motion for the body of negligible mass, known as Hill's regions.

Charles-Eugéne Delaunay was another well-known contributor to the theory of the problem. Performing a huge number of calculations, after two decades of a hard working, Delaunay's methods were published in 1846, but it was not until 1860 and 1867 that a serious publication including two large volumes of over nine hundred pages each came out. His method involved complex expressions and a slow convergence, and that's one of the reasons why it was impractical at that time. Nevertheless, the theoretical work has been well-regarded and highly influential in a large variety of fields, from lunar theory to quantum theory.

Henri Poincaré marked the end of the classical period of work on the three body problem. King Oscar II of Sweden, in the late 19th century, stated the problem as follows, and offered a price for solving the N-body problem, which is the problem with N rather than three masses, on the advice of Gösta Mittag-Leffler, Karl Weierstrass and Charles Hermite:

> *"Given a system of arbitrary many mass points that attract each other according to Newton's law, under the assumption that no two points collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly".*

Poincaré's work was so progressive and important that although he did not solve the problem, he won the price regardless. A mathematical error was found for $N = 3$ in his initial submission, that's why he destroyed the original paper and published a corrected paper which rectified the mistake. This new paper contained several ideas which would open a new approach to the problem and would lead to the development of the theory of mathematical chaos. Poincaré went to invent new mathematical methods that produced the modern field of differential geometry and topology in order to answer the stability question using geometry rather than analytic methods.

Poincaré's ideas were so influential that his foundations caused a pronounced progress in the 20th century on many different fronts. Solutions with power series were found by Sundman in 1912 and by Qiudong Wang in 1991 for $N = 3$ and for N bodies respectively. However, in both cases, the series constructed converged so slowly that they were essentially useless in practice.

As no analytical solutions were found, simplified versions of the three body problem were analysed. Different approaches began to take form, for instance the *Copenhagen problem*, which assumes the masses of the two other bodies to be equal. Thanks to Poincaré's demonstration that the system is chaotic, the possibility of systematically analysing orbits based on statistical distributions was brought up. Mathematicians Valtonen and Karttunen looked at the problem from the statistical point of view, analysing the scattering of the escape orbits, i.e. one mass leaving the two other bodies permanently, and the other two forming a binary system.

Hénon's contribution to the restricted three body problem consisted on classifying the possible motions in what he called *families* of orbits by perturbing the mass of one of the bodies taking that the parameter $\mu = 0$, which was an idea that Poincaré had already explored (see [3]).

3.2 Formal Approach

Naturally following from the two-body problem, first solved by Newton in his *Principia*, the three-body problem is a classical astronomical problem. Many mathematicians and scientists have focused on its general statement, but despite centuries of exploration, there is no solution to the three-body problem, as there are no coordinate transformations that can simplify the problem. Unlike the two-body problem or the restricted three-body problem, which will be introduced in the next section, the full three-body problem has no analytical solution (see [18]). It can be stated as follows:

> *"Given at any time the positions and velocities of three massive particles moving under their mutual gravitational force, the masses also being known, calculate their position and velocities for any other time".*

Let's consider three bodies with vector positions $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ and masses m_1, m_2 and m_3.

As the three-body problem is set in an inertial reference frame, we can apply equations 1.7, 1.9 and 1.10. Under the same hypothesis that we mentioned in the two-body problem, i.e. that the gravitational force is the only force interacting between the bodies and that the bodies are symmetrically spherical, we get:

$$
\begin{aligned}
m_1\ddot{\vec{r}}_1 &= \frac{Gm_1m_2}{\|\vec{r}_2-\vec{r}_1\|^3}(\vec{r}_2-\vec{r}_1)+\frac{Gm_1m_3}{\|\vec{r}_3-\vec{r}_1\|^3}(\vec{r}_3-\vec{r}_1);\\
m_2\ddot{\vec{r}}_2 &= -\frac{Gm_1m_2}{\|\vec{r}_2-\vec{r}_1\|^3}(\vec{r}_2-\vec{r}_1)+\frac{Gm_2m_3}{\|\vec{r}_3-\vec{r}_2\|^3}(\vec{r}_3-\vec{r}_2);\\
m_3\ddot{\vec{r}}_3 &= -\frac{Gm_1m_3}{\|\vec{r}_3-\vec{r}_1\|^3}(\vec{r}_3-\vec{r}_1)-\frac{Gm_2m_3}{\|\vec{r}_3-\vec{r}_2\|^3}(\vec{r}_3-\vec{r}_2).
\end{aligned}
\tag{3.1}
$$

The three punctual particles are shown at Figure 3.1 with their respective masses, their position vectors and the attractive forces acting between them:

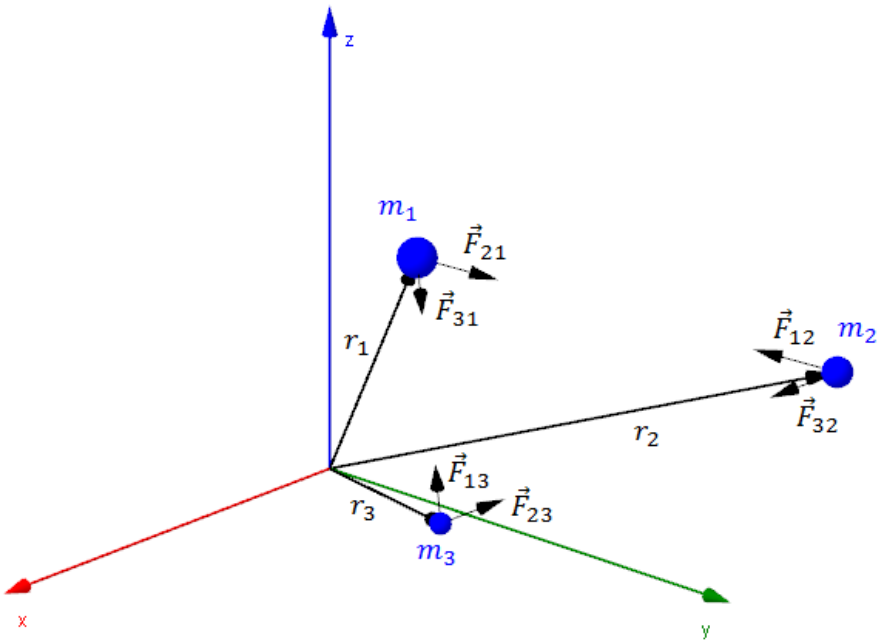

Figure 3.1: 3-D sheme of the three-body problem.

As we did in the previous chapter, some observations can be extracted from the system of equations

1. It's an **autonomous system**: the position, velocity and acceleration vectors are time-dependent, but the system does not depend explicitly on the independent variable (the time).

2. We have 3 equations, each of them three-dimensional, and each of them of second order. Hence, we have **18 unknowns**.

It has been clear that it does not exist a general solution for the three-body problem, but despite of this fact, some particular solutions have been worked out. The ones that give periodic solutions give rise to a particular dynamic system. This is a good point to mention Poincaré's famous *dictuum*:

> *"...what makes these (periodic) solutions so precious to us, is that they are, so to say, the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible".*

These periodic solutions are classified in three big families:

1. The first family of analytical solutions dates back to the eighteenth century, and is called the Lagrange-Euler one. It has been supplemented by one recent orbit due to C. Moore.

2. Secondly we have the Broucke-Henon-Hadjidemetriou family, dating to the mid-1970s with periodic rediscoveries of certain members of this family.

3. And finally the Figure-8 family, discovered by Moore in 1993, rediscovered in 2000, and extended to the rotating case.

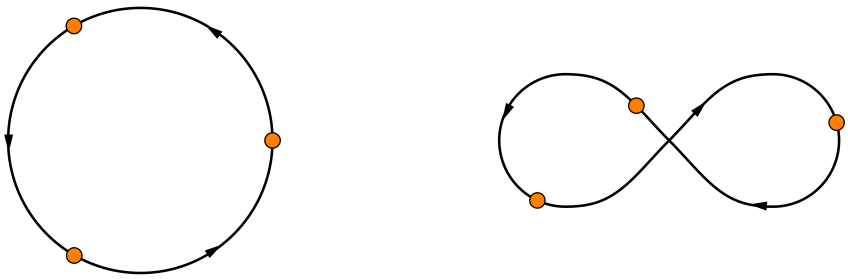

Figure 3.2: Two solutions for the 3BP when the three bodies have equal mass.

3.3 The restricted problem

In the last section we made the observation that the system of equations (3.1) contains 18 unknowns. It is also known that there are 10 constants of motion (6 from the center of mass, 1 of the energy and 3 from the angular momentum). In spite of this fact, there are still 8 unknowns remaining, so a number of simplifications are made.

The first and most prominent simplification is that we will make the mass of the third body m_3 tend to zero, so that it is negligible. So with this restriction, simplifying the system (3.1) we obtain:

$$\begin{aligned}
\ddot{\vec{r}}_1 &= \frac{Gm_2}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1); \\
\ddot{\vec{r}}_2 &= -\frac{Gm_1}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1); \\
\ddot{\vec{r}}_3 &= -\frac{Gm_1}{\|\vec{r}_3 - \vec{r}_1\|^3}(\vec{r}_3 - \vec{r}_1) - \frac{Gm_2}{\|\vec{r}_3 - \vec{r}_2\|^3}(\vec{r}_3 - \vec{r}_2).
\end{aligned} \tag{3.2}$$

In this way, the third body has no influence on the movement of the other two bodies, which we will now call *primary bodies*, and so, it is only necessary to study how this movement will influence the one of the body with an infinitesimal mass. This simplified system is called the restricted three-body problem, and was proposed by Euler. From now on, we will focus on the planar circular restricted three-body problem (R3BP), that's why we will make two more suppositions:

1. The movement of both primary bodies is circular.
2. The movement of the third body is produced in the same plane that contains the primaries.

In order to make notation easier and without losing generality, we will introduce the following simplifications:

- $m_1 + m_2 = 1$.
- $\|\vec{r}_1 - \vec{r}_2\| = 1$.
- We set the origin at the centre of mass, and define the *mass parameter* $\mu = m_2$.

Under these simplifications we get that the period of the orbit of the primary bodies is 2π, and that the vector positions of the primaries follow the following equations:

$$\begin{aligned}
\vec{r}_1 &= \mu(\cos t, \sin t); \\
\vec{r}_2 &= (\mu - 1)(\cos t, \sin t).
\end{aligned} \tag{3.3}$$

We will take the new coordinates $(\hat{x}, \hat{y})$ so that it is a simpler system, as it doesn't depend explicitly of t (see Figure 3.3), defined as:

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

The primaries with masses $1-\mu$ and μ are at rest and located at the points $(\mu,0)$ and $(\mu-1,0)$ respectively.

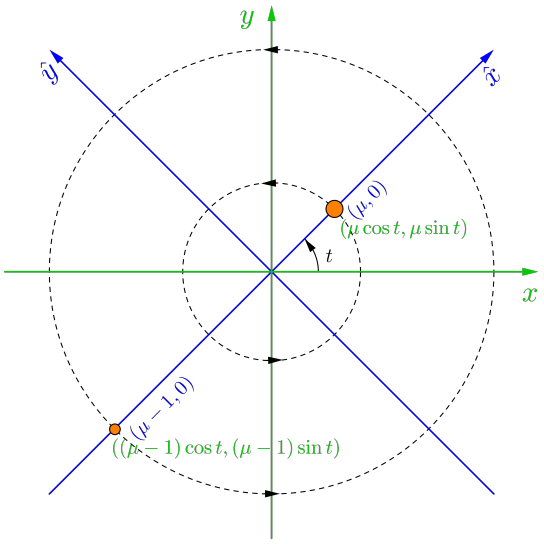

Figure 3.3: Positions of the primaries in the actual reference frame (represented in green); versus their positions in a new one (labelled in blue).

In order to clarify the notation we will change the name of the new variables $(\hat{x},\hat{y})$ to (x,y). We can make some **observations** from the new coordinates. **(1)**A clear disadvantage is that this new reference frame that has been set is not inertial, so Newton's Laws cannot be applied. **(2)** Nevertheless, it has the advantage that it is still an autonomous system in which the following equations describe the movement of the body with negligible mass (see [16]):

$$\begin{aligned}\ddot{x}-2\dot{y}&=D_x\Omega(x,y)\,;\\ \ddot{y}+2\dot{x}&=D_y\Omega(x,y),\end{aligned}\tag{3.4}$$

where:

$$\Omega(x,y)=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{d_1}+\frac{\mu}{d_2}+\frac{1}{2}\mu(1-\mu);\tag{3.5}$$

with $d_1=\sqrt{(x-\mu)^2+y^2}$ and $d_2=\sqrt{(x-\mu+1)^2+y^2}$. Where d_1 and d_2 are the distances from the first primary to the third body and from the second primary to the third body respectively.

The equations of motion for the third body (3.4) has a constant of motion, named *Jacobi's integral*, which is given by:

$$C = 2\Omega(x, y) - \dot{x}^2 - \dot{y}^2. \tag{3.6}$$

3.4 Equilibrium points

Imagine two primaries orbiting one around another. c These equilibrium points can be separated in two groups:

1. **Triangular points** ■

   The equilibrium points L_4 and L_5 are called *triangular points*. They lie at equal distance from the two primaries, and each of them forms the third vertex of an equilateral triangle with the primaries (see the yellow points represented at Figure **??**). These points are stable, and at that place the gravity forces exerted by the primaries cancels with the centrifugal force, which is directed away of the centre of mass, also called the *barycentre* of the system.

   As we said, L_4 is situated at the third vertex of an equilateral triangle, with the two primaries forming the other vertices. We can easily obtain L_5 by a mirror reflection of L_4 about the x-axis. The fourth and fifth Lagrange points have coordinates:

$$\begin{aligned} L_4 &: \left(\mu - \frac{1}{2}, \frac{\sqrt{3}}{2}\right); \\ L_5 &: \left(\mu - \frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \end{aligned} \tag{3.7}$$

2. **Collinear points** ■

   L_1, L_2 and L_3 are called the *colinear equilibrium points*, as they are located on the x-axis (see the magenta points represented at Figure **??**). Unlike the triangular ones, they're not stable. In order to determine the equations, we have to remember the Jacobian's constant, see (3.6). As L_1, L_2 and L_3 lie in a straight line on the x-axis, $y = 0$, and as they are equilibrium points, i.e. they are if we put a third body of negligible mass, from it we would see that the points are at rest, $\dot{x}, \dot{y} = 0$. Hence, letting $\Omega'(x, y = 0) = \Omega'(x)$, we want the points that follow this form:

$$\Omega'(x) = 0. \tag{3.8}$$

   Newton's method can be used to solve equation (3.8). The *Newton-Raphson Method* or *Newton's Method* is a powerful technique to find zeros of nonlinear equations

numerically (see [8]). It is based on the simple idea of linear approximation, and it consists of an iterative procedure.

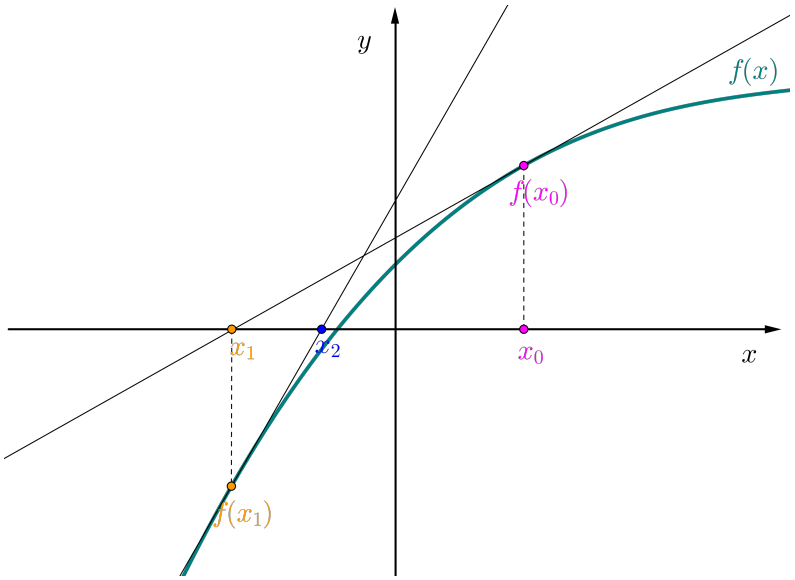

Figure 3.4: Graphical representation of Newton's Method.

Let the function $f : \mathbb{R} \to \mathbb{R}$ be a differential function and have a zero in s (i.e. $f(s) = 0$). We start with an initial estimate x_0 of s, called a "guess". Newton's method tends to converge if our initial estimate is close to s, and will have a low probability to converge if it is not. If x_n is the current estimate, then the next estimate x_{n+1} is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3.9}$$

We can use the geometric interpretation to explain the method. Figure 3.4 represents two iterations of Newton's Method to find a zero x_s of the function $f(x)$.

In our problem, we want to calculate the zeros of $\Omega'(x)$, so we will change $f(x)$ in equation 3.9 to $\Omega'(x)$ and $f'(x)$ to $\Omega''(x)$, where (see [16]):

$$\begin{aligned} \Omega'(x) &= x - \frac{(1-\mu)(x-\mu)}{((x-\mu)^2)^{3/2}} - \frac{\mu(x-\mu+1)}{((x-\mu+1)^2)^{3/2}} \\ \Omega''(x) &= 1 + \frac{2(1-\mu)}{((x-\mu)^2)^{3/2}} + \frac{2\mu}{((x-\mu+1)^2)^{3/2}} \end{aligned} \tag{3.10}$$

The result for the x-coordinate location of the three collinear points depending on the mass parameter μ is represented in Figure 3.5. The code that has been implemented in order to solve the problem is shown in section A.2.

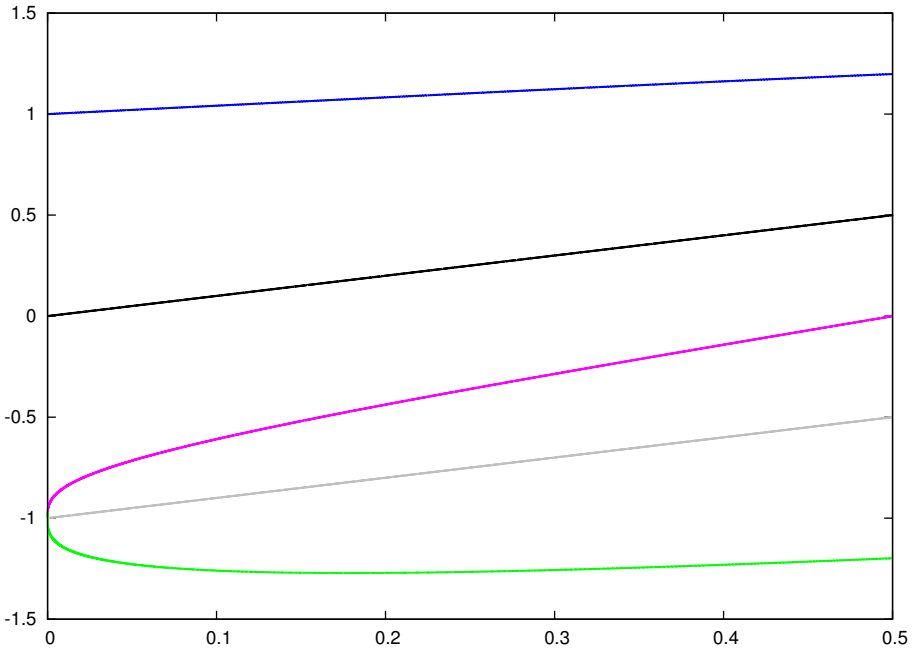

Figure 3.5: Variation of the x-coordinate of the collinear equilibrium points with respect to the mass parameter μ.

In Figure 3.5:

- The **grey** line represents the movement for the second primary.
- The **black** line represents the path of the first primary.
- The **pink** line is for the equilibrium point L_1.
- The **blue** one for L_2.
- The **green** line sketches the movement of L_3.

3.5 Hill Region

We defined the Jacobi's constant c at equation (3.6) (see [20]). From this equation, we can set the inequality $2\Omega(x, y) \geq C$, which places a constraint on the variable position of x for each value of C, and if x satisfies this condition, then a solution of the restricted problem through the point x for that concrete value of the constant can be found. The associated Hill region can be defined as:

$$\mathcal{R}(C) = \{(x, y) \in \mathbb{R}^2 \mid 2\Omega(x, y) \geq C\}. \tag{3.11}$$

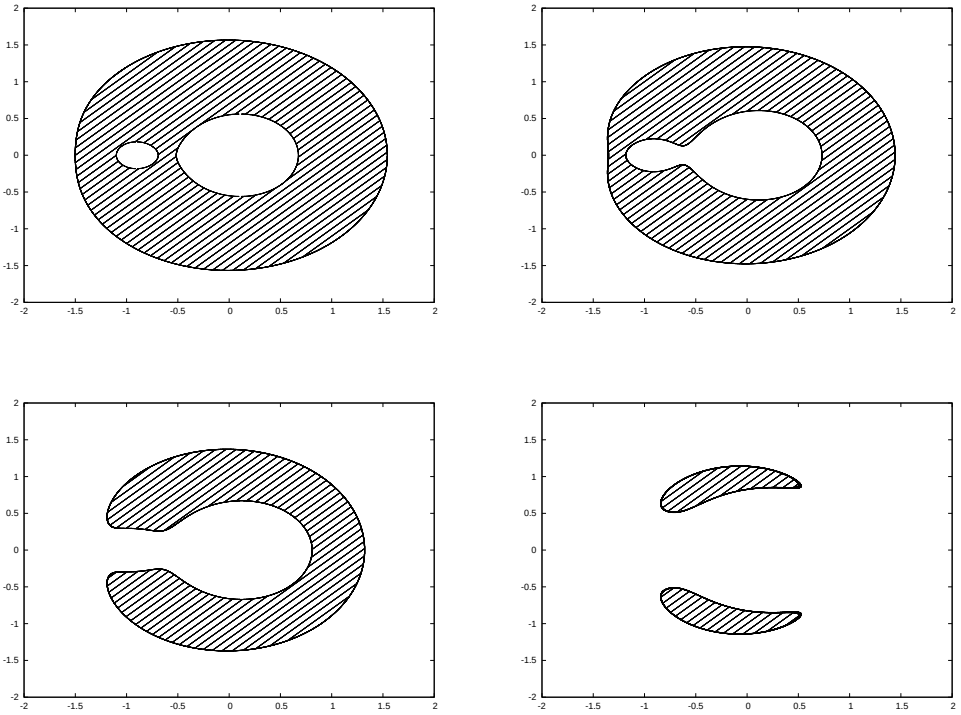

Figure 3.6: Plot of the Hill Region with different Jacobi constants for $\mu = 0.1$.

Lemma 3.5.1. *The Hill Region, defined by equation* (3.11)*, gives the enclosed area where the orbits of the third body can live.*

Proof. We consider the inequality $2\Omega(x, y) \geq C$. When $2\Omega(x, y) = C$, we are at the border of the Hill region, and this equality can only be satisfied if the velocity of the third body is zero. Therefore, the Hill region is a sector in space x, y where movement can take place: all the orbits of the third body satisfy the inequality stated at the beginning of the paragraph. If this last statement weren't true, i.e. the orbits of the third body could live at a region out of the area given by the inequality, it would mean that the modulus of the velocity of the third body is negative, and that is not possible. □

I have created a program shown at appendix A.3 in order to depict the Hill Region with four different Jacobi Constants $C = 3.1, 3.4, 3.6, 3.8$ respectively (see Figure 3.6 from left to right and from the top to the bottom). As it can be seen from the cited figure, the higher the Jacobi Constant, the smaller the Hill Region, as the third body will only be able to be located in regions of higher energy.

Chapter 4

Numerical Methods for ODEs

As it was said in section 1.5, given a differential equation, the solution of such problems in analytical form can be found in very limited cases, only for some rather special right-hand side function. In this chapter we will study several methods which consist of a finite number of steps, including one-step and multistep methods, in which we will not look for a closed-form expression for the desired solution of an ODE but for an approximation to this unknown solution at certain points, where the solution exists, by means of a numerical analysis. The aim in this chapter is to define these numerical methods, i.e. to give the description of these algorithms. We will only deal with explicit numerical methods, i.e. methods that calculate the state of a system at a later time from the state of the system at the current time. For having clear the explained information and additional information, see [6] [11].

Our objective is to find the numerical solution of the problem

$$\begin{cases} \dfrac{\partial x}{\partial t} = f(t,x), t \in [0,T]; \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $T > 0$ is s.t. the IVP (4.1) a unique solution on the time interval $[0,T]$ exists. This means that we want an approximation of the solution of this problem at a finite number of points in the time interval, denoted by $\{0 = t_0 < t_1 < ... < t_n = T\}$.

This chapter will help us to choose the best method, that is, the one whose error is the lowest. Therefore, every time that we introduce a numerical method to solve ODEs, we will expose its error, and we will refer to it as the *order* of that particular method. In numerical analysis, the *order of accuracy* quantifies how good a numerical approximation of the numerical solution of a differential equation converges to its solution (see [2, ch. 4]). A numerical solution to a differential equation is said to be n-th order accurate if

$$\varepsilon(h) = \mathcal{O}(h^n), \tag{4.2}$$

where

- ε is the global error, i.e. accumulation of the local error (the error caused during a single iteration) over all of the iterations.
- h is the step-size (the increment of time).
- $\mathcal{O}(h^n)$ is written in big $\mathcal{O}$ notation, and n is the order of the numerical method.

Nevertheless, before starting with the description and some examples of these methods, a section is going to be introduced whose aim is to give a first approach to the differentiation of functions between euclidean spaces of finite dimension, as a good number of the methods that are going to be explained require these calculations.

4.1 Differentiation of functions

The concept of *differential* has its origins in the calculation of the planes tangent to a surface. In 1911, the french mathematician M. Fréchet defined it as Given the function $f : \mathbb{R} \to \mathbb{R}$, f is differentiable at the point a when there exists the

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{4.3}$$

This limit is represented as $f'(a)$, and is named the derivative of f at point a.

The geometrical meaning of the existence of the derivative at point a means that there exists a tangent line at the point $(a, f(a))$ to the curve represented by the graph of f, and the slope of the mentioned tangent line is $f'(a)$. Hence, the equation of the tangent line is given by:

$$y = f(a) + f'(a)(x - a).$$

Now, we want to extend the previous results to functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Given a function $f(\vec{x})$, a point $\vec{a} = (a_1, ..., a_n)$, and a director vector $\vec{v} = (v_1, ..., v_n)$, we define the *directional derivative* of f in $\vec{a}$ and the direction $\vec{v}$ as:

$$\begin{aligned} f'_v(\vec{a}) &= \lim_{h\to 0} \frac{f(\vec{a}+h\cdot\vec{v}) - f(\vec{a})}{h} \\ &= \lim_{h\to 0} \frac{f(a_1+h\cdot v_1, ..., a_n+h\cdot v_n) - f(a_1, ..., a_n)}{h} \end{aligned}$$

The *partial derivative* of a function f in a point $\vec{a} = (a_1, ..., a_n)$ with respect to the i^{th} component is the directional derivative at the point and in the direction of the vector $\vec{e_i} = (0, ..., 0, 1^{(i)}, 0..., 0)$. It is denoted by:

$$f'_{e_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$$

The *gradient vector* of a function $f(\vec{x}), x \in \mathbb{R}^n$ is the vector formed by all the partial derivatives (see [12]):

$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(\vec{x}), ..., \frac{\partial f}{\partial x_n}(\vec{x})\right)$$

4.2 One-Step Methods

The theorems and definitions considered in section 1.5 inform us about the existence and uniqueness of the solution of the IVP or Cauchy problem, but when we ask ourselves how to find its solution, there is no answer to the question.

This section focuses on *one-step methods*, i.e. those methods in which the value of the approximated solution to the problem (4.1) at a given t_n is defined only by the approximation at the time t_{n-1}.

4.2.1 Euler's method

A first numerical method for the solution of the initial value problem defined at equation (4.1) can be worked out by the following observation: as we want that our solution is of the form $y'(t) = f(t, y(t))$ with $t \in [0, T]$, $f(t, y(t))$ is the slope of the desired exact solution $y(t)$, then:

$$\begin{aligned} &\frac{y(t+h) - y(t)}{h} \approx f(t, y(t)); \\ &y(t+h) \approx y(t) + hf(t, y(t)). \end{aligned}$$

We chose a steplenght $h \neq 0$, and starting with the given initial values $(t_0, y(t_0))$, we obtain at equidistant points (given that h is constant) $t_i = t_0 + ih, i = 1, ..., n$ approximations ψ_i to the values $y_i = y(t_i)$ of the exact solution $y(x)$ follow the subsequent iterative algorithm:

$$\begin{aligned} \psi_0 &= y_0; \\ \text{for } i &= 0, ..., n : \\ \psi_{i+1} &= \psi_i + hf(t_i, \psi_i), \\ t_{i+1} &= t_i + h. \end{aligned}$$

It is important to remark that Euler's method is an algorithm of order 1, that is, $\varepsilon \propto \mathcal{O}(h)$.

4.2.2 Taylor's method

Euler's method can be sharpened and expanded into what we call *Taylor's method*. From the solution $y(t)$ of the Cauchy problem (see 1.5.3), which satisfies that:

$$y'(t) = f(t, y(t)), t \in [0, T], \tag{4.4}$$

we assume that f has continuous partial derivatives of any order. Differentiating equation 4.4, using the chain rule, we get the following relation at some point $\tilde{t} \in [0, T]$:

$$\begin{aligned} y'(\tilde{t}) &= f(\tilde{t}, y(\tilde{t})), \\ y''(\tilde{t}) &= f_1(\tilde{t}, y(\tilde{t})) + f_2(\tilde{t}, y(\tilde{t}))y'(\tilde{t}), \\ y'''(\tilde{t}) &= f_{11}(\tilde{t}, y(\tilde{t})) + f_{12}(\tilde{t}, y(\tilde{t}))y'(\tilde{t}) + f_{22}(\tilde{t}, y(\tilde{t}))(y'(\tilde{t}))^2 + f_2(\tilde{t}, y(\tilde{t}))y''(\tilde{t}). \end{aligned}$$

All these derivatives can be compounded exactly, as $y(\tilde{t})$ is a known value. Higher order derivatives can be computed in the same way, but their corresponding formulae become increasingly complicated.

Let $t > \tilde{t}$ such that $[\tilde{t}, t] \subset [0, T]$. Then, inside the domain of convergence, the following relation holds:

$$y(t) = \sum_{k=0}^{\infty} \frac{y^k(\tilde{t})}{k!}(t - \tilde{t})^k. \tag{4.5}$$

As it is not possible to compute partial derivatives of any order of the function f, and to compute the exact value of the solution at some fixed point, equation (4.5) requires the

summation of an infinite series (which is typically not possible), the computation of the exact value $y(t)$ is not possible, and therefore we aim to define only its approximation. Taylor's method is an algorithm of second order, hence: $\varepsilon \propto \mathcal{O}(h^2)$.

4.3 Multistep Methods

While in the previous section we considered such numerical methods in which the new value of the approximation solution is only defined by the previous point; in the sequel, the new value of the approximation is defined by several previous approximations. These methods in which the unknown value y_{n+k} is in function of some previous values $y_{n+k-1}, y_{n+k-2}, ..., y_n$. In this case, we obtain a method of k steps and, in particular, if $k = 1$, we get a one-step method.

Formally, in a multistep method for the solution of the IVP ((1.35)) one computes a value a_{n+r} of $y(x_{n+r})$ from $r \geq 2$ given approximate values a_k of $y(x_k), k = n, n+1, ..., r-1$, at equidistant points $x_k = x_0 + kh$.

Definition: Let $a_0, a_1, ..., a_m$ and $b_0, b_1, ..., b_m$ be given numbers. The iteration of the form

$$a_0 y_i + a_1 y_{i-1} + ... + a_m y_{i-m} = h[b_0 f_i + b_1 f_{i-1} + ... + b_m f_{i-m}],\ i = m, m+1, ...; \quad (4.6)$$

is called a linear m-step method.

In order to initiate multistep methods, the r starting values $a_0, ..., a_{r-1}$ must be our disposal, and thus, we can obtain them, for example, with the aid of a one-step method.

4.3.1 Adams-Bashforth methods

Adams methods are obtained when in the general formula (4.6) the parameters a_i are defined as:

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -1, \\ a_2 &= a_3 = ... = a_m = 0 \end{aligned}$$

The parameters $b_0, b_1, ..., b_m$ are free parameters. The Adams method with $b_0 = 0$ is called *Adams-Bashforth* method, and is going to be studied in this subsection.

In 1833, Bashforth and Adams firstly proposed the idea of extending Euler's method. They brought up the idea of allowing the approximate solution at a point to depend on

the solution values and the derivative values not at the immediate previous step but at several previous step values so that the method gives a better approximation solution, and moreover, they wanted to avoid the use of derivatives.

Particularly, we will use *Adams-Bashforth 3*, which uses the three previous points in order to calculate the following approximation solution. The approximation solution y_{i+1} is given by:

$$y_{i+1} = y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}], \tag{4.7}$$

where $f(y_i) = f(y)$.

The Adam-Bashforth 3 method is an algorithm of order three, that is $\varepsilon \propto \mathcal{O}(h^3)$, which is much lower than the method studied in the first section (see [8]).

4.4 Runge Kutta Methods

In 1895, Runge, came up with the idea of generalising the Euler method by allowing for a number of evaluations of the derivative to take place in a step. Further contributions were proposed by Kutta in 1901, and the latter completely characterised the set of the called the called *Runge-Kutta* methods.

This is one of the methods most thoroughly used, and is particularly appropriate for functions whose calculation of derivatives of greater order is complex. It can be used for equations of arbitrary order transforming them in a system of equations of first order. Nevertheless, its major inconvenient resides in the difficulty of the error estimation. The method is given by

$$y_{n+1} = y_n + h\sum_{i=1}^{p} b_i k_i, \tag{4.8}$$

where:

$$k_p = f(t_n + c_s h, y_n + h(a_{s1}k_1 + a_{s2}k_2 + ... + a_{s,s-1}k_{s-1}))$$

It is convenient to represent the Runge-Kutta methods by a partitioned tableau of the form:

$$\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$$

Which, in extended form, corresponds to:

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1r} \\ \vdots & \vdots & & \vdots \\ c_r & a_{r1} & \dots & a_{rr} \\ \hline & b_1 & \dots & b_r \end{array}$$

where

- The vector c indicates the positions, within the step, of the stage values
- The matrix A indicates the dependence of the stages on the derivatives found at other stages
- $b^\top$ is a vector of quadrature weights, showing how the final approximation solution depends on the derivatives, computed at the various stages.

Since the advancements in digital computers, mathematicians have focused their interest on the called *Runge-Kutta* methods, and a large number of research workers have contributed to the development of particular methods. One of them is the called *Dormand–Prince method*, and it is the one that we will use and program. The *Dormand–Prince method* has seven stages or function evaluations, and it calculates both fourth and fifth order accurate solutions. The error of the algorithm is taken to be the difference between the two solutions.

Dormand–Prince method chooses some specific conditions [7], for example uses the coefficients a, b, c that make the error of the solution is of fifth order. The Butcher tableau for the method is:

$$\begin{array}{c|ccccccc} 0 & & & & & & & \\ \frac{1}{5} & \frac{1}{5} & & & & & & \\ \frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\ \frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\ \frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\ 1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\ 1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\ \hline & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \\ & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40} \end{array}$$

Table 4.1: The Butcher tableau for Dormand&Prince method.

In the section of b coefficients, the first row gives the fifth-order accurate solution and the second row gives the fourth-order accurate solution. *Dormand&Prince* method's code, programmed by myself, can be seen at the appendix A.7.

4.5 Comparative analysis of the methods

Keeping in mind the algorithms stated in the previous sections from this chapter, we will compare them in order to ratify the theoretical results, which say that Runge-Kutta 45 method approximates the best the solution of an equation given that it has the greater order.

We known that the solution of the pendulum system (1.33) taking $a = 1$ and the time-interval being $t \in [0, 7]$, is the following:

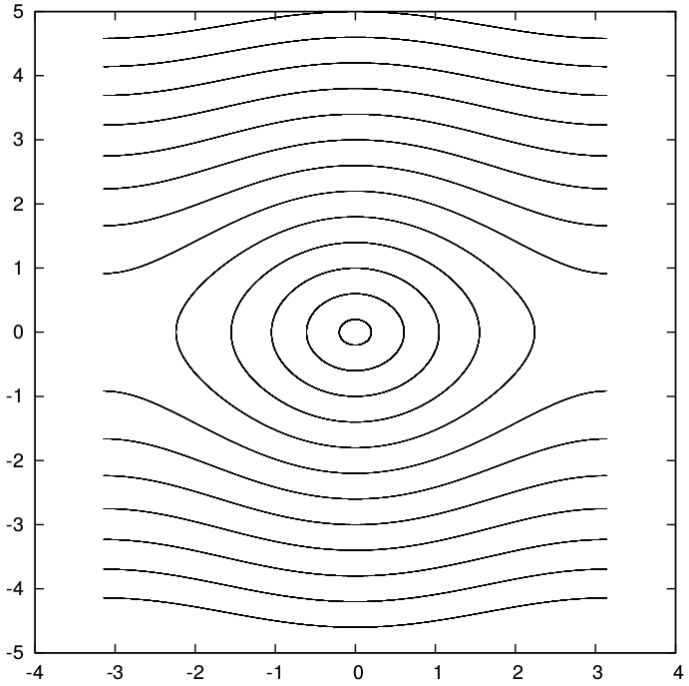

Figure 4.1: Solution to the Pendulum System integrated with Dormand&Prince method.

Euler and Taylor 2 are methods that have order 1 and 2 respectively; that means that in order to acquire an error under the imposed tolerance a huge number of steps are required. In Figure 4.2, we can observe the solution of the pendulum system (1.33) has been approximated with these two methods. Remembering the problem:

$$\begin{cases} \theta''(t) = -\sin\theta \\ \theta(0) = 1 \\ \theta'(0) = 0 \\ \quad t \in [0, 7] \end{cases}$$

The solution obtained with the given initial conditions and the size-step control for both methods $h_{new} = 0.9 \cdot h \cdot \frac{\text{TOL}}{\varepsilon}$, where TOL is the tolerance and ε the local error, has been:

From Figure 4.2 we can make the following comments. A lot of integrations are needed

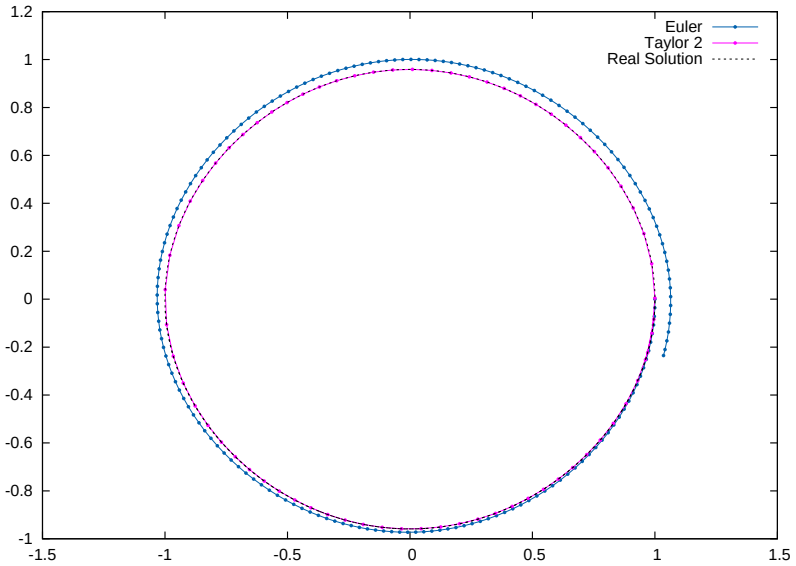

Figure 4.2: Comparison between the solution of the pendulum system with Euler and Taylor methods, using a tolerance of 10^{-3}.

so that the local error is lower than the tolerance. This is the main drawback of the algorithms. Moreover, as Euler method requires more steps than Taylor's method, the local error keeps accumulating giving a remarkable global error. The C++ code of both methods can be seen in A.4 and A.5.

We have also found the solution of the pendulum equation with Adams-Bashforth 3 method:

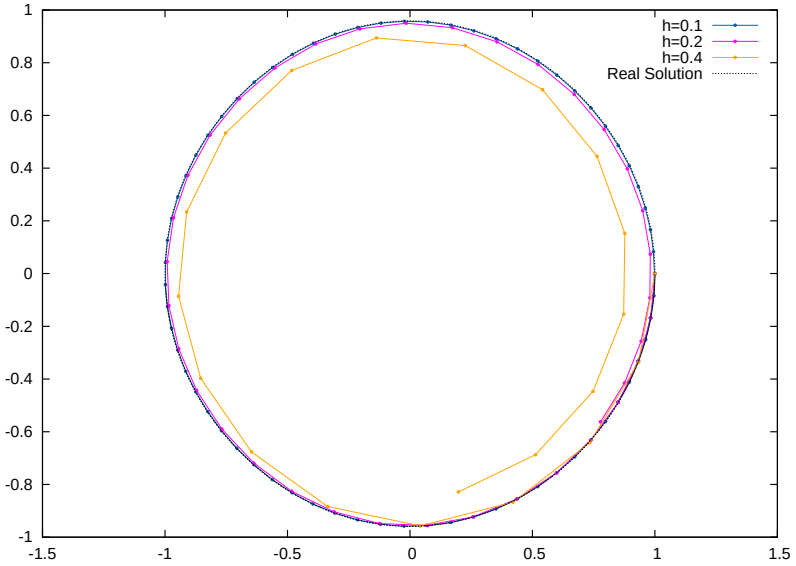

Figure 4.3: Adams-Bashforth approximate solutions of the Pendulum System

As we can observe, several solutions for distinct values of h are drawn at Figure 4.3.

This is because the step-size is constant for each solution, as it is difficult to implement a step-size control in multistep methods. This last comment is the main drawback of the algorithm. We could implement a step-size control, but we would need to solve a linear system of equations, and this is work that, using other methods, can be saved. The C++ program of Adams-Bashforth method can be seen in A.6.

Also from Figure 4.3, it can be observed that the bigger the h, the greater the error is (or, in other words, the worse the approximation solution is). This makes sense, as the error depends proportionally on the size-step.

With Runge-Kutta 45 method, given that the size-step control is $h_{new} = 0.9h(\frac{TOL}{\varepsilon})^{1/6}$, where TOL is the tolerance and ε is the local error, the solution is the following:

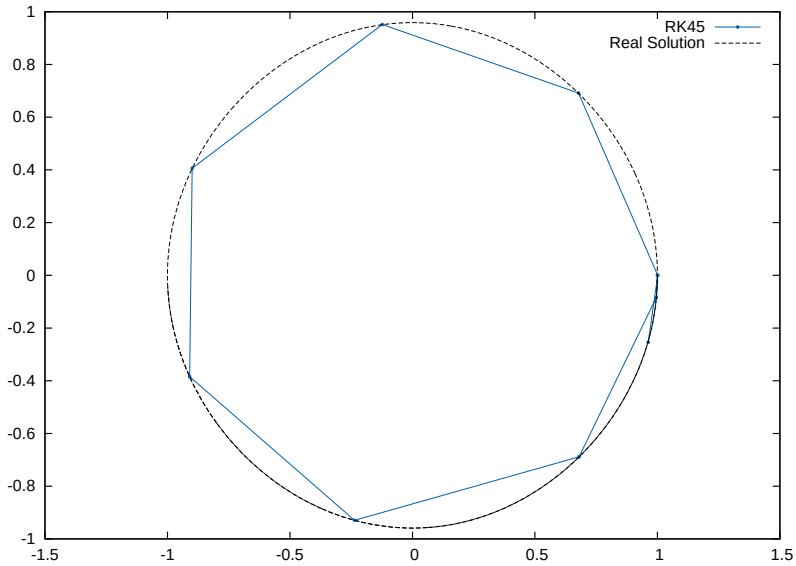

Figure 4.4: Approximation solution to the Pendulum System with Dormand&Prince method.

Dormand&Prince method approximates perfectly the solution with only eight steps. A big size-step h still gives an error under the imposed tolerance 10^{-3}, so it is a very good method.

A Figure showing all the methods in one has been created in order to do a final comparison between them with the following system:

$$\begin{cases} \dot{x} = -x - y + x(x^2 + y^2) \\ \dot{y} = x - y + y(x^2 + y^2) \\ x\,(0) = 1 \\ y\,(0) = 0 \end{cases} \tag{4.9}$$

It is necessary to remark that we have taken $h = 0.02$ and the time interval is $t \in [0, 7]$:

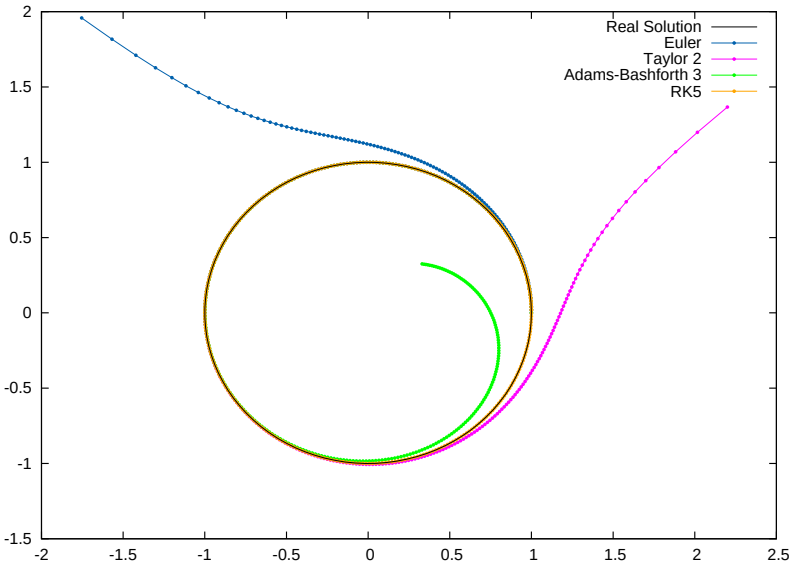

Figure 4.5: Final comparison between all the studied methods with a fixed step-size.

From Figure 4.5 it can be clearly seen that the only integration method that gives a good approximation solution is Runge-Kutta 45. This result coincides with the theoretical result, which said that the higher the order of a numerical method, the better the approximation solution is.

Before continuing, we will try to integrate the famous Lorenz system with our Runge-Kutta method:

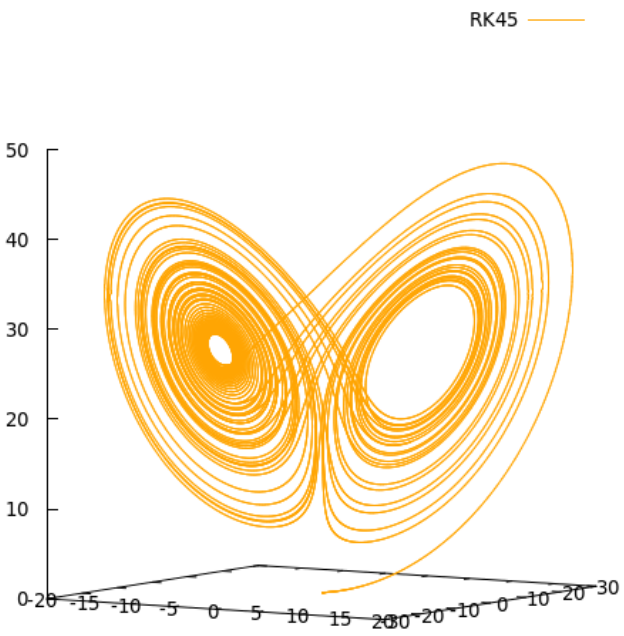

Figure 4.6: Runge-Kutta's 45 approximated solution to the well-known Lorenz system.

Chapter 5

Objective L_4

5.1 About L_4 in the Earth-Moon system

Remembering some concepts explained at section 3.4, talking about the Earth-Moon system, L_4 is an equilibrium point (i.e. a region where the gravitational forces cancel out with the centrifugal force) that forms the third vertex of an equilateral triangle with the primaries (the Earth and the Moon).

Studies showed that there were bodies located at L_4 in many systems. For example, in the Sun-Jupiter system, a huge number of asteroids (more than 2000) called Trojans, orbit near its triangular equilibrium points. Another example can be found in the Sun-Saturn system. One of Saturn's moon called Dione is located at L_4, and in the Saturn-Dione system, a satellite called Helena is found at its L_4 equilibrium point.

Nevertheless, no objects were found near this point (and nor near L_5) in the Earth-Moon system. Astronomers decided to create a simulation including hypothetical small particles near L_4 and L_5 to understand this absence. Their only explanation for this observed curious fact was that additional gravitational forces of more distant bodies, such as the Sun and the other planets of the Solar System made these regions unstable. They divided the experiment in two simulations, during which they tested Trojan particles that were initially located near the L_4 Earth-Moon point in either the Earth-Moon-Sun system, and a system including all 8 planets of the Solar System with the Moon and the Sun.

The results extracted from the first simulation showed that, in the Earth-Moon system, taking into account the gravity of the Sun, the "Trojans" artificially located at L_4 lasted more than 10^9 years. On the other hand, a different result was obtained in the second simulation. When the other 7 planets were added, no particles survived beyond a few million years. This fact can be explained because, although the force of gravity exerted by the other planets on the Earth and on the Moon is much smaller than the Sun's gravity, they change the eccentricity of the Earth's orbit around the Sun.

5.2 Objective

The aim of this Chapter is to determine the regions in space where a satellite can be found if we want it to pass through the equilibrium point L_4 of the Earth and Moon's orbits, i.e. we want to solve the R3BP 3.3 where the two primaries are the Earth and the Moon and the body of infinitesimal mass is our satellite. The R3BP gives us the system of differential equations (3.4) that describes the movement of the third body, so we have to solve an IVP 1.5.3 using the numerical method of integration *Dormand&Prince* 4.4.

Firstly we want our data (masses of the primaries, distances and radi) to follow the simplifications stated at section 3.3. Knowing that, in S.I. units

	Earth	Moon
Distance	384400 km (d_{12})	
Mass	$5.972 \cdot 10^{24}$ kg (m_1)	$7.348 \cdot 10^{22}$ kg (m_2)
Radius	6371 km (r_1)	1737 km (r_2)

Table 5.1: Data of the Earth and the Moon

Now, using the values exposed in table 5.1 we get the following data in our simplifications:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{7.348 \cdot 10^{22}}{5.972 \cdot 10^{24} + 7.348 \cdot 10^{22}} = 0.01215;$$
$$d = \frac{r_1}{d_{12}} = \frac{6371}{384400} = 0.01657.$$

where μ is the mass parameter and d is the distance between the point from which we launch the satellite and the centre of the first primary (the Earth).

5.3 Method and procedure

In this section it is going to be explained the technique that will be applied concerning how are we planning to calculate the regions where a third body can be found with the condition that we want it to pass through L_4 of the Earth-Moon system.

In the first place have to chose a point on the Earth's surface, i.e. at a distance of 0.01657 from the centre of our planet. Once the launching point is set, we have to calculate the modulus of the limiting velocity that makes our satellite pass through L_4 between the region of angles that is going to be determined in each specific case. Then, we do multiple launches of the satellite with this velocity (in modulus), and calculate which is the angle that makes the distance between our satellite and L_4 minimal. After saving this angle, the number of that particular vector and the time of integration (which is the time that it

takes our satellite launched from the Earth's surface to the triangular point); we increase the modulus of the velocity and perform in a parallel way until with 100 different initial velocities.

With the data assembled, taking the two limiting trajectories that go from Earth to L_4, we will have determined the region that we wanted, i.e. we will have reached our objective.

5.4 Results

In this section I will show the images and explain the program that I have developed in order to sketch the regions in space where a third body can be found if we throw it from a point on the surface of Earth (actually I have done the experiment with three different launching points) to the equilibrium point L_4.

5.4.1 My C++ programs

Having the objective of my work in mind, I have programmed two codes which are shown at appendix A.8 and A.9 and will be explained in the following lines in order to run my simulation.

In the first program, firstly I have created a *.dat* file in which we type the initial conditions of the satellite. The file is read by the program and then it begins to proceed with the algorithm. The mentioned *.dat* file contains the following information:

1. The initial launching position of the satellite.
2. t_0, which is the initial time of the launching.
3. The final time of integration, common for all the different iterations.
4. The initial angle at the launching point from which we fire off the satellite.
5. The modulus of its initial velocity.

This file is created in order to save work and time having only to modify the numbers on it in case we want to change the launching point instead of recompiling the whole program.

Once the program reads the *.dat* file, given a maximum and a minimum angle, we subdivide the segment formed by both maximum and minimum angles in 21 equidistant points, i.e. we launch 21 vectors, integrate the system of equations (3.4) with *Dormand&Prince* method for each launch and save the number of the vector such that the distance between the approximation solution and the point L_4 is minimum. Then, we save the vector on

the right and the one on the left of the selected vector. The inclination of these vectors will give us two new minimum and maximum angles, and we will use them in the next iteration of the algorithm. Figure 5.1 shows this process with the first 2 iterations and 7 vectors on each one.

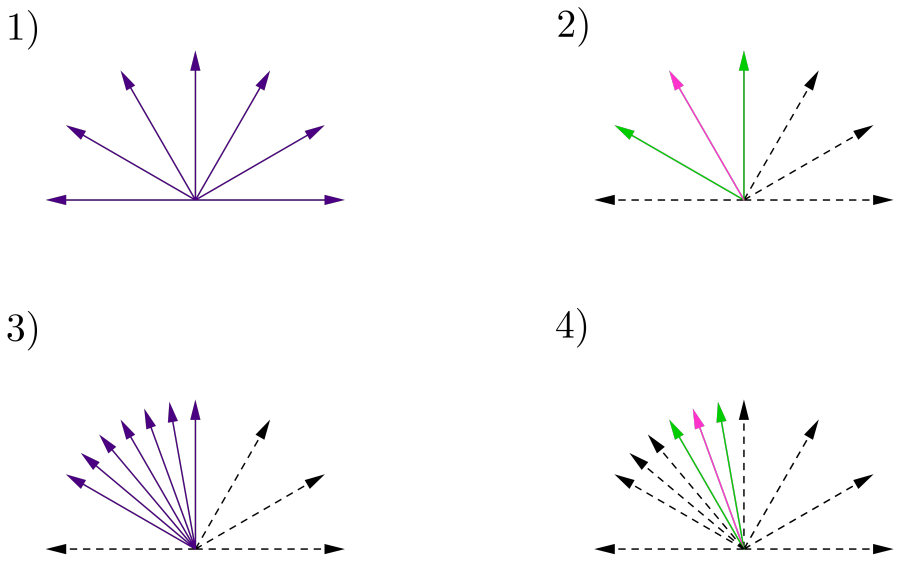

Figure 5.1: Illustrative representation of the first two iterations of the part of code A.8.

Where:

- → are all the launches with the same velocity in modulus and different angles.
- → represent the selected vectors on each iteration, i.e. the ones that make the distance between the equilibrium point and my approximation solution minimum.
- → are the limit vectors that define the region where we are going to fire off our satellite in the next iteration.

After these steps, we will have a good approximation for the vector whose solution passes through L_4 for that particular velocity. Finally, we repeat this iterative process for 99 more different velocities. Before changing the modulus of the initial velocity, I open another *.dat* file, where the following elements are saved:

1. The modulus of the initial velocity v_0, which I have selected in order that the satellite goes through L_4.
2. The angle that makes the distance from my approximation solution to the equilibrium point minimum for each velocity.
3. The time of integration required until the satellite reaches L_4.

The second program reads the mentioned file and repeats the integrations of the function, saving all the necessary data in order to be able to plot the different trajectories of my satellite with different velocities that go through L_4. The difference between the first and the second program is that in the second one, the time of integration is particular for each velocity, it is the one calculated and saved in the first program, so that when I plot the results, all the fire offs start and and at the same points (start at Earth and arrive at L_4).

5.4.2 P_1: 180 degrees with the horizontal

The first point from which we will launch our satellite is obviously located at the surface of Earth, and makes an angle of 180 degrees with the horizontal. We consider that Earth has a volume, so the satellite can't cross the planet during its trajectory to L_4. For that reason, it is clear that all the possible initial angles θ of firing off are contained in the interval $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, as shown at Figure 5.2.

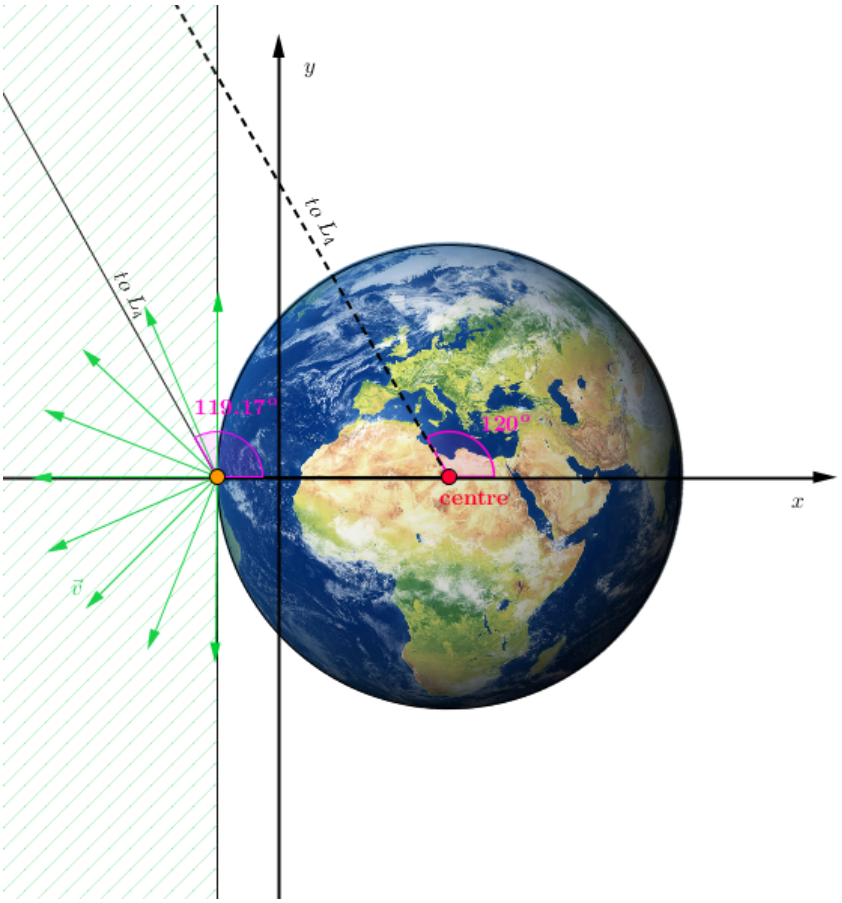

Figure 5.2: Initial setup illustrating the different launching vectors from the point making 180^o with the horizontal.

Many comments can be made from figure 5.2. In the first term, I am going to identify the different symbols and objects:

a) The initial position of the satellite is represented by •.

b) The valid region for firing off the third body is delimited by the green-lined section.

c) The different launches are labelled as →.

d) - - - represents the straight-line trajectory that the satellite would follow if it would have infinite velocity from the centre of Earth.

e) — is the trajectory that would define the satellite if it had infinite velocity form the launching point.

Analysing the image, it is clear that launching our satellite with infinite velocity, it would directly go to L_4 in a straight line, thus the launching angle would be 120^o with respect to the horizontal. Now, talking about our initial position, as it is a little bit at the left from the centre of Earth, if we launch the third body with infinite velocity, the angle will be a little bit lower. Calculating it by trigonometry, I have found that it would have the value of 119.17^o rounding it up to two decimal places.

With this initial concepts clear, I had to execute the programs explained in the previous section. Before that, I needed to calculate the lower-limiting initial velocity (in modulus) that makes our satellite go through the triangular point L_4. Taking into consideration that the launching point needs to be at the surface of Earth and that the distance from any point of the circumference to the centre of Earth is 0.01657 in our scaled units, giving the angle with the horizontal at which we fire off our satellite to the program, it calculates the third body's x and y initial positions. In order to calculate the minimum initial velocity I have made several trials executing the program and then plotting the data with *GNUplot*. Rounding the number up to four significant figures, I have set the modulus of the minimum launching initial velocity to $v_0 = 2.850$ (in our scaled units). As there are 99 more different velocities and $v_i = 0.1 + v_{i-1}$, $v_{99} = 12.75$.

Furthermore, I considered important to plot the angle of launching with respect to the velocity in order to show that, as calculated theoretically, as the modulus of the initial velocity is larger, the angle of launching decreases until it reaches a minimum angle of 119.17 degrees when the velocity tends to infinity (see Figure 5.3).

The purple curve in Figure 5.3 is the plotting of the data obtained from my programs. As it can be observed, the greater the initial velocity, the lower the launching angle. This makes sense, because the faster the satellite is fired off, the less the gravitational forces will affect to it. If the initial velocity is infinity, the influence of the gravitational forces exerted by the two primaries on the satellite is null. The dark blue line represents the minimum angle at which we would fire off the satellite supposing that it would have infinite velocity(119.17 degrees). The orange and the light blue lines delimit the region where theoretically we would fire off the satellite without crossing the Earth. This section, as I said before, is enclosed for the values $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, and is represented by the green-lined region.

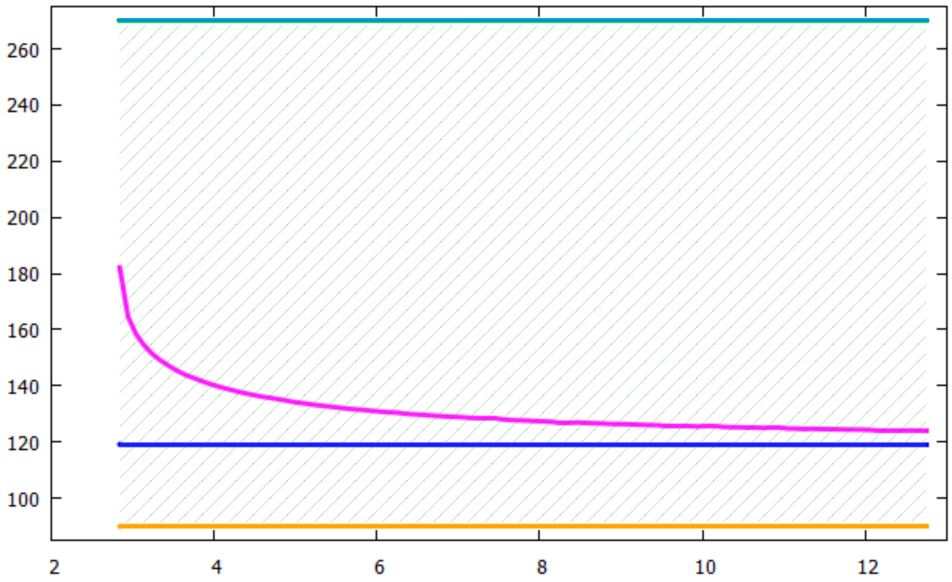

Figure 5.3: Angle of launching with respect to the modulus of the initial velocity of the satellite at P_1.

Finally, with the data extracted from my second program, I made the plot portrayed at Figure 5.4, which shows the solution of my paper at P_1, i.e. it depicts the region where a third body can live if we throw it from the mentioned point and want it to pass through L_4:

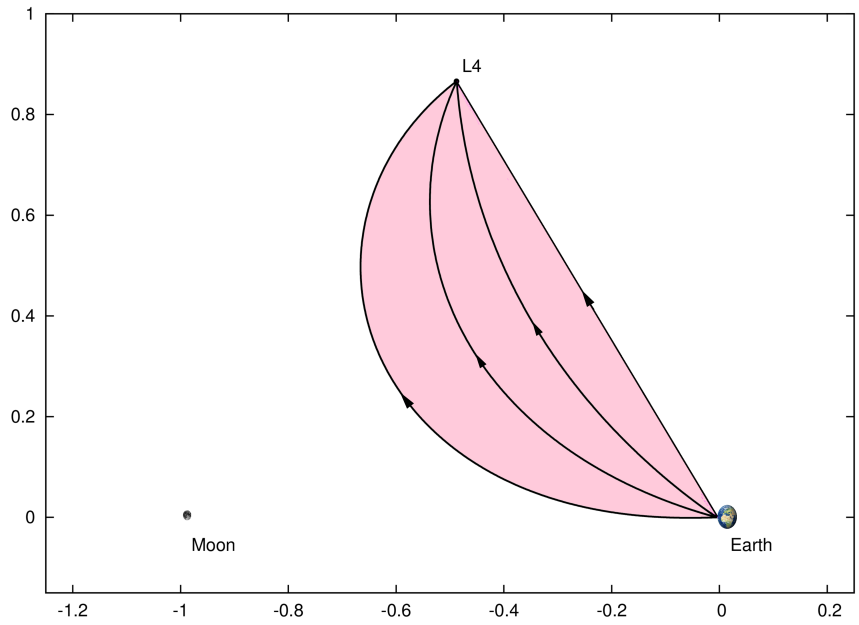

Figure 5.4: Solution at the point P_1.

The solution region is shaped in pink at Figure 5.4. From left to right, the vectors at this

Figure are the approximation solutions of the system of equations (3.4) of the trajectories of the satellite for the first, second, seventh and last iterations of my second program.

5.4.3 P_2: 135 degrees with the horizontal

I decided to find a solution for my research question also for the launching point P_2, which makes 135^o with the horizontal and is also located at the surface of our planet. As in the first simulation, we are considering the Earth as a planet with its own volume, so it is not possible to cross it as we cannot cross a rock neither. Similarly to the first case, the solutions that are enclosed in a range of 180 degrees are valid for the launching of our satellite, as it can be seen in Figure 5.5. The interval valid for the angle θ in this case is $\theta \in [\frac{\pi}{4}, \frac{5\pi}{4}]$.

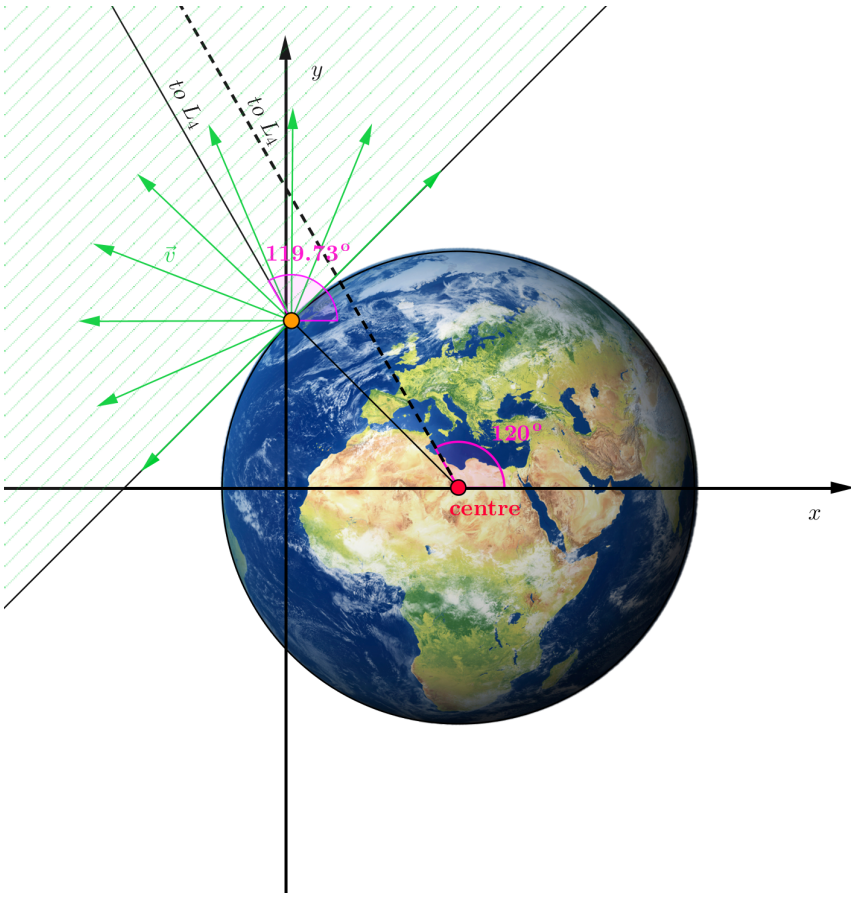

Figure 5.5: Initial setup for the launching of our satellite at point P_2.

The nomenclature of Figure 5.5 is exactly the same as the one used in Figure 5.2. In this case, the minimum angle from which we would launch the satellite if it had an infinite velocity would be 119.73^o rounding the value to two decimal places.

In order to execute my programs, I needed to introduce the angle with respect to the horizontal so that it could calculate the initial coordinates of P_2, and also the modulus of

the initial velocity. Following the same procedure as the explained in the previous section and rounding the value up to four significant figures, I set it as v_0 =[FALTA], as always in the scaled units.

With the data gathered after executing my programs, I have depicted the launching angle (which is the one that makes the distance between L_4 and my approximation solution minimum for that particular velocity), with respect to the modulus of the initial velocity (see Figure 5.6):

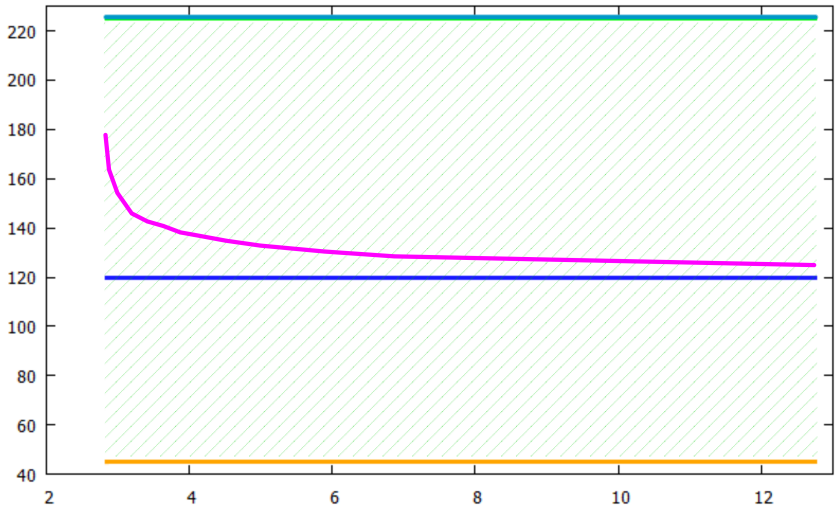

Figure 5.6: Angle of launching with respect to the modulus of the initial velocity of the satellite at P_2.

It can bee seen from Figure 5.6 that, as I said before, the theoretical range of angles where we could fire off the satellite (delimited by the orange and the light blue lines and represented by the thin green lines) is given by $\theta \in [\frac{\pi}{4}, \frac{5\pi}{4}]$. Nevertheless, with the simulation we can observe that practically, there's an asymptote at $\theta = 119.73^o$ when $\lim_{v\to\infty}$. This is due to the fact that the fastest way in order to arrive from one point to another in space is in a straight line. If we want our satellite to follow a straight-line trajectory it means that the gravitational forces exerted by the primaries on it would have no effect, i.e. we will have to launch it with infinite velocity.

5.4.4 P_3: 225 degrees with the horizontal

The third and last simulation that I did was to find the solution of my research question firing the satellite off from P_3, which is a point forming an angle of 225 degrees with the horizontal located at the surface of Earth.

Imposing the condition that the satellite cannot cross the Earth as it would be as crossing a hard rock, in this third case we have a problem. If we launched the third body with infinite velocity, it would go in a straight line to L_4, i.e. the launching angle would be of 119.09 degrees, while the theoretical 180^o range of angles from which we can launch

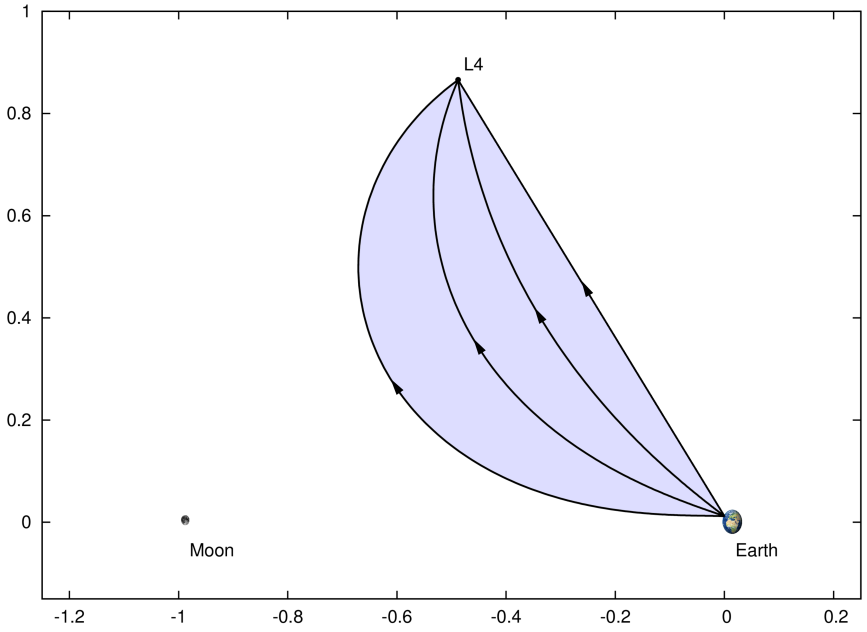

Figure 5.7: Solution at the point P_2.

the satellite without crossing Earth goes from $\theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$. This means that there will be some initial velocities whose trajectories will cross our planet before arriving to L_4, i.e. there will be a region that will not be valid for our solution.

The comment that I made in the last lines of the previous paragraph is clearly illustrated at Figure 5.8. The line that goes directly from P_3 to L_4 forms an angle of 119.09^o with the horizontal, i.e. if we threw the satellite with infinite velocity from that point and wanted it to pass through L_4, the launching angle would be 119.09^o. Nevertheless, this angle does not belong to the valid region of angles where the third body can be launched without crossing Earth. At Figure 5.8 this region is marked with thin green lines. This means that, when executing the program and plotting the results, we will have to calculate the velocity that separates the valid from the invalid region and make a difference between the valid and the non valid regions where the third body can live. The calculation for the mentioned velocity can be easily performed, as the limiting valid angle is known to be 135 degrees.

As in the previous simulations, I considered it would be interesting to plot the variation of the launching angle with respect to the velocity at which the satellite is launched. The difference between Figure 5.9 and Figures 5.3 and 5.6 is that the first one is divided in two regions representing a valid angle of launching (green-lined region) and a non-valid one (illustrated with thin red lines); while the other Figures only have a valid zone of launching angles in their respective 180^o range of possible firing off angles from which the satellite does not cross Earth. It can be clearly seen from the Figure 5.9 that the minimum angle of launching (represented by the dark blue line) is bigger than the angle

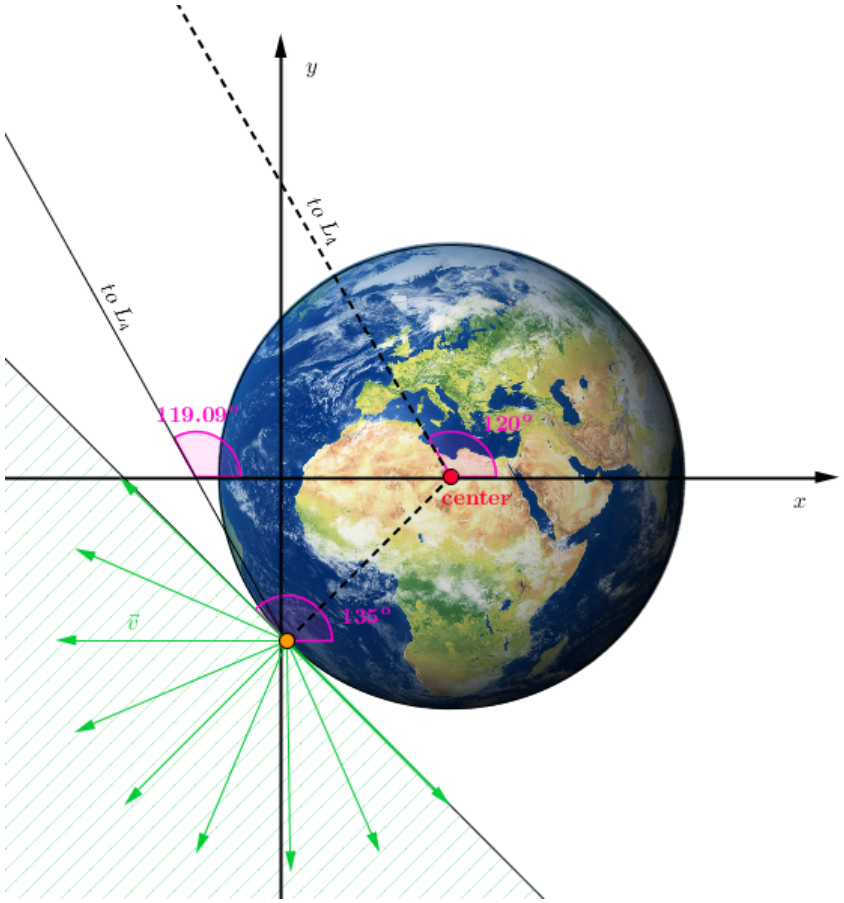

Figure 5.8: Basic geometric information for the launching of our satellite from P_3.

at which the satellite would be thrown directly to L_4 if it had infinite velocity (orange line). Thus, the region between the mentioned lines is not valid, as it would mean that the satellite crosses our planet during its trip to the triangular point. At the same time, the vertical line in the Figure separates the valid initial velocities, which are enclosed in the interval $FALTA \leq v_0 \leq FALTA2$ from the non-valid ones, which are encolsed in the interval $FALTA2 < v < +\infty$.

Finding the approximate solution of the system of equations (**??**), the valid region for the satellite is represented in pale green at Figure 5.10.

5.5 Comparing the results

In order to structure the work, I have created Table [5.2 so that the comparison between the data obtained from the C++ programs for the 3 simulations is easier to perform:

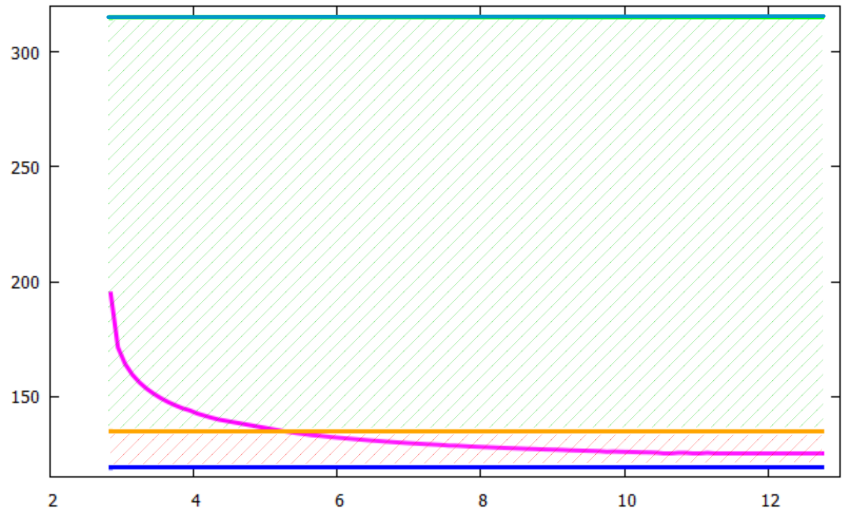

Figure 5.9: Bla bla bla.

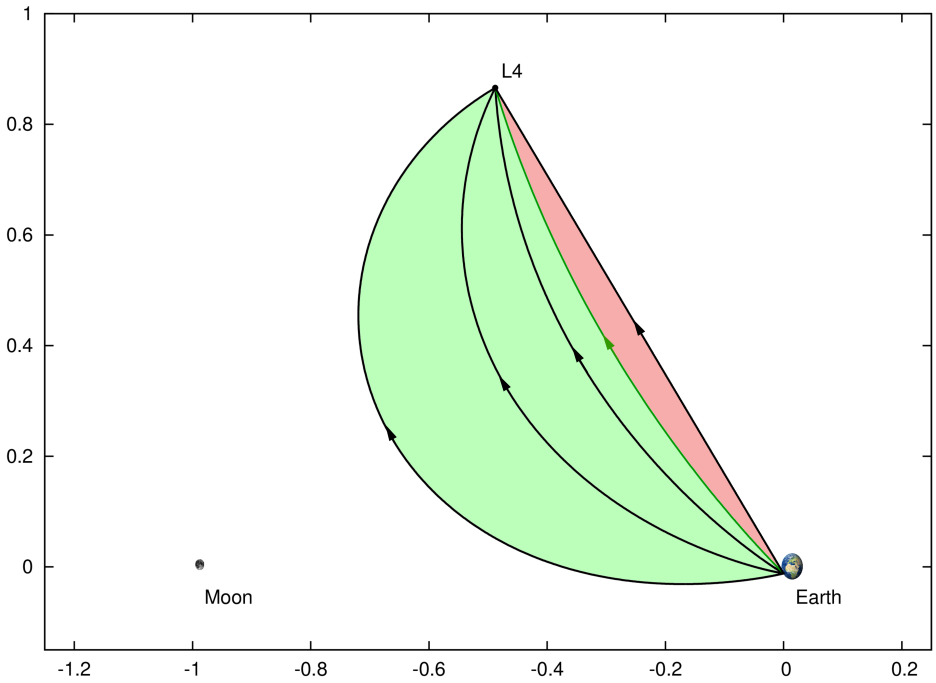

Figure 5.10: Solution at the point P_3.

5.6 Future research directions

In order to accomplish my objective, it has been necessary for me to explore and learn a lot of physical and in particular mathematical concepts. These last were the most difficult for me due to the fact that they are majorly abstract. What happened to me, was that the more I read books, the more possible ways to continue my work opened, so I had to do the tough decision of which route to take in a lot of moments.

	P_1	P_2	P_2
Coordinates	$(-0.00442, 0)$	$(0.000433, 0.0117)$	$(0.000433, -0.0117)$
Possible angles	$\theta \in [90^o, 270^o]$	$\theta \in [45^o, 225^o]$	$\theta \in [135^o, 315^o]$
Direct angle	119.17^o	119.73^o	119.09^o
Possible velocities	$v \in [2.850, \infty)$	$v \in [2.838, \infty)$	$v \in [2.854, 5.220]$

Table 5.2: My caption

A huge number of ways to continue my exploration are possible. What I would like to do next would be to focus on periodic orbits and study the stability of the Lagrangian points, particularly of the one that I have been working with, L_4. Then, I could study what varieties are and analyse the stable and unstable ones.

I could also continue by doing something more practical and applied to real life. For instance, it would be interesting to find the orbit from Earth to L_4 such that the energy consumed by our satellite would be minimal, and maybe model our system by introducing another variable parameter, the mass of the satellite, which would not be constant but would be every time smaller, simulating the real uncoupling of some parts of a satellite.

Appendices

Appendix A

Programs

A.1 Auxiliary functions

```
#include <iostream>
#include "math.h"
#include "stdlib.h"
#include "stdio.h"
#include <cmath>

using namespace std;

//Second derivative of function Omega(x)
double omegaxx(double mu, double x){
   double r1 = fabs(x-mu);
   double r2 = fabs(x-mu+1.);
   return 1.+(2.*(1-mu))/pow(r1,3)+(2.*mu)/pow(r2,3);
}

//First derivative of Omega(x)
double omegax(double mu, double x){
   double r1 = fabs(x-mu);
   double r2 = fabs(x-mu+1.);
   return x-((1.-mu)*(x-mu))/pow(r1,3)-((1.+x-mu)*mu)/pow(r2,3);
}

//Function Omega(x)
double omega(double mu, double x){
   double r1 = fabs(x-mu);
   double r2 = fabs(x-mu+1.);
   return 0.5*x*x + (1.-mu)/r1 + mu/r2 + 0.5*mu*(1.-mu);
```

```
28 }
29
30 //Function Omega(x,y)
31 double Omega(double mu, double x, double y){
32     double r1 = sqrt(pow(x-mu,2)+y*y);
33     double r2 = sqrt(pow(x-mu+1.,2)+y*y);
34     return 0.5*(x*x+y*y) + (1.-mu)/r1 + mu/r2 + 0.5*mu*(1.-mu);
35 }
36
37 // Sum of vectors
38 VEC suma(VEC a, VEC b){
39    int n = a.size();
40    VEC c(n);
41    for (int i = 0; i < n; ++i){
42       c[i] = a[i]+b[i];
43    }
44    return c;
45 }
46
47 //Multiplication of between a constant and a vector
48 VEC multiplica(double c, VEC v){
49    int n = v.size();
50    for (int i = 0; i < n; ++i){
51       v[i] = c*v[i];
52    }
53    return v;
54 }
55
56 //Multiplication between square matrices and a vector
57 VEC multiplicaMV(MAT A, VEC b){
58    int n = b.size();
59    VEC c(n);
60    for(int i =0; i < n; ++i){
61       c[i] = 0;
62       for(int j = 0; j < n; ++j){
63          c[i] = c[i]+A[i][j]*b[j];
64       }
65    }
66    return c;
67 }
68
69 //System function
70 VEC function(VEC y){ // y' = f(y)
71    int n = y.size();
72    VEC f(n);
```

```
73      f[0] = y[1];
74      f[1] = -sin(y[0]);
75      return f;
76  }
77
78  MAT Dfunction(VEC y){
79      int n = y.size();
80      MAT Df(n, VEC(n));
81      Df[0][0] = 0;
82      Df[0][1] = 1;
83      Df[1][0] = -cos(y[0]);
84      Df[1][1] = 0;
85      return Df;
86  }
87
88  // Euclidean distance between two points
89  double norm(VEC a, VEC b){
90      int n = a.size();
91      double norma = 0;
92      for (int i = 0; i < n; ++i){
93          norma = norma + pow(a[i]-b[i],2);
94      }
95      return pow(norma,0.5);
96  }
```

A.2 Newton's Method for the collinear equilibrium points (L_1, L_2, L_3)

```
1   #include <iostream>
2   #include "math.h"
3   #include "stdlib.h"
4   #include "stdio.h"
5   #include <cmath>
6   FILE *fitxer;
7
8   using namespace std;
9
10
11  double omegaxx(double mu, double x);
12
13  double omegax(double mu, double x);
14
```

```
//Newton's Method to find the zeros of Omegax
double Newton(double mu, double x0){
   double x1 = x0 - omegax(mu,x0)/omegaxx(mu,x0);
   while(fabs(x1-x0)>1.e-15){
      x0 = x1;
      x1 = x0 - omegax(mu,x0)/omegaxx(mu,x0);
   }
   return x1;
}


int main(){
   fitxer = fopen("L1L2L3.dat","w");
   for(double mu = 0; mu <= 0.5; mu = mu + 0.000001){
      fprintf(fitxer,"%12.15e %12.15e %12.15e %12.15e %12.15e\n",
      mu, Newton(mu,mu-0.1),Newton(mu,mu-1.1),Newton(mu,mu+0.1),mu-1);
   }
   fclose(fitxer);
}
```

A.3 Hill Region

```
#include "stdio.h"
#include <cmath>
#include <vector>
#include <string.h>
#include <sstream>
#include <string>
FILE *fitxer;

using namespace std;

typedef vector<vector<double> > MAT;
typedef vector<double> VEC;


double Omega(double mu, double x, double y);


int main(){
   fitxer = fopen("hill.dat","w");
   for (double i = 0; i <= 4000; ++i){
      double x = -2.+i/1000.;
```

```
22          for (double j = 0; j <= 4000; ++j){
23             double y = -2.+j/1000.;
24             if (pow(Omega(0.1, x, y)-1.55,2)<0.000001){
25                fprintf(fitxer,"%12.15e %12.15e  \n", x, y);
26             }
27          }
28       }
29       for (double i = 0; i <= 80; ++i){
30          for (double j = 0; j <= 4000; ++j){
31             double x = -6.+i/10.+j/1000.;
32             double y = -2.+j/1000.;
33             if (Omega(0.1, x, y)-1.55 <0){
34                fprintf(fitxer,"%12.15e %12.15e  \n", x, y);
35             }
36          }
37       }
38       fclose(fitxer);
39    }
```

A.4 Euler method

```
1  #include <iostream>
2  #include "math.h"
3  #include "stdlib.h"
4  #include "stdio.h"
5  #include <cmath>
6  #include <vector>
7  FILE *fitxer;
8
9  using namespace std;
10
11 typedef vector<vector<double> > MAT;
12 typedef vector<double> VEC;
13
14
15 VEC suma(VEC a, VEC b);
16
17 VEC multiplica(double c, VEC v);
18
19 VEC function(VEC y);
20
21 double norm(VEC a, VEC b);
22
```

```
23
24 VEC euler(double h, VEC y){
25    int n = y.size();
26    VEC f(n);
27    f = function(y);
28    return suma(y,multiplica(h,f));
29 }
30
31
32 int main(){
33    int n = 2;
34    double h = 0.1;
35    double t0 = 0;
36    double tf = 5;
37    double TOL = 0.001;
38    VEC Y0(n);
39    Y0[0] = 1;
40    Y0[1] = 0;
41    fitxer = fopen("euler.dat","w");
42    fprintf(fitxer,"%12.15e %12.15e \n", Y0[0], Y0[1]);
43    double hnew = h;
44    while (t0 < tf){
45       VEC Y(n);
46       VEC Y1(n);
47       VEC Y2(n);
48       double error = 10;
49       while (error > TOL){
50          h = hnew;
51          Y = euler(h, Y0);
52          Y1 = euler(h/2, Y0);
53          Y2 = euler(h/2,Y1);
54          error = norm(Y,Y2);
55          hnew = 0.9*h*TOL/error; //Step-size control
56       }
57       if (t0 + h > tf){
58          Y = euler(tf-t0,Y0);
59          fprintf(fitxer,"%12.15e %12.15e \n", Y[0], Y[1]);
60          t0 = tf;
61       }
62       else{
63          t0 = t0+h;
64          //fprintf(fitxer,"%12.15e %12.15e \n", Y1[0], Y1[1]);
65          fprintf(fitxer,"%12.15e %12.15e \n", Y2[0], Y2[1]);
66          Y0 = Y2;
67       }
```

```
68      }
69      fclose(fitxer);
70  }
```

A.5 Taylor Second Order method

```
1   #include <iostream>
2   #include "math.h"
3   #include "stdlib.h"
4   #include "stdio.h"
5   #include <cmath>
6   #include <vector>
7   FILE *fitxer;
8
9   using namespace std;
10
11  typedef vector<vector<double> > MAT;
12  typedef vector<double> VEC;
13
14  VEC suma(VEC a, VEC b);
15
16  VEC multiplica(double c, VEC v);
17
18  VEC multiplicaMV(MAT A, VEC b);
19
20  VEC function(VEC y);
21
22  MAT Dfunction(VEC y);
23
24  double norm(VEC a, VEC b);
25
26  VEC taylor(double h, VEC y){
27      int n = y.size();
28      VEC f(n);
29      MAT Df(n, VEC(n));
30      f = function(y);
31      Df = Dfunction(y);
32      return suma(suma(y,multiplica(h,f)),multiplica((h*h/2),
33      multiplicaMV(Df,f)));
34  }
35
36
37
```

```
38 int main(){
39    int n = 2;
40    double h = 0.1;
41    double hmax = 0.3;
42    double t0 = 0;
43    double tf = 5;
44    double TOL = 0.001;
45    VEC Y0(n);
46    Y0[0] = 1;
47    Y0[1] = 0;
48    fitxer = fopen("resultattaylor.dat","w");
49    double hnew = h;
50    while (t0 < tf){
51       VEC Y(n);
52       VEC Y1(n);
53       VEC Y2(n);
54       double error = 10;
55       while (error > TOL){
56          h = hnew;
57          Y = taylor(h, Y0);
58          Y1 = taylor(h/2, Y0);
59          Y2 = taylor(h/2,Y1);
60          error = norm(Y,Y2);
61          hnew = 0.9*h*TOL/error;
62       }
63       if (t0 + h > tf){
64          Y = taylor(tf-t0,Y0);
65          fprintf(fitxer,"%12.15e %12.15e \n", Y[0], Y[1]);
66          t0 = tf;
67       }
68       else{
69          t0 = t0+h;
70          //fprintf(fitxer,"%12.15e %12.15e \n", Y1[0], Y1[1]);
71          fprintf(fitxer,"%12.15e %12.15e \n", Y2[0], Y2[1]);
72          for (int i = 0; i < n; ++i){
73             Y0[i] = Y2[i];
74          }
75       }
76       if (hnew > hmax) hnew = hmax;
77    }
78    fclose(fitxer);
79 }
```

A.6 Adams-Bashforth three-step method

```
#include <iostream>
#include "math.h"
#include "stdlib.h"
#include "stdio.h"
#include <cmath>
#include <vector>
FILE *fitxer;

using namespace std;

typedef vector<vector<double> > MAT;
typedef vector<double> VEC;

VEC suma(VEC a, VEC b);

VEC multiplica(double c, VEC v);

VEC multiplicaMV(MAT A, VEC b);

VEC function(VEC y);

MAT Dfunction(VEC y);

VEC taylor(double h, VEC y);


VEC AdamBashforth3(double h, VEC y1, VEC y2, VEC y3){
   int n = y1.size();
   VEC f1(n);
   VEC f2(n);
   VEC f3(n);
   f1 = function(y1);
   f2 = function(y2);
   f3 = function(y3);
   return suma(y3,multiplica(h/12,suma(multiplica(23,f3),
   suma(multiplica(-16,f2),multiplica(5,f1)))));
}


int main(){
   int n = 2;
   double h = 0.2;
   double t0 = 0;
```

```
    double tf = 7;
    //double ERR = 0.1;
    VEC Y0(n);
    Y0[0] = 1;
    Y0[1] = 0;
    VEC Y1(n);
    VEC Y2(n);
    fitxer = fopen("adams.dat","w");
    Y1 = taylor(h, Y0);
    Y2 = taylor(h, Y1);
    fprintf(fitxer,"%12.15e %12.15e \n", Y0[0], Y0[1]);
    fprintf(fitxer,"%12.15e %12.15e \n", Y1[0], Y1[1]);
    fprintf(fitxer,"%12.15e %12.15e \n", Y2[0], Y2[1]);
    while (t0 < tf){
       VEC Y3(n);
       Y3 = AdamBashforth3(h, Y0, Y1, Y2);
       fprintf(fitxer,"%12.15e %12.15e \n", Y3[0], Y3[1]);
       Y0 = Y1;
       Y1 = Y2;
       Y2 = Y3;
       t0 = t0+h;
    }
    fclose(fitxer);
}
```

A.7 Dormand & Prince method

```
#include <iostream>
#include "math.h"
#include "stdlib.h"
#include "stdio.h"
#include <cmath>
#include <vector>
FILE *fitxer;

using namespace std;

typedef vector<vector<double> > MAT;
typedef vector<double> VEC;

VEC suma(VEC a, VEC b);

VEC multiplica(double c, VEC v);
```

```
17
18 VEC multiplicaMV(MAT A, VEC b);
19
20 VEC function(VEC y);
21
22 double norm(VEC a, VEC b);
23
24
25 MAT RK45(double h, VEC y){
26    int n = y.size();
27    VEC k1(n),k2(n),k3(n),k4(n),k5(n),k6(n),k7(n);
28    VEC aux(n),Y4(n),Y5(n);
29    MAT YY(2, VEC(n));
30
31    k1 = function(y);
32    for (int i=0; i < n; ++i){
33       aux[i] = y[i] + (1./5.)*h*k1[i];
34    }
35    k2 = function(aux);
36    for (int i=0; i < n; ++i){
37       aux[i] = y[i] + ((3./40.)*k1[i] + (9./40.)*k2[i])*h;
38    }
39    k3 = function(aux);
40    for (int i=0; i < n; ++i){
41       aux[i] = y[i] + ((44./45.)*k1[i] - (56./15.)*k2[i]
42       + (32./9.)*k3[i])*h;
43    }
44    k4 = function(aux);
45    for (int i=0; i < n; ++i){
46       aux[i] = y[i] + ((19372./6561.)*k1[i] - (25360./2187.)*k2[i]
47       + (64448./6561.)*k3[i] - (212./729.)*k4[i])*h;
48    }
49    k5 = function(aux);
50    for (int i=0; i < n; ++i){
51       aux[i] = y[i] + ((9017./3168.)*k1[i] - (355./33.)*k2[i]
52       + (46732./5247.)*k3[i] + (49./176.)*k4[i] - (5103./18656.)*k5[i])*h;
53    }
54    k6 = function(aux);
55    for (int i=0; i < n; ++i){
56       aux[i] = y[i] + ((35./384.)*k1[i] + (500./1113.)*k3[i]
57       + (125./192.)*k4[i] - (2187./6784.)*k5[i] + (11./84.)*k6[i])*h;
58    }
59    YY[0] = aux;
60    k7 = function(aux);
61
```

```
62      for (int i=0; i < n; ++i){
63         YY[1][i] = y[i] + ((5179./57600.)*k1[i] + (7571./16695.)*k3[i]
64         + (393./640.)*k4[i] - (92097./339200.)*k5[i] + (187./2100.)*k6[i]
65         + (1./40.)*k7[i])*h;
66      }
67
68      return YY;
69   }
70
71
72
73   int main(){
74      int n = 2;
75      double h = 0.1;
76      double hmax = 10;
77      double t0 = 0;
78      double tf = 7;
79      double TOL = 0.001;
80      VEC Y0(n);
81      Y0[0] = 1;
82      Y0[1] = 0;
83      fitxer = fopen("rk45pendul.dat","w");
84      fprintf(fitxer,"%12.15e %12.15e \n", Y0[0], Y0[1]);
85      double hnew = h;
86      while (t0 < tf){
87         MAT Y(2, VEC(n));
88         double error = 10; //forcem q entri dins del while
89         while (error > TOL){
90            h = hnew;
91            if (h > hmax) h = hmax;
92            Y = RK45(h, Y0);
93            error = norm(Y[0],Y[1]);
94            hnew = 0.9*h*pow(TOL/error,1./6.);
95         }
96         if (t0 + h > tf){
97            Y = RK45(tf-t0,Y0);
98            fprintf(fitxer,"%12.15e %12.15e \n", Y[0][0], Y[0][1]);
99            t0 = tf;
100        }
101        else{
102           t0 = t0+h;
103           fprintf(fitxer,"%12.15e %12.15e \n", Y[0][0], Y[0][1]);
104           Y0 = Y[1];
105        }
106     }
```

```
107     fclose(fitxer);
108 }
```

A.8 Launching angles

```
1  #include <iostream>
2  #include "math.h"
3  #include "stdlib.h"
4  #include "stdio.h"
5  #include <cmath>
6  #include <vector>
7  #include <string.h>
8  #include <sstream>
9  #include <string>
10 FILE *fitxer;
11
12 using namespace std;
13
14 typedef vector<vector<double> > MAT;
15 typedef vector<double> VEC;
16
17
18 VEC suma(VEC a, VEC b);
19
20 VEC multiplica(double c, VEC v);
21
22 VEC multiplicaMV(MAT A, VEC b);
23
24 VEC function(VEC y){ // y' = f(y)
25     double mu = 0.01215;
26     int n = y.size();
27     VEC f(n);
28     f[0] = y[2];
29     f[1] = y[3];
30     f[2] = 2.*y[3] - ((1.-mu)*(y[0]-mu))/pow(pow(mu-y[0],2)+pow(y[1],2),3/2)
31     + mu*(mu-y[0]-1.)/pow(pow(-mu+y[0]+1.,2)+pow(y[1],2),3/2) + y[0];
32     f[3] = -2.*y[2] + (mu-1.)*y[1]/pow(pow(mu-y[0],2)+y[1]*y[1],3/2)
33     - mu*y[1]/pow(pow(-mu+y[0]+1.,2)+y[1]*y[1],3/2)+y[1];
34     return f;
35 }
36
37 double norm(VEC a, VEC b);
38
```

```
39 MAT RK45(double h, VEC y);
40
41
42
43 int main(){
44    //Data:
45    fitxer = fopen("entrada.dat","r");
46    double initial_time, final_time, angle_sortida, vel_inicial;
47    fscanf(fitxer, "%lf %lf %lf %lf\n", &inital_time, &final_time,
48    &angle_sortida, &vel_inicial);
49    fclose(fitxer);
50
51    //L4:
52    const VEC L4(n);
53    L4[0]= 0.01215-0.5;
54    L4[1]= sqrt(3.)/2.;
55    L4[2]= 0.;
56    L4[3]= 0.;
57
58    fitxer = fopen("angle3.dat","w");
59    double distancia_max = 0.;
60    for (double j; j < 100; ++j){
61       double anglebo = -1.;
62       double minimdistL4= 1000.;
63       double modulvel = vel_inicial+j/10.;
64       double angle_inf = min(atan((L4[1]-0.01657*sin(angle_sortida))/(L4[0]
65       -(0.01215+0.01657*cos(angle_sortida)))), angle_sortida-M_PI/2.);
66       double angle_sup = angle_sortida+M_PI/2.;
67       double temps_final=0.;
68       double num = -1;
69       for (double k = 0; k < 10; ++k){
70          double sec = angle_sup - angle_inf;
71          for (double l = 0; l < 21; ++l){
72             double angle = angle_inf + sec/20.*l;
73
74             int n = 4;
75             double h = 0.001;
76             double hmax = 0.01;
77             double t0 = initial_time;
78             double tf = final_time;
79             double TOL = 0.00000000000001;
80
81             VEC Y0(n);
82             Y0[0] = 0.01215+0.01657*cos(angle_sortida);
83             Y0[1] = 0.01657*sin(angle_sortida);
```

```
84              Y0[2] = modulvel*cos(angle);
85              Y0[3] = modulvel*sin(angle);
86
87              double hnew = h;
88              while (t0 < tf){
89                 MAT Y(2, VEC(n));
90                 double error = 10; //forcem q entri dins del while
91                 while (error > TOL){
92                    h = hnew;
93                    if (h > hmax) h = hmax;
94                    Y = RK45(h, Y0);
95                    error = norm(Y[0],Y[1]);
96                    hnew = 0.9*h*pow(TOL/error,1./6.);
97                 }
98                 if (t0 + h > tf){
99                    Y = RK45(tf-t0,Y0);
100                   double aux = pow(pow(Y[0][0]-L4[0],2)
101                   +pow(Y[0][1]-L4[1],2),0.5);
102                   if (minimdistL4 > aux){
103                      minimdistL4 = aux;
104                      anglebo = angle;
105                      num = l;
106                      temps_final = tf;
107                   }
108                   Y0 = Y[0];
109                   t0 = tf;
110                }
111                else{
112                   t0 = t0+h;
113                   double aux = pow(pow(Y[0][0]-L4[0],2)
114                   +pow(Y[0][1]-L4[1],2),0.5);
115                   if (minimdistL4 > aux){
116                      minimdistL4 = aux;
117                      anglebo = angle;
118                      num = l;
119                      temps_final = t0;
120                   }
121                   Y0 = Y[0];
122                }
123                if (hnew > hmax) hnew = hmax;
124             }
125          }
126          angle_inf = angle_inf + sec/20.*(num-1);
127          angle_sup = angle_inf + sec/20.*(num+1);
128          if (angle_inf<1.*M_PI/4.) angle_inf=1.*M_PI/4.;
```

```
129             if (angle_sup>1.*M_PI/4.) angle_sup=5.*M_PI/4.;
130         }
131         //cout <<modulvel << " "<< anglebo << " "<< minimdistL4 << endl;
132         fprintf(fitxer,"%12.15e %12.15e  %12.15e\n", modulvel, anglebo,
133         temps_final);
134         if (distancia_max < minimdistL4) distancia_max = minimdistL4;
135     }
136     fclose(fitxer);
137     cout << distancia_max << endl;
138 }
```

A.9 PVI angles

```
1  #include <iostream>
2  #include "math.h"
3  #include "stdlib.h"
4  #include "stdio.h"
5  #include <cmath>
6  #include <vector>
7  #include <string.h>
8  #include <sstream>
9  #include <string>
10 FILE *fitxer;
11
12 using namespace std;
13
14 typedef vector<vector<double> > MAT;
15 typedef vector<double> VEC;
16
17 VEC suma(VEC a, VEC b);
18
19 VEC multiplica(double c, VEC v);
20
21 VEC multiplicaMV(MAT A, VEC b);
22
23 VEC function(VEC y);
24
25 double norm(VEC a, VEC b);
26
27 MAT RK45(double h, VEC y);
28
29
30
```

```
31 int main(){
32    fitxer = fopen("data.dat","r");
33    VEC modulvel(100);
34    VEC anglebo(100);
35    VEC tf(100);
36    for (int i=0; i < 100; ++i){
37       fscanf(fitxer, "%lf %lf %lf\n", &modulvel[i],&anglebo[i],&tf[i]);
38    }
39    fclose(fitxer);
40    for (int i=0; i < 100; ++i){
41       stringstream ss;
42       ss << "2difsolucions" << i << ".dat";
43       string s = ss.str();
44       char *cstr = &s[0u];
45       fitxer = fopen(cstr,"w");
46
47       int n = 4;
48       double h = 0.001;
49       double hmax = 0.01;
50       double t0 = 0;
51       double TOL = 0.00000000000001;
52
53       VEC Y0(n);
54       Y0[0] = 0.01215+0.01657*cos(5.*M_PI/4.);
55       Y0[1] = 0.01657*sin(5.*M_PI/4.);
56       Y0[2] = modulvel[i]*cos(anglebo[i]);
57       Y0[3] = modulvel[i]*sin(anglebo[i]);
58
59       fprintf(fitxer,"%12.15e %12.15e  \n", Y0[0], Y0[1]);
60       double hnew = h;
61       while (t0 < tf[i]){
62          MAT Y(2, VEC(n));
63          double error = 10; //forcem q entri dins del while
64          while (error > TOL){
65             h = hnew;
66             if (h > hmax) h = hmax;
67             Y = RK45(h, Y0);
68             error = norm(Y[0],Y[1]);
69             hnew = 0.9*h*pow(TOL/error,1./6.);
70          }
71          if (t0 + h > tf[i]){
72             Y = RK45(tf[i]-t0,Y0);
73             fprintf(fitxer,"%12.15e %12.15e  \n", Y[0][0], Y[0][1]);
74             Y0 = Y[0];
75             t0 = tf[i];
```

```
76            }
77            else{
78               t0 = t0+h;
79               fprintf(fitxer,"%12.15e %12.15e  \n", Y[0][0], Y[0][1]);
80               Y0 = Y[0];
81            }
82            if (hnew > hmax) hnew = hmax;
83         }
84         fclose(fitxer);
85      }
86   }
```

Appendix B

Time-lined Contributions to the 3BP

Tycho-Brahé (1546-1601)

Was a Danish nobleman who observed stellar and planetary positions noteworthy both for their accuracy and quantity. He also developed the *Tychonic geo-heliocentric system*, in which the Sun and the Moon orbit around Earth, while the other planets orbit the Sun.

Johannes Kepler (1571-1630)

Considered a key figure of the 17^{th} Century, the German mathematician, astronomer and astrologer was the first to formulate the three laws that describe planetary motion. He proposed a problem which results to be a special case of the two-body problem, named in his honour the *Kepler Problem*, and since then, scientists endeavoured to solve for the equation of motion of the planets.

Isaac Newton (1642-1727)

He is the well-known physicist and mathematician that described the three laws of motion and the law of universal gravitation (also known as the law of *universal attraction*), which was accurate enough to lead to the discovery of unknown planets such as Neptune and Pluto. Moreover, he showed that Kepler's laws of planetary motion correspond to the 2BP, and lead to the modern form of the 3BP.

Leonhard Euler (1707 – 1783)

This important Swiss mathematician and physicist attempted to solve for the motion of a particle that is acted upon by the gravitational field of two other point masses that are fixed in space; also known as *Euler's Three-body problem.* Further on, he raised a deeper question and wondered if the small contributions from the gravitational interactions of all the planets make the planetary system unstable over long periods of time. A lot of mathematicians examined this question and tried to give an answer to it, some of which are mentioned below in this list.

Alexis Clairaut (1713 – 1765)

Born in France, in 1747 he worked out a series approximation to the 3BP. After winning the St. Petersburg Academy prize for his work on the problem, in 1759 the

value of his approximations was amply demonstrated when Halley's comet passed Earth within a month of what his equations had predicted, the margin of error he himself had prescribed.

Joseph-Louis Lagrange (1736 – 1813)
Attempting to solve the general 3BP, this Italian mathematician and astronomer managed to discover the five special-case solutions of the restricted 3BP, known as the *Lagrangian points*. As we know, the libration points are positions in which a third body of negligible mass can remain in a stable position with respect to the primaries.

Pierre-Simon Laplace (1749 – 1827)
This influential French scholar made an important contribution to celestial mechanics using lagrangian conceptions to better explain the dynamics of bodies. He spent a good time of his life working in mathematical astronomy, and his work culminated with the verification of the dynamical stability of the Solar System.

Carl Gustav Jacob Jacobi (1804 – 1851)
Was a German mathematician who worked on planetary theory and other particular dynamical problems that occupied his attention from time to time. While working in celestial mechanics, he introduced the *Jacobi integral*, which is the only known conserved quantity for the CR3BP (Circular restricted three-body problem).

William Rowan Hamilton (1805 – 1865)
Born in Dublin, he used Delaunay's work in order to develop the equations of motion in Hamiltonian form. *Hamiltonian's equations* derive from *Hamiltonian's mechanics*, and this equations can be used to solve the R3BP system over time.

Joseph Liouville (1809 – 1882)
The French mathematician created the known *Liouville dynamical system*. If we treat the 3BP as a Liouville dynamical system, its exact solution can be expressed in terms of elliptic integrals.

Urbain Jean Joseph Le Verrier (1811 – 1877)
He was a French mathematician that is best known for predicting the existence and position of Neptune using only mathematics. He also worked on planetary perturbations. Talking about the Sun-Mercury system, he noted that the orbital precession (i.e. the change of orientation of the rotational axis) of Mercury was faster than the predicted by the Newtonian theory.

Charles-Eugène Delaunay (1816 – 1872)
Born in France, he made a very important theoretical work related to the 3BP. His contribution concerned a huge number of calculations, and consisted on taking g repeated canonical transformations of the problem, known as Delaunay's methods.

John Couch Adams (1819 – 1892)
The British mathematician predicted the existence and position of Neptune. He also developed the well-known *Adams methods*, which are multistep numerical integration methods for differential equations.

Simon Newcomb (1835 – 1909)

This Canadian mathematician worked on the research for trigonometric series solutions for the 3BP. proved that the differential equations describing the motion of the planets could formally be satisfied by trigonometric series.

George William Hill (1838 – 1914)

Born in the US, he focused on the mathematics describing the 3BP. Particularly, he discovered the *Hill region*, which is a very useful application of Jacobi's integral, describing the regions of possible motion for the body of negligible mass.

Jean Gaston Darboux (1842 – 1917)

The French mathematician made a particular case of the 2BP, known as *Darboux problem*. He also did some research in potential applications, which are intermediate steps in order to solve the 3BP.

Jules Henri Poincaré (1854 – 1912)

The French genius has been one of the most important contributors to the 3BP. When Oscar II of Sweden celebrated his 60th birthday in 1887, he established a price for anyone who could find the solution to the n-body problem but, if finding its solution was not possible, the price would be given to the most important contribution. Henri Poincaré won this price despite the fact of not having found the exact solution of the problem. Karl Weierstrass, one of the judges, said that "[Poincarre's] *publication is of such importance that it will inaugurate a new era in the history of celestial mechanics*".

Edmund Taylor Whittaker (1873 – 1956)

In 1898 he was asked ti guve a report on the current state of planetary theory by the British Association for the Advancement of Science. In his work, he discussed about the 3BP and gave a concise and clear description of many ideas discussed by other mathematicians such as invariant integrals, stability, periodic and asymptotic solutions to the 3BP...

Karl Frithiof Sundman (1873 – 1949)

He was born in Finland, and proved that there existed a convergent infinite series solution to the 3BP by means of analytic methods. Further on, Q. Wang would generalise Sundman's solution to the $n-$body problem.

George David Birkhoff (1884 – 1944)

Born in America, he attempted to solve the four color problem; and in 1913 he proved Poincaré's special case of the 3BP known as the *Last Geometric Theorem*.

Conclusion

Arrived at this point, in the first term I will focus on my work itself and make several observations about my objective and the results that I have obtained. Secondly I will make a few comments about my impressions towards my research project including a little bit of reflexion.

Firstly we will verify if the objectives stated at the introduction have been accomplished:

Talking about the first purpose, a specific chapter has been designed in order to explain briefly the topics that I have reviewed by myself and were not part of the school syllabus. Moreover, when a definition or a proposition was not clear or trivial for me, I developed a lemma including the mentioned idea and its proof.

Secondly, an analysis about the areas of mathematics relative to the calculus bloc and the links between them are going to be made. Integral calculus, differential equations, and numerical calculus are the three main areas that come to my mind which I have studied accurately throughout my work. The system of differential equations of the R3BP cannot be solved exactly (analytically), so we need to find its solution approximately (numerically). There are several numerical methods used to solve this equations. This methods solve differential equations by integrating them and finding approximate solutions, so a review of integral calculus must be made.

In the third place, the answer of my question consists on integrating the differential system of equations (3.4), which is an IVP, with the most powerful of the studied numerical methods (Runge-Kutta method) to find the numerical approximation solution for the movement of a satellite.

Last but not least, the degree of difficulty increases in order to reach our principal objective, i.e. to find a solution to my research question. A Figure showing the region where the third body can live given the conditions that I stated previously has been developed by myself launching the satellite from three different points at the surface of Earth; including a final table comparing all the results obtained. All in all, our objectives have been accomplished ideally.

Passing to the second part of my conclusion and talking about my impressions after the realisation of this project, the principal difficulty that I have come up with has been the level of mathematics required to be able to understand the bibliography. Due to the lack

of previous mathematical knowledge, a big part of my work has consisted on the learning of calculus, which I have included in a summarised way. Another important difficulty has been that the books that I read showed an incomplete reasoning in the sense that the results of the theorems and some indications to understand the proofs were exposed but these were not explained in a detailed way anywhere. Contrarily, I have tried to structure and make this paper a little more easier to understand so that the propositions and the lemmas appeared always with their respective reasoning and proofs. This has been really complicated because a lot of the proofs have been elaborated by myself, and I have managed to widely develop a lot of images so that the understanding of the concepts was clear.

I have found necessary to remark that not only has it provided me with a huge number of concepts about which I had literally no idea such as the eccentricity, differential equations, or Runge-Kutta method; but I have also travelled through history and realised how men could manage to solve really complex questions with three resources: a pencil, a paper and their privileged brains. Reading and learning about the 3BP I learnt, of course, the mathematical part, but I also noticed how all the authors that I have mentioned at appendix B and a lot more that I haven't named have been working on this unsolvable problem for centuries.

It has impressed me a lot the fact that in the last chapter I can't hardly find a number keeping in mind that this is a mathematical exploration. Nevertheless, all the figures and me being able to understand and comment them have required a hard and solid mathematical background. So I would refer as my fifth chapter, which, as I have said before, is the part where me and my knowledge have played a role; as the result and the culmination of a huge task of research and understanding, shown in the rest of the chapters.

All in all, the realisation of my research project has given me a completely different vision of mathematics, a lot more deductive, abstract and formal. Finally, one of the key ideas that can be extracted from the well-known *Dunning-Kruger effect* (Nobel Price in psychology in 2000) is that the more that we learn about something, the more quickly we discover a world of subtlety about things that we have yet to know. This is the main conclusion to which I have arrived after doing this research project.

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